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LARGE DEGREE VERTICES IN LONGEST CYCLES OF GRAPHS, I

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Abstract

In this paper, we consider the least integer d such that every longest cycle of a k-connected graph of order n (and of independent number α) contains all vertices of degree at least d.

Keywords: longest cycle, large degree vertices, order, connectivity, independent number.

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1. Introduction

1.1. Basic notation and terminology

All graphs considered here are simple and finite. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [2]. Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ and $d_H(v)$ to denote the set and the number of neighbors of v in H, respectively. We call $N_H(v)$ the *neighborhood* of v in H and $d_H(v)$ the *degree* of v in H. We use $d_H(u, v)$ to denote the distance between two vertices $u, v \in V(H)$ in H. For two subgraphs H and L of a graph G, we set $N_L(H) = \bigcup_{v \in V(H)} N_L(v)$. When no confusion occurs, we will denote $N_G(v)$ and $d_G(v)$ by $N(v)$ and $d(v)$, respectively. We set $N[x] = N(x) \cup \{x\}$.

Throughout this paper, we denote the order, the connectivity and the independent number of a graph G, by $n(G)$, $\kappa(G)$ and $\alpha(G)$, respectively.

1.2. Motivation and main results of this paper

By the definition every Hamilton cycle of a graph passes through every vertex of the graph. Thus, in non-Hamiltonian graphs, a (longest) cycle through some special vertices should be also interesting for the same topic. There are many results on the problem whether a graph has a (longest) cycle through some special vertices, for example, any given vertex set [8]; large degree vertices, see [1, 7, 9]. Unlike most research of the existence of some (longest) cycle passing through special vertices in the literature, we put our attention to the problem to determine the least integer d such that every longest cycle of a graph passes all vertices of degree at least d, using some additional conditions of order, of connectivity or of independence number.

The following known result gave a partly answer for the above problem.

Theorem 1 (Li and Zhang [6])**.** *Let G be a 2-connected graph of order $n \geq 8$. Then every longest cycle of G contains all vertices of degree at least $n - 4$.*

We firstly extend Theorem 1 to k-connected graphs for any $k \geq 2$ and shall give a complete answer for the above problem by using the order of a graph and its connectivity.

Theorem 2. *Let G be a graph of connectivity $\kappa(G) \geq k \geq 2$ and of order $n \geq 6k - 4$. Then every longest cycle of G contains all vertices of degree at least $n - 3k + 2$.*

The bound on the degree in Theorem 2 is sharp. We construct a graph as follows. Let $R = 2K_2 \cup (k-2)P_3$, $S = kK_1$ and $T = (n - 4k + 1)K_1$ are vertex-disjoint. Let R' be the subset of $V(R)$ each vertex of which is either a vertex of a K_2 or a center of a P_3 in R, and let s' be a fixed vertex of S and x a vertex not in $R \cup S \cup T$. Let $L(k,n)$ be the graph with $V(L(k,n)) = \{x\} \cup V(R) \cup V(S) \cup V(T)$, and $E(L(k,n)) = E(R) \cup \{r's', rs, s'x, sx, st, xt : r' \in R', r \in V(R), s \in V(S)\backslash\{s'\}, t \in V(T)\}$. One can check that $L(k,n)$ is k-connected and the degree of x is $n-3k+1$, but there is a longest cycle (in the subgraph induced by $V(R) \cup V(S)$) excluding x.

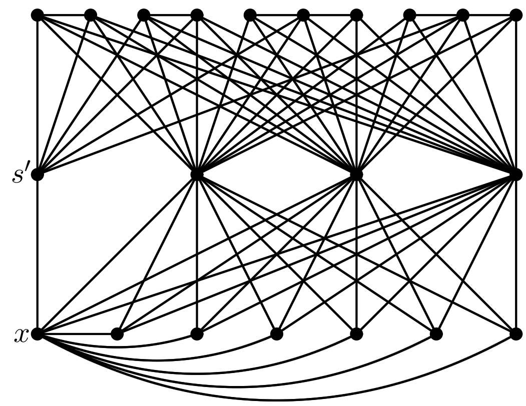

Figure 1. Graph $L(4,21)$.

The bound $n \geq 6k-4$ is also sharp. This can be seen from the complete bipartite graph $K_{3k-3,3k-2}$ of order $6k-5$. However, the longest cycles of $K_{3k-3,3k-2}$ exclude some vertices of degree $3k-3 = n-3k+2$.

Now we define $\varphi(k,n)$ to be the least integer such that every longest cycle of a k-connected graph G of order n contains all vertices of degree at least $\varphi(k,n)$ in G.

To avoid the discussions of the petty cases, we put our considerations on 2-connected graphs, i.e., we always assume that $k \geq 2$. Note that if $n \leq k$, then there are no k-connected graphs of order n. Hence $\varphi(k,n)$ will be meaningless. Is $\varphi(k,n)$ well-defined for all pairs (k,n) with $n \geq k+1$? No. Under the condition that it holds "every k-connected graph on n vertices is Hamiltonian" (e.g., $n = k+1$), $\varphi(k,n)$ does not exist (or we may say $\varphi(k,n) = -\infty$). So we should take the pair (k,n) such that there exist some k-connected graphs of order n which are not Hamiltonian. This implies that $n \geq 2k+1$ from the well-known Dirac's theorem [4]. On the other hand, there indeed exist such graphs when $n \geq 2k+1$ (for example, complete bipartite graphs $K_{k,n-k}$). So $\varphi(k,n)$ is well-defined if and only if $n \geq 2k+1$.

