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THE EXISTENCE OF QUASI REGULAR AND BI-REGULAR SELF-COMPLEMENTARY 3-UNIFORM HYPERGRAPHS

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Abstract

A k-uniform hypergraph $H = (V; E)$ is called self-complementary if there is a permutation $\sigma : V \to V$, called a complementing permutation, such that for every k-subset e of V, $e \in E$ if and only if $\sigma(e) \notin E$. In other words, H is isomorphic with $H' = (V; V^{(k)} - E)$. In this paper we define a bi-regular hypergraph and prove that there exists a bi-regular self-complementary 3-uniform hypergraph on n vertices if and only if n is congruent to 0 or 2 modulo 4. We also prove that there exists a quasi regular self-complementary 3-uniform hypergraph on n vertices if and only if n is congruent to 0 modulo 4.

Keywords: self-complementary hypergraph, uniform hypergraph, regular hypergraph, quasi regular hypergraph, bi-regular hypergraph.

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1. Introduction

Sachs [8] and Ringel [7] proved that a graph of order n is self-complementary if and only if n is congruent to 0 or 1 modulo 4. They also proved that a regular graph of order n is self-complementary if and only if n is congruent to 1 modulo 4.

Szymański and Wojda [9] proved that "A self-complementary 3-uniform hypergraph of order n exists if and only if n is congruent to 0 or 1 or 2 modulo 4."

Potočnik, and Šajana [6] raised the following question strengthening Hartman's conjecture [2, 3] about the existence of large sets of (not necessarily isomorphic) designs.

Question [6]**.** *Is it true that for every triple of integers $t < k < n$ such that $\binom{n-i}{k-i}$ is even for all $i = 0, \ldots, t$, there exists a self-complementary t-subset-regular k-uniform hypergraph of order n?*

The answer to the above question is affirmative for $k = 2$ and $t = 1$ (see [8]). The answer was proved affirmative also for the case $k = 3$ and $t = 1$ (see [6]). And in [4] it is shown that the answer to the question above is affirmative for the remaining case of 3-uniform hypergraphs, namely for the case $k = 3$, $t = 2$.

In this paper we digress a little from the case $k = 3$ and $t = 1$ to prove that a quasi-regular self-complementary 3-uniform hypergraph of order n exists if and only if $n \geq 4$ and n is congruent to 0 modulo 4, and a bi-regular self-complementary 3-uniform hypergraph of order n exists if and only if n is congruent to 0 or 2 modulo 4.

2. Preliminary Definitions and Results

Definition (k-uniform hypergraph)**.** Let V be a finite set with n vertices. By $V^{(k)}$ we denote the set of all k-subsets of V. A k-uniform hypergraph is a pair $H = (V; E)$, where $E \subset V^{(k)}$. V is called the vertex set, and E the edge set of H.

Definition (Degree of a vertex)**.** The degree of a vertex v in a hypergraph H is the number of edges containing the vertex v and is denoted as $d_H(v)$.

Definition (Regular hypergraph)**.** A hypergraph H is said to be regular if all vertices have the same degree.

Definition (Bi-regular hypergraph)**.** A hypergraph H is said to be bi-regular if there exist two distinct positive integers d_1 and d_2 such that the degree of each vertex is either d_1 or d_2.

Definition (Quasi regular hypergraph)**.** A hypergraph H is said to be quasi regular if the degree of each vertex is either r or $r - 1$ for some positive integer r.

It is clear that every quasi regular hypergraph is bi-regular.

Definition (Self-complementary k-uniform hypergraph)**.** A k-uniform hypergraph $H = (V; E)$ is called self-complementary if there exists a permutation $\sigma : V \to V$, called a complementing permutation, such that for every k-subset e of V, $e \in E$ if and only if $\sigma(e) \notin E$.

In other words, H is isomorphic to $H' = (V; V^{(k)} - E)$.

Definition (Tournament)**.** A tournament is a directed graph (V, A) with the property that for all pairs of distinct vertices $u, v \in V$, either $(u, v) \in A$ or $(v, u) \in A$.

Further, a tournament is said to be *self-converse* if there exists a bijection $\varphi : V \to V$ such that for all distinct $u, v \in V$, we have $(u, v) \in A$ if and only if $(\varphi(u), \varphi(v)) \notin A$.

Kocay [5] proved the following result on complementing permutations of self-complementary 3-uniform hypergraphs.