From Theorem 2 and the construction of $L(k,n)$, we have

$$\varphi(k,n) = n - 3k + 2, \text{ for } n \geq 6k-4.$$

How about the cases when $2k+1 \le n \le 6k-5$? First we construct a graph as follows: if n is odd, then let $L(k,n) = K_{(n-1)/2,(n+1)/2}$; if n is even, then let $L(k,n) = K_{n/2-1,n/2+1}$. Note that every longest cycle of $L(k,n)$ excludes some vertices of degree $\lceil n/2 \rceil - 1$. This shows that $\varphi(k,n) \ge \lceil n/2 \rceil$. On the other hand, we have the following result (one may compare it with the results in [1] and [9] where they replaced "every cycle" with "there exists some cycle" under the condition that "G is 2-connected").

Theorem 3. *Let G be a k-connected graph on $n \le 6k-5$ vertices. Then every longest cycle of G contains all vertices of degree at least $\lceil n/2 \rceil$.*

Instead of Theorems 2 and 3, we shall prove the following theorem in Section 3.

Theorem 4. *Let G be a graph of connectivity $\kappa(G) \ge k \ge 2$ and of order $n \ge 2k+1$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \max\left\{ \left\lceil \frac{n}{2} \right\rceil, n-3k+2 \right\}.$$

Now we have a complete formula

$$\varphi(k,n) = \max\left\{ \left\lceil \frac{n}{2} \right\rceil, n-3k+2 \right\}, \text{ for all } n \ge 2k+1.$$

In the following we consider the same problem by using an additional condition of independent number. We use $\varphi(k,\alpha,n)$ to denote the least integer such that for every k-connected graph G of order n and of independent number α, every longest cycle of G contains all vertices of degree at least $\varphi(k,\alpha,n)$. As the analysis above, we should take the triple (k,α,n) such that there exists a k-connected graph of order n and independent number α that is not Hamiltonian. This requires $\alpha \ge k+1$ from Chvátal-Erdös's theorem [3]; and $\alpha \le n-k$, since every k connected graph of order n has independent number at most $n-k$ (note that an independent set excludes the k neighbors of some vertex). On the other hand, for triple (k,α,n) with $k+1 \le \alpha \le n-k$, the graph $kK_1 \vee ((\alpha-1)K_1 \cup K_{n-k-\alpha+1})$ is a k-connected graph of order n and independent number α that is not Hamiltonian. Thus $\varphi(k,\alpha,n)$ is well-defined if and only if $k+1 \le \alpha \le n-k$.

By the definition of $\varphi(k,n)$, we can see that

$$\varphi(k,n) = \max\{\varphi(k,\alpha,n) : k+1 \le \alpha \le n-k\}, \text{ for all } n \ge 2k+1.$$

Using a result in [10], we can prove the following result.

Theorem 5. *Let G be a k-connected graph of order n and of independent number α. Then every longest cycle of G contains all vertices of degree more than*

$$d_0 = \frac{(\alpha-k)n - k\alpha + k^2 + \alpha^2 - 2\alpha}{\alpha}.$$

Taking $\alpha = k+1$ in the above theorem, we can obtain the following correspondence.

Theorem 6. *Let G be a graph of connectivity $\kappa(G) \geq k \geq 2$, of order $n \geq 2k+1$ and of independent number $k+1$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1.$$

The bound on d in Theorem 6 is sharp. We construct a graph $L(k, k+1, n)$ by joining each vertex of $R = kK_1$ to all vertices of $S = rK_{q+1} \cup (k+1-r)K_q$, where

$$n - k = q(k+1) + r,\ 0 \leq r \leq k.$$

Note that $L(k, k+1, n)$ has a longest cycle excluding some vertices of degree

$$q + k - 1 = \left\lfloor \frac{n-k}{k+1} \right\rfloor + k - 1 = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 2.$$

By Theorem 6, the above equality implies that

$$\varphi(k, k+1, n) = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1, \text{ for all } n \geq 2k+1.$$

Thus, in the following we will assume that $\alpha \geq k+2$. For the case $k = 2$, we have the following result.

Theorem 7. *Let G be a 2-connected graph of order $n \geq 8$ and independent number $\alpha \geq 4$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \left\lfloor \frac{n-5}{\alpha} \right\rfloor (\alpha - 2) + \max\left\{3, n - 4 - \left\lfloor \frac{n-5}{\alpha} \right\rfloor \alpha\right\}$$

i.e.,

$$d = \begin{cases} q(\alpha-2)+3, & 0 \leq r \leq 2, \\ q(\alpha-2)+r+1, & 3 \leq r < \alpha, \end{cases}$$

where

$$n - 5 = q\alpha + r,\ 0 \leq r < \alpha.$$

The bound on d in Theorem 7 is sharp when $q \geq 1$ (i.e., when $n \geq \alpha + 5$). We construct extremal graphs as follows. If $0 \leq r \leq 2$, then let $R = rK_{q+2} \cup (2-r)K_{q+1}$ and $T = (\alpha-2)K_q$; if $3 \leq r < \alpha$, then let $R = 2K_{q+2}$ and $T = (r-2)K_{q+1} \cup (\alpha - r)K_q$. Let s', s, x be three vertices not in $R \cup T$. Let $L(2, \alpha, n)$

be a graph with the vertex set $V(L(2,\alpha,n)) = \{s', s, x\} \cup V(R) \cup V(T)$ and the edge set

$$E(L(2,\alpha,n)) = E(R) \cup E(T) \cup \{s'r, sr, s'x, sx, st, xt : r \in V(R), t \in V(T)\}.$$

One can check that $L(2,\alpha,n)$ is a 2-connected graph of order n and of independent number α, and x has degree $d-1$. But there is a longest cycle of G excluding x. By Theorem 7, this implies that for $n \geq \alpha + 5$,

$$\varphi(2,\alpha,n) = \begin{cases} q(\alpha-2)+3, & 0 \leq r \leq 2, \\ q(\alpha-2)+r+1, & 3 \leq r < \alpha, \end{cases}$$

where

$$n - 5 = q\alpha + r, 0 \leq r < \alpha.$$

For the case $q = 0$, the above construction does not give the exact value of $\varphi(2,\alpha,n)$, since the independent number of the constructed graph is less than α. What is its exact values for this case?

Note that $n \leq \alpha+4$ in this case. Also note that in our assumption $n \geq \alpha+2$. We have three cases: $n = \alpha+2$, $n = \alpha+3$ and $n = \alpha+4$.

Theorem 8. *Let G be a 2-connected graph of independent number $\alpha \geq 4$ and of order n such that $\alpha+2 \leq n \leq \alpha+4$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \begin{cases} n-\alpha+1, & n-\alpha = 2, 3, \\ \alpha, & n-\alpha = 4. \end{cases}$$

Now we will show the sharpness of the bound in Theorem 8. For the case $n = \alpha+2$, consider the graph $L(2,\alpha,\alpha+2) = K_{2,\alpha}$. Note that every longest cycle of $L(2,\alpha,\alpha+2)$ excludes some vertices of degree 2.

For the case $n = \alpha+3$, consider the graph $L(2,\alpha,\alpha+3) = K_{3,\alpha}$. Note that every longest cycle of $L(2,\alpha,\alpha+3)$ excludes some vertices of degree 3.

Now we consider the case $n = \alpha+4$. We construct the graph $L(2,\alpha,\ \alpha+4)$ by combining a cycle C_6 and a star $K_{1,\alpha-3}$ in such a way: choosing two vertices u, v in C_6 with distance 3, joining the center x of the star to u and v, and joining all the end-vertices of the star to v. Note that one longest cycle of $L(2,\alpha,\alpha+4)$ excludes x of degree $\alpha-1$.

Therefore, we give formulas for $\alpha \geq 4$,

$$\varphi(2,\alpha,\alpha+2) = 3,$$
$$\varphi(2,\alpha,\alpha+3) = 4,$$
$$\varphi(2,\alpha,\alpha+4) = \alpha.$$

The bound of d_0 in Theorem 5 seems not sharp for $\alpha \geq k+2$ (at least it is not sharp when $k = 2$). We propose the following conjecture.

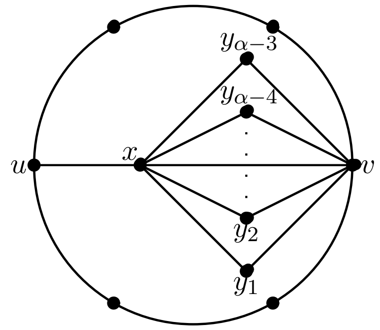

Figure 2. Graph $L(2, \alpha, \alpha + 4)$.

Conjecture 9. *Let G be a k-connected graph, $k \geq 3$, of independent number $\alpha \geq k + 2$ and of order $n \geq \max\{2\alpha + 1, \alpha + 3k + 1\}$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \begin{cases} q(\alpha - k) + k + 1, & 0 \leq r \leq k, \\ q(\alpha - k) + k + 2, & k + 1 \leq r \leq 2k + 1, \\ q(\alpha - k) + r - k + 1, & 2k + 2 \leq r < \alpha + k, \end{cases}$$

where

$$n - 2k - 1 = q(\alpha + k) + r,\ 0 \leq r < \alpha + k.$$

We remark that if the conjecture is true, then the bound on d is sharp. We construct a graph as follows. If $0 \leq r \leq k$, then let $R = rK_{2q+2} \cup (k-r)K_{2q+1}$ and $T = (\alpha-k)K_q$; if $k+1 \leq r \leq 2k+1$, then let $R = (r-k-1)K_{2q+3} \cup (2k+1-r)K_{2q+2}$ and $T = K_{q+1} \cup (\alpha - k - 1)K_q$; if $2k + 2 \leq r < \alpha + k$, then let $R = kK_{2q+3}$ and $T = (r - 2k)K_{q+1} \cup (\alpha + k - r)K_q$, and let $S = kK_1$. Let x be a vertex not in $R \cup S \cup T$. Let $L(k, \alpha, n)$ be a graph with $V(L(k, \alpha, n)) = \{x\} \cup V(R) \cup V(S) \cup V(T)$ and

$$E(L(k, \alpha, n)) = E(R) \cup E(T) \cup \{sr, sx, st, xt : r \in V(R), s \in V(S), t \in V(T)\}.$$

One can check that $L(k, \alpha, n)$ is a 2-connected graph of order n and of independent number α, and x has degree $d - 1$. But there is a longest cycle of G excluding x.