Proposition 1 [5]**.** *A permutation σ is a complementing permutation of a self-complementary 3-uniform hypergraph if and only if*

(i) *every cycle of σ has even length, or*

(ii) *σ has 1 or 2 fixed points, and the length of all other cycles is a multiple of 4.*

Szymański and Wojda [9] proved the following result on the order of a self-complementary uniform hypergraph.

Proposition 2 [9]**.** *Let k and n be positive integers, $k \leq n$. A k-uniform self-complementary hypergraph of order n exists if and only if $\binom{n}{k}$ is even.*

Remark 3. For 3-uniform self-complementary hypergraph the Proposition 2 can be stated as "A 3-uniform self-complementary hypergraph of order n exists if and only if $n \equiv 0$ or 1 or 2 (mod 4).

The following remark is obvious and hence is stated without proof.

Remark 4. If H is a self-complementary 3-uniform hypergraph of order n with complementing permutation σ, then

(i) for any vertex v in H, $d_H(v) + d_H(\sigma(v)) = \binom{n-1}{2}$,

(ii) for any vertex v in H, $d_H(v) = d_H(\sigma^2(v)) = d_H(\sigma^4(v)) = \cdots$ and
$$d_H(\sigma(v)) = d_H(\sigma^3(v)) = d_H(\sigma^5(v)) = \cdots$$

Further, if x is a fixed point of σ, then $d_H(x) = \frac{1}{2}\binom{n-1}{2}$.

Lemma 5. *If H is a self-complementary 3-uniform hypergraph on n vertices, where n is congruent to 1 modulo 4 and $n \geq 5$, then H cannot be bi-regular.*

Proof. Let H be a self-complementary 3-uniform hypergraph on n vertices where n is congruent to 1 modulo 4, i.e., $n = 4m + 1$, $m \in \mathbb{N}$. Let $\sigma : V(H) \to V(H)$ be its complementing permutation. By Proposition 1, σ necessarily has one fixed point, say x.

From Remark 4(ii) $d_H(x) = m(4m - 1)$. For H to be bi-regular either $d_1 = m(4m-1)$ or $d_2 = m(4m-1)$. Without loss of generality let $d_1 = m(4m-1)$. Since there are only two types of degrees d_1 and d_2, for any other vertex v, $d_v(H)$ is d_1 or d_2. By Remark 4(i) we have, $d_1 + d_2 = \frac{4m(4m-1)}{2}$ which gives $d_2 = 2m(4m-1) - m(4m-1) = m(4m-1) = d_1$. Hence H cannot be bi-regular. ■

3. Existence of a Quasi Regular and Bi-Regular Self-Complementary 3-Uniform Hypergraph

The following theorem gives a necessary and sufficient condition on the order n of a quasi regular self-complementary 3-uniform hypergraph. This theorem actually gives a construction of a quasi regular self-complementary 3-uniform hypergraph of desirable order.

Theorem 6. *There exists a quasi regular self-complementary* 3*-uniform hypergraph of order* n *if and only if* $n \geq 4$ *and* $n \equiv 0 \pmod 4$*.*

Proof. Let H be a quasi regular self-complementary 3-uniform hypergraph on n vertices such that degree of each vertex is either r or $r - 1$ for some positive integer r.

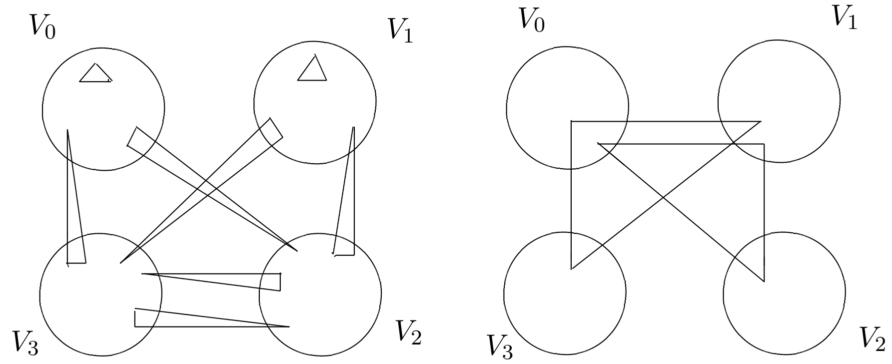

Figure 1. The types of triples making up the edge set of a quasi regular self-complementary 3-uniform hypergraph on $n = 4m$ vertices.

Let $\sigma : V(H) \to V(H)$ be a complementing permutation of H. By Proposition 1, σ has (i) every cycle of even length, or (ii) 1 or 2 fixed points and the length of all the other cycles is a multiple of 4. By Remark 3, we know that a self-complementary 3-uniform hypergraph exists if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, or $n \equiv 2 \pmod 4$. Lemma 5 shows that n is not congruent to 1 modulo 4.