2. Preliminaries

Let G be a graph and $x, y \in V(G)$. An x-path is a path with x as one of its end vertices; an (x, y)-path is one connecting x and y. If Y is a subset of $V(G)$, then an (x, Y)-path is one connecting x and a vertex in Y with all internal vertices in

$V(G)\backslash Y$; a Y-path is one connecting two vertices in Y with all internal vertices in $V(G)\backslash Y$. For a subgraph H of G, we use the notations (x,H)-path and H-path instead of $(x,V(H))$-path and $V(H)$-path, respectively. It is convenient to denote a path P with end-vertices x,y by $P(x,y)$.

For a cycle C with a given orientation and a vertex x on C, we use x^+ to denote the successor, and x^- the predecessor of x on C. In the following, we always assume that C has an orientation, $\overrightarrow{C}$. For two vertices x,y on C, $\overrightarrow{C}[x,y]$ or $\overleftarrow{C}[y,x]$ denotes the path from x to y along $\overrightarrow{C}$. Similarly, if x,y are two vertices in a path P, $P[x,y]$ denotes the subpath of P between x and y. For an arbitrary path P or cycle C, we use $l(P)$ or $l(C)$ to denote the length (the number of edges) of it.

We first give some lemmas on longest cycles of graphs.

Lemma 10. *Let C be a longest cycle of a graph G, and $P=P(u,v)$ be a C-path. Then $l(\overrightarrow{C}[u,v])\geq l(P)$.*

Proof. Otherwise, $\overrightarrow{C}[v,u]uPv$ is a cycle longer than C, a contradiction. ■

Lemma 10 can be extended to the following.

Lemma 11. *Let C be a longest cycle of a graph G, H be a component of $G-C$ and $P=P(u,v)$ be a C-path of length at least 2 with all internal vertices in H. Then*

$$l\left(\overrightarrow{C}[u,v]\right)\geq l(P)+2\left|N_C(H)\cap V\left(\overrightarrow{C}[u^+,v^-]\right)\right|.$$

Proof. We use induction on $|N_C(H)\cap V(\overrightarrow{C}[u^+,v^-])|$. If $N_C(H)\cap V(\overrightarrow{C}[u^+,v^-]) =\emptyset$, then we are done by Lemma 10. Now we suppose that $N_C(H)\cap V(\overrightarrow{C}[u^+,v^-]) \neq\emptyset$.

Let x be a vertex in $N_C(H)\cap V(\overrightarrow{C}[u^+,v^-])$. Let $P'=P'(x,x')$ be an $(x,P-\{u,v\})$-path with all internal vertices in $H-P$. Then $P_1=P[u,x']x'P'$ and $P_2=P'x'P[x',v]$ are two C-paths with end-vertices u,x and x,v, respectively, and with all internal vertices in H. Clearly, the length of P_1 and P_2 are at least 2. By induction hypothesis,

$$l\left(\overrightarrow{C}[u,x]\right)\geq l(P_1)+2\left|N_C(H)\cap V\left(\overrightarrow{C}[u^+,x^-]\right)\right|,$$

$$l\left(\overrightarrow{C}[x,v]\right)\geq l(P_2)+2\left|N_C(H)\cap V\left(\overrightarrow{C}[x^+,v^-]\right)\right|.$$

Note that $l(P_1)+l(P_2)=l(P)+2l(P')\geq l(P)+2$, and

$$\begin{aligned}\left|N_C(H)\cap V\left(\overrightarrow{C}[u^+,x^-]\right)\right|+\left|N_C(H)\cap V\left(\overrightarrow{C}[x^+,v^-]\right)\right|&\\ =\left|N_C(H)\cap V\left(\overrightarrow{C}[u^+,v^-]\right)\right|-1.&\end{aligned}$$

We have

$$l\left(\overrightarrow{C}[u,v]\right) = l\left(\overrightarrow{C}[u,x]\right) + l\left(\overrightarrow{C}[x,v]\right) \geq l(P) + 2\left|N_C(H) \cap V\left(\overrightarrow{C}[u^+,v^-]\right)\right|.$$

The assertion is proved. ■

Lemma 12. *Let G be a graph, C be a longest cycle of G and H be a component of $G - C$.*

(1) *If $u \in N_C(H)$, then $u^+, u^- \notin N_C(H)$.*

(2) *If $u, v \in N_C(H)$, then $u^+v^+, u^-v^- \notin E(G)$.*

Proof. The assertion (1) can be deduced from Lemma 10. Now we prove the assertion (2).

Suppose that $u, v \in N_C(H)$. Then let P be a (u,v)-path of length at least 2 with all internal vertices in H. If $u^+v^+ \in E(G)$, then

$$C' = \overrightarrow{C}[u^+, v]vPu\overleftarrow{C}[u, v^+]v^+u^+$$

is a cycle longer than C, a contradiction. Thus we conclude that $u^+v^+ \notin E(G)$, and similarly, $u^-v^- \notin E(G)$. ■

Let G be a graph and $yz \in E(G)$, we define the *contraction of G at yz*, denoted by $G \cdot yz$, as the graph with $V(G \cdot yz) = V(G) \backslash \{y\}$, and $E(G \cdot yz) = E(G - y) \cup \{xz : xy \in E(G) \text{ and } x \neq z\}$.

Lemma 13. *Let G be a graph and $yz \in E(G)$. If there is a cycle C in $G \cdot yz$, then there is a cycle C' in G with length at least $l(C)$ such that $V(C') \subseteq V(C) \cup \{y\}$.*

Proof. If C does not contain z, then C is also a cycle of G and we are done. So we assume that $z \in V(C)$. By the definition of contraction, $zz^+ \in E(G)$ or $yz^+ \in E(G)$, and $zz^- \in E(G)$ or $yz^- \in E(G)$. Let

$$C' = \begin{cases} C, & \text{if } zz^+ \in E(G) \text{ and } zz^- \in E(G), \\ \overrightarrow{C}[z, z^-]z^-yz, & \text{if } zz^+ \in E(G) \text{ and } zz^- \notin E(G), \\ zyz^+\overrightarrow{C}[z^+, z], & \text{if } zz^+ \notin E(G) \text{ and } zz^- \in E(G), \\ yz^+\overrightarrow{C}[z^+, z^-]z^-y, & \text{if } zz^+ \notin E(G) \text{ and } zz^- \notin E(G). \end{cases}$$

Then C' is a required cycle. ■

We will use the following theorems from [3, 10].

Theorem 14 (Chvátal and Erdös [3])**.** *If G is a graph of order $n \geq 3$ with $\alpha(G) \leq \kappa(G)$, then G is Hamiltonian.*

Theorem 15 (O, West and Wu [10])**.** *If G is a graph of order n with $\alpha(G) \geq \kappa(G)$, then G has a cycle of length at least*

$$\frac{\kappa(G)(n+\alpha(G)-\kappa(G))}{\alpha(G)}.$$

Theorem 16 (O, West and Wu [10])**.** *If G is separable, then $V(G)$ admits a partition (V_1, V_2) such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$.*

Now we prove some more lemmas.

Lemma 17. *Let G be a nonseparable graph. Then for any two distinct vertices u, v of G, G contains a (u,v)-path of order at least $\lceil n(G)/\alpha(G)\rceil$.*

Proof. If G is complete, then the result is trivially true. Now we assume that G is not complete, i.e., $\alpha(G) \geq 2$. So G is 2-connected. If $\alpha(G) \leq \kappa(G)$, then by Theorem 14, G has a Hamilton cycle C, and either $\overrightarrow{C}[u,v]$ or $\overleftarrow{C}[u,v]$ is a required path. Now we assume that $\alpha(G) > \kappa(G)$.

Let C be a longest cycle of G. By Theorem 15, $l(C) \geq 2n(G)/\alpha(G)$. Since G is 2-connected, we may choose a (u,C)-path P_1 and a (v,C)-path P_2 such that they are vertex-disjoint. Let u' and v' be the end-vertices of P_1 and P_2, respectively, on C (possibly $u = u'$ or $v = v'$, or both). Then $P_1u'\overrightarrow{C}[u',v']v'P_2$ or $P_1u'\overleftarrow{C}[u',v']v'P_2$ is a required path. ∎

Lemma 18. *Let G be a 2-connected graph. Let C be a subgraph of G with at least two vertices, and H be an induced subgraph of $G - C$. Then G contains a C-path P such that*

$$|V(P) \cap V(H)| \geq \left\lceil \frac{n(H)}{\alpha(H)} \right\rceil.$$

Proof. We use induction on $n(H)$. If H has only one vertex, say x, then $n(H) = \alpha(H) = 1$. Since G is 2-connected, there is a C-path passing through x, which is a required path. Now we assume that H has at least two vertices.

Suppose first that H is nonseparable. Let $P_1(u,u')$ and $P_2(v,v')$ be two vertex-disjoint paths between H and C with all internal vertices in $G-(H \cup C)$, where $u, v \in V(H)$ and $u', v' \in V(C)$. By Lemma 17, H contains a (u,v)-path P' of order at least $\lceil n(H)/\alpha(H)\rceil$. Thus $P = P_1uP'vP_2$ is a required path.

Now we suppose that H is separable. By Theorem 16, there is a partition (V_1, V_2) of $V(H)$ such that $\alpha(H) = \alpha(G[V_1]) + \alpha(G[V_2])$. Let $H_1 = G[V_1]$ and $H_2 = G[V_2]$. Note that $n(H) = n(H_1) + n(H_2)$. It is not hard to see that

$$\frac{n(H)}{\alpha(H)} \leq \max\left\{\frac{n(H_1)}{\alpha(H_1)}, \frac{n(H_2)}{\alpha(H_2)}\right\} \overset{\text{say}}{=\!=} \frac{n(H_1)}{\alpha(H_1)}.$$

By induction hypothesis, there is a C-path P such that

$$|V(P) \cap V(H)| \geq |V(P) \cap V(H_1)| \geq \left\lceil \frac{n(H_1)}{\alpha(H_1)} \right\rceil \geq \left\lceil \frac{n(H)}{\alpha(H)} \right\rceil,$$

and P is a required path. ■

Let L_1 be the graph obtained from C_6 by adding a new vertex x, and by adding three edges from x to three pairwise nonadjacent vertices of the C_6, and let $L_2 = K_3 \vee 4K_1$. Set

$$\mathcal{L} = \{G : L_1 \subseteq G \subseteq L_2\}.$$

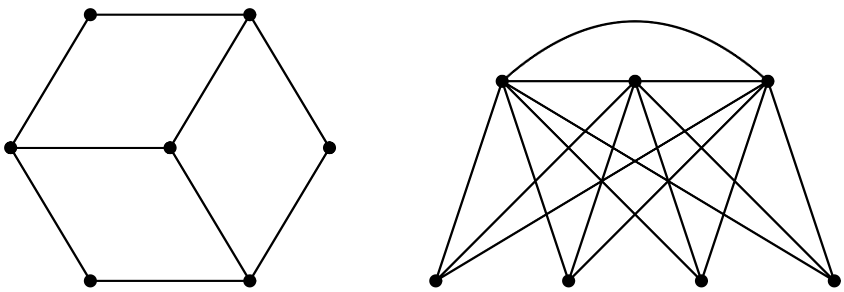

Figure 3. Graphs L_1 and L_2.