If $n \equiv 2 \pmod 4$, i.e., $n = 4m + 2,\ m \in \mathbb{N}$, then either σ has 2 fixed points and the length of all other cycles is a multiple of 4 or σ has all cycles of even length.

If σ has 2 fixed points, then both must have the same degree and for some other vertex v, $d_H(v) \neq d_H(\sigma(v))$ otherwise H will be regular. Since there are only two possible degrees r and $r-1$, from Remark 4 we get that $r + r - 1 = \binom{n-1}{2} = \binom{4m+1}{2}$, i.e., $2r - 1 = 2m(4m + 1)$, a contradiction.

If σ has all cycles of even length, then again we get the same contradiction.

Hence, if there exists a quasi regular self-complementary 3-uniform hypergraph on n vertices, then $n \equiv 0 \pmod 4$.

For the converse, we construct a quasi regular self-complementary 3-uniform hypergraph on n vertices where $n \equiv 0 \pmod 4$.

Let m be a positive integer such that $n = 4m$ and $V = V_0 \cup V_1 \cup V_2 \cup V_3$, where $V_i = \{v_j^i : j \in \mathbb{Z}_m\}$, $i \in \mathbb{Z}_4$.

For every pairwise distinct triple $i, i', i'' \in \mathbb{Z}_4$ we define the following subsets of $V^{(3)}$:

$$\begin{aligned} E_i &= V_i^{(3)}, \\ E_{(i,i')} &= \{\{v_{j_1}^i, v_{j_2}^i, v_{j'}^{i'}\} : j_1, j_2, j' \in \mathbb{Z}_m, j_1 \neq j_2\}, \\ E_{(i,i',i'')} &= \{\{v_j^i, v_{j'}^{i'}, v_{j''}^{i''}\} : j, j', j'' \in \mathbb{Z}_m\}. \end{aligned}$$

Let us denote

$$E = E_0 \cup E_1 \cup E_{(2,1)} \cup E_{(2,3)} \cup E_{(3,0)} \cup E_{(3,2)} \cup E_{(1,3)} \cup E_{(0,2)} \cup E_{(0,1,3)} \cup E_{(0,1,2)}.$$

Let H be the 3-uniform hypergraph with vertex set V and edge set E. Figure 1 explains the construction of the hypergraph H. We show that H is quasi regular. Take any vertex v_j^i.

Case (i) If $i \in \{0, 1\}$, then the vertex v_j^i lies in $\binom{m-1}{2}$ triples of E_i, $(m-1)m$ triples of $E_{(i,i')}$, $\binom{m}{2}$ triples of $E_{(i',i)}$ and $2m^2$ triples of $E_{(i,i',i'')}$. Hence, for every vertex v_j^i in H with $i \in \{0, 1\}$, we have

$$d_H(v_j^i) = \binom{m-1}{2} + \binom{m}{2} + m(m-1) + 2m^2 = 4m^2 - 3m + 1.$$

Case (ii) If $i \in \{2, 3\}$, then the vertex v_j^i lies in $2(m-1)m$ triples of $E_{(i,i')}$, $2\binom{m}{2}$ triples of $E_{(i',i)}$ and m^2 triples of $E_{(i,i',i'')}$. Hence for every vertex v_j^i in H with $i \in \{2, 3\}$, we obtain

$$d_H(v_j^i) = 2(m-1)m + 2\binom{m}{2} + m^2 = 4m^2 - 3m.$$

Thus H is quasi regular with degrees $r = 4m^2 - 3m + 1$ and $r - 1 = 4m^2 - 3m$. To prove that H is self-complementary, we define a permutation $\phi : V \to V$ by $\phi(v_j^0) = v_j^3$, $\phi(v_j^1) = v_j^2$, $\phi(v_j^2) = v_j^1$ and $\phi(v_j^3) = v_j^0$, for all $j \in \mathbb{Z}_m$. Then ϕ is a complementing permutation of H and H is self-complementary. ■

In the next theorem we give a necessary and sufficient condition on the order n of a bi-regular 3-uniform hypergraph to be self-complementary. In this theorem we shall use the following result by Alspach [1] on existence of a regular self-converse tournament.

Theorem 7 (Alspach [1])**.** *There exists a regular self-converse tournament with n vertices for every odd integer n.*

Theorem 8. *There exists a bi-regular self-complementary 3-uniform hypergraph of order n if and only if either $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$ and $n \geq 4$.*

Proof. Necessity follows from Lemma 5 and Remark 3. Conversely, let $n \equiv 0 \pmod 4$. The self-complementary 3-uniform hypergraph constructed in Theorem 6 is quasi regular and hence biregular.