We prove the following lemma to complete this section.

Lemma 19. *Let G be a 2-connected graph with $n(G) \leq 7$ and x be a vertex of G with $d(x) \geq 3$. If there is a longest cycle of G excluding x, then $G \in \mathcal{L}$.*

Proof. Let C be a longest cycle of G excluding x and let H be the component of $G - C$ containing x.

We first suppose that H has at least two vertices. By the 2-connectedness of G, there are two independent edges from H to C. Let yy' and zz' be such two edges, where $y, z \in V(H)$ and $y', z' \in V(C)$. Let P be a (y, z)-path of H. Then $P' = y'yPzz'$ is a path with length at least 3. By Lemma 10, $l(\overrightarrow{C}[y', z']) \geq 3$ and $l(\overleftarrow{C}[y', z']) \geq 3$. Thus $l(C) \geq 6$. Note that $n(H) \geq 2$, we have $n(G) \geq 8$, a contradiction.

Thus we conclude that H has the only one vertex x. By Lemma 12, x cannot be adjacent to two successive vertices on C. Since $d(x) \geq 3$, there will be at least three vertices on C adjacent to x, and at least three vertices on C nonadjacent to x. Thus $l(C) \geq 6$. Since $n(G) \leq 7$ and $x \notin V(C)$, we have $l(C) \leq 6$. Thus C

has exactly 6 vertices and x is adjacent to three pairwise nonadjacent vertices of C. This implies that $L_1 \subseteq G$.

Let $C = y_1z_1y_2z_2y_3z_3y_1$ such that $N(x) = \{y_1, y_2, y_3\}$. By Lemma 12, $\{x, z_1, z_2, z_3\}$ is an independent set. This implies that $G \subseteq L_2$. ■

3. Proofs of main results

In this section, we shall present the proof of main results.

***Proof of Theorem* 4.** Let C, with an orientation $\overrightarrow{C}$, be a longest cycle of G. We assume on the contrary that there is a vertex x in $V(G-C)$ with $d(x) \geq d = \max\{\lceil n/2\rceil, n-3k+2\}$.

An (x,C)-fan is a collection of (x,C)-paths such that they have the only vertex x in common. Since G is k-connected, there is an (x,C)-fan with $s \geq k$ paths $P_i = P_i(x, z_i)$, $1 \leq i \leq s$, where $z_i \in V(C)$. We choose the (x,C)-fan such that s is as large as possible. We suppose that $z_1, z_2, \dots, z_s$ appear in this order along $\overrightarrow{C}$. Thus

$$l(C) = \sum_{i=1}^{s} l(\overrightarrow{C}[z_i, z_{i+1}]), \tag{1}$$

where the subscripts are taken modulo s.

By Menger's theorem, there is a vertex $y_i \in V(P_i - x)$ such that $S = \{y_i : 1 \leq i \leq s\}$ is a vertex-cut of G separating x and $C - S$. We choose y_i in such a way that $d_{P_i}(x, y_i)$ is as small as possible (note that y_i is possibly equal to z_i). Clearly

$$N_C(x) \subseteq S. \tag{2}$$

Let H be the component of $G - S$ containing x. Then

Claim 20. *For every vertex $y_i \in S$, either $N_H(y_i) = \{x\}$ or $|N_H(y_i)| \geq 2$.*

Proof. Suppose on the contrary that $|N_H(y_i)| = 1$ and $y_i' \neq x$ is the vertex in $N_H(y_i)$. Then y_i' is the neighbor of y_i on $P_i[x, y_i]$. Let $S' = (S\backslash\{y_i\}) \cup \{y_i'\}$. Then S' is a vertex-cut of G separating x and $C - S'$ such that $d_{P_i}(x, y_i') < d_{P_i}(x, y_i)$, contradicting the choice of S. □

If H has only one vertex x, then $d(x) = |S| = s$. By Lemma 12, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 2$ for all $i \in \{1, 2, \dots, s\}$. By (1), $l(C) \geq 2s = 2d(x)$ and

$$n \geq l(C) + 1 \geq 2d(x) + 1 \geq n + 1,$$

a contradiction.