Let $n \equiv 2 \pmod 4$. Then $n = 4m + 2 = 2k$ where $k = 2m + 1$ is odd. Let $V = V_0 \cup V_1$, where $V_i = \{v_j^i : j \in \mathbb{Z}_k\}$, $i \in \mathbb{Z}_2$. By Theorem 7, there exists a regular self-converse tournament $T = (\mathbb{Z}_k, A)$.

For $i \in \mathbb{Z}_2$, we define the following subsets of $V^{(3)}$:

$$\begin{aligned} E_i &= V_i^{(3)}, \\ E_{(i,i+1)} &= \{\{v_{j_1}^i, v_{j_2}^i, v_j^{i+1}\} : j_1, j_2, j \in \mathbb{Z}_k, j_1, j_2, j \text{ pairwise distinct}\}, \\ E_A &= \{\{v_{k_1}^i, v_{k_2}^i, v_{k_1}^{i+1}\} : (k_1, k_2) \in A, i \in \mathbb{Z}_2\}. \end{aligned}$$

Let

$$E = E_0 \cup E_{(0,1)} \cup E_A.$$

Let H be the 3-uniform hypergraph with vertex set V and edge set E. Figure 2 explains the construction of the hypergraph H. We show that H is bi-regular. Let v_j^i be an arbitrary vertex of H.

Case (i) If $i = 0$, then the vertex v_j^0 lies in $\binom{k-1}{2}$ triples of E_0, $(k-1)(k-2)$ triples of $E_{(0,1)}$ and $\frac{3(k-1)}{2}$ triples of E_A. Hence

$$d_H(v_j^0) = \binom{k-1}{2} + (k-1)(k-2) + \frac{3(k-1)}{2} = \frac{3(k-1)^2}{2}.$$

Case (ii) If $i = 1$, then the vertex v_j^1 lies in $\binom{k-1}{2}$ triples of $E_{(0,1)}$, $\frac{3(k-1)}{2}$ triples of E_A. Therefore,

$$d_H(v_j^1) = \binom{k-1}{2} + \frac{3(k-1)}{2} = \frac{k^2-1}{2}.$$

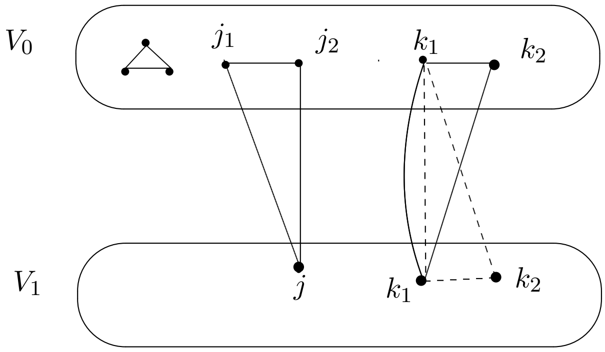

Figure 2. The types of triples making up the edge set of a bi-regular self-complementary 3-uniform hypergraph on $n = 4m+2$ vertices.

This proves that H is bi-regular with degrees $d_1 = \frac{3(k-1)^2}{2}$ and $d_2 = \frac{k^2-1}{2}$.

Let $\varphi : \mathbb{Z}_k \to \mathbb{Z}_k$ be an arc-reversing mapping of the tournament T; that is, φ is a bijection on $\mathbb{Z}_k$ such that $\varphi(a) \notin A$ for all $a \in A$.

To prove that H is self-complementary, we define a permutation $\phi : V \to V$ by $\phi(v_j^i) = v_{\varphi(j)}^{i+1}$ for $i \in \mathbb{Z}_2$ and $j \in \mathbb{Z}_k$. ϕ interchanges the sets E_1 and E_0, and also the sets $E_{(0,1)}$ and $E_{(1,0)}$. Furthermore, for all $(k_1, k_2) \in A$ and $i \in \mathbb{Z}_2$, since φ is arc-reversing, ϕ maps the triple $\{v_{k_1}^i, v_{k_2}^i, v_{k_1}^{i+1}\} \in E_A$ to the triple $\{v_{\varphi(k_1)}^{i+1}, v_{\varphi(k_2)}^{i+1}, v_{\varphi(k_1)}^i\} \notin E_A$. It follows that ϕ is a complementing permutation of H and therefore H is self-complementary. ■

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