If H has exactly two vertices, then let x' be the vertex in $V(H)\backslash\{x\}$. By Claim 20, every vertex y_i in S is adjacent to x. Hence $N(x) = S \cup \{x'\}$ and $s = d(x) - 1$. Note that $d(x') = d_S(x') + 1$ and $d(x') \geq k$, since G is k-connected. We have $d_S(x') \geq k - 1$. By Lemma 12, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 2$ for all i. Moreover, if $x'y_i \in E(G)$, then $P = P_i[z_i, y_i]y_i x' x y_{i+1} P_{i+1}[y_{i+1}, z_{i+1}]$ is a C-path of length at least 3, by Lemma 10, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 3$. This implies that $l(C) = \sum_{i=1}^{s} l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 3d_S(x') + 2(s - d_S(x')) \geq 2s + d_S(x') \geq 2d(x) + k - 3 \geq n + k - 3$, and $n \geq l(C) + 2 \geq n + k - 1 \geq n + 1$, a contradiction.

Now it remains to consider the case when H has at least three vertices.

By $b(x)$ we denote the number of vertices in $V(G)\backslash N[x]$. Then $b(x) = n - 1 - d(x) \leq 3k - 3$. Hence, by (2),

$$l(C) \leq s + b(x) \leq s + 3k - 3. \tag{3}$$

Claim 21. *Every vertex in $V(H)\backslash\{x\}$ is not a cut-vertex of H.*

Proof. Suppose, otherwise, that $x' \neq x$ is a cut-vertex of H. Let H_1 and H_2 be two components of $H - x'$ such that $x \in V(H_1)$.

We claim that for every vertex $y_i \in S$, $N_{H_1}(y_i) \neq \emptyset$. Otherwise, every (x, y_i)-path with all internal vertices in H will pass through x', so is $P_i[x, y_i]$. Let $S' = (S\backslash\{y_i\}) \cup \{x'\}$. Then S' is a vertex-cut of G separating x and $C - S'$ such that $d_{P_i}(x, x') < d_{P_i}(x, y_i)$, a contradiction. Thus as we claimed, $N_{H_1}(y_i) \neq \emptyset$.

For every $y_i \in S$, let w_i be a vertex in $N_{H_1}(y_i)$. Now we claim that $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 4$ for those i such that $N_{H_2}(y_i) \neq \emptyset$. Suppose $N_{H_2}(y_i) \neq \emptyset$. Let w'_i be a neighbor of y_i in H_2. Then H has a (w'_i, w_{i+1})-path P of length at least 2. Thus $P' = P_i[z_i, y_i]y_i w'_i P w_{i+1} y_{i+1} P_{i+1}[y_{i+1}, z_{i+1}]$ is a path of length at least 4 with all internal vertices in $G - C$. By Lemma 10, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 4$.

Note that $|N_S(H_2)| \geq k - 1$, since G is k-connected and $N_S(H_2) \cup \{x'\}$ is a vertex-cut. Therefore,

$$l(C) \geq 4(k-1) + 2(s - k + 1) = 2s + 2k - 2 \geq s + 3k - 2,$$

contradicting (3). $\square$

Claim 22. *H is a star with center x.*

Proof. Suppose, otherwise, H has an x-path $xx'x''$ (say) of length 2. Then there is an (x'', S)-fan with k internally disjoint paths $Q_i = Q_i(x'', y_{j_i})$, $1 \leq j_1 < j_2 < \cdots < j_k \leq s$, such that they have the only vertex x'' in common. We set $S' = \{y_{j_i} : 1 \leq i \leq k\}$.

Note that at most one path of Q_i passes through x. We will prove that $l(\overrightarrow{C}[z_{j_i}, z_{j_i+1}]) \geq 4$ for those j_i such that $y_{j_i} \in S'$ and Q_i does not pass through x.

Suppose that $y_{j_i} \in S'$ and Q_i does not pass through x. Let w_{j_i} be the neighbor of y_{j_i} on Q_i. Then $w_{j_i} \neq x$. If $l(Q_i) \geq 2$, then let v_{j_i} be a neighbor of w_{j_i} on the path $Q_i[x'', w_{j_i}]$; if $l(Q_i) = 1$, then ($w_{j_i} = x''$ and) we let $v_{j_i} = x'$. Then $v_{j_i} \neq x$. By Claim 20, y_{j_i+1} has a neighbor w'_{j_i+1} in H other that w_{j_i}. We claim that H has a (w_{j_i}, w'_{j_i+1})-path of length at least 2. Otherwise $w_{j_i}w'_{j_i+1} \in E(G)$ and $w_{j_i}w'_{j_i+1}$ is a cut-edge of H. By Claim 21, every vertex of $V(H)\backslash\{x\}$ is not a cut-vertex of H. This implies that $w'_{j_i+1} = x$ and w_{j_i} has only one neighbor x in H, contradicting the fact that $v_{j_i} \in N_H(w_{j_i})$ and $v_{j_i} \neq x$. Thus as we claimed, H has a (w_{j_i}, w'_{j_i+1})-path P of length at least 2. Thus $P' = P_{j_i}[z_{j_i}, y_{j_i}]y_{j_i}w_{j_i}Pw'_{j_i+1}y_{j_i+1}P_{j_i+1}[y_{j_i+1}, z_{j_i+1}]$ is a path of length at least 4 with all internal vertices in $G - C$. By Lemma 10, $l(\overrightarrow{C}[z_{j_i}, z_{j_i+1}]) \geq 4$.

Thus we conclude that there are at least $k-1$ segments $\overrightarrow{C}[z_i, z_{i+1}]$ of length at least 4. Hence

$$l(C) \geq 4(k-1) + 2(s-k+1) = 2s + 2k - 2 \geq s + 3k - 2,$$

a contradiction. □

By Claim 22, $H = K_{1,n(H)-1}$. Let

$$S_0 = \{y_i \in S : N_H(y_i) = \{x\}\}, \qquad S_2 = S\backslash(S_0 \cup S_1),$$
$$S_1 = \{y_i \in S : |N_H(y_i)\backslash\{x\}| = 1\}, \quad s_i = |S_i|, \; i \in \{0,1,2\}.$$

Thus $s = s_0 + s_1 + s_2$.

Let y_{j_i}, $1 \leq j_1 < j_2 < \cdots < j_{s_1+s_2} \leq s$, be the vertices in $S_1 \cup S_2$. Since G is k-connected,

$$s_1 + s_2 \geq |N_S(x')| \geq k-1 \tag{4}$$

for any $x' \in V(H)\backslash\{x\}$, and

$$s_1 + (n(H)-1)s_2 \geq |E(H-x, S)| \geq (k-1)(n(H)-1). \tag{5}$$

If $s_1 + s_2 = 1$, then without loss of generality we assume that $x'y_1 \in E(G)$, where $x' \in V(H)\backslash\{x\}$ and $y_1 \in S_1 \cup S_2$. Note that $\{x, y_1\}$ is a vertex-cut of G, implying that $k = 2$. Since $z_1P_1[z_1, y_1]y_1x'xy_2P_2[y_2, z_2]$ is a path of length at least 3, by Lemma 10, $l(\overrightarrow{C}[z_1, z_2]) \geq 3$ and by symmetry, $l(\overrightarrow{C}[z_1, z_s]) \geq 3$. Thus

$$l(C) \geq 3 + 3 + 2(s-2) = 2s + 2 > s + 3k - 3,$$

a contradiction. Now we conclude that $s_1 + s_2 \geq 2$.

Claim 23. *For every vertex $y_{j_i} \in S_1 \cup S_2$,*

$$l(\overrightarrow{C}[z_{j_i}, z_{j_{i+1}}]) \geq \begin{cases} 3 + 2|N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])|; & y_{j_i} \in S_1, \\ 4 + 2|N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])|; & y_{j_i} \in S_2, \end{cases}$$

where the subsubscripts are taken modulo $s_1 + s_2$.

Proof. For any $y_{j_i} \in S_1 \cup S_2$, we let w_{j_i} be a vertex in $N_H(y_{j_i}) \setminus \{x\}$. If $y_{j_i} \in S_1$, then by Claim 20, $y_{j_i}x \in E(G)$. Thus

$$P = P_{j_i}[z_{j_i}, y_{j_i}]y_{j_i}xw_{j_{i+1}}y_{j_{i+1}}P_{j_{i+1}}[y_{j_{i+1}}, z_{j_{i+1}}]$$

is a C-path of length at least 3. If $y_{j_i} \in S_2$, then let w'_{j_i} be a vertex in $N_H(y_{j_i}) \setminus \{x, w_{j_{i+1}}\}$. Thus $P = P_{j_i}[z_{j_i}, y_{j_i}]y_{j_i}w'_{j_i}xw_{j_{i+1}}y_{j_{i+1}}P_{j_{i+1}}[y_{j_{i+1}}, z_{j_{i+1}}]$ is a C-path of length at least 4. Note that $N_C(H) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-]) = N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])$. By Lemma 11, we have the assertion. □

Note that $\sum_{i=1}^{s_1+s_2} |N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])| = s_0$. By Claim 23,

$$l(C) = \sum_{i=1}^{s_1+s_2} l(\overrightarrow{C}[z_{j_i}, z_{j_{i+1}}]) \geq 2s_0 + 3s_1 + 4s_2 = 2s + s_1 + 2s_2.$$

By (4) and (5), we have

$$\begin{aligned} l(C) &\geq 2s + s_1 + 2s_2 = 2s + \frac{n(H)-3}{n(H)-2}(s_1 + s_2) + \frac{1}{n(H)-2}(s_1 + (n(H)-1)s_2) \\ &\geq 2s + \frac{n(H)-3}{n(H)-2}(k-1) + \frac{n(H)-1}{n(H)-2}(k-1) = 2s + 2k - 2 \geq s + 3k - 2, \end{aligned}$$

a contradiction.

The proof is complete. ■

Proof of Theorem 5. If $\alpha \leq \kappa(G)$, then G is Hamiltonian by Theorem 14 and we are done. Now suppose that $\alpha > \kappa(G)$. Let C be a longest cycle of G with an orientation, $\overrightarrow{C}$. Assume for contradiction that there exists a vertex x of degree more than d_0 such that $x \notin V(C)$. Let H be the component of $G - C$ containing x. Then $|N_C(H)| \geq k$, since G is k-connected. Let $N_C(H) = \{z_1, z_2, \ldots, z_s\}$, where $s = |N_C(H)|$. Hence

$$d(x) \leq |V(H - x)| + |N_C(H)| \leq n + s - l(C) - 1. \tag{6}$$

By Lemma 12, $\{x, z_1^+, z_2^+, \ldots, z_s^+\}$ is an independent set of G. Thus, we obtain that $s + 1 \leq \alpha$. Therefore, by (6) and by the hypothesis of $d(x) > d_0$ and by Theorem 15,

$$\begin{aligned} d_0 < d(x) \leq n + s - l(C) - 1 &\leq n + \alpha - 1 - \frac{\kappa(G)(n + \alpha - \kappa(G))}{\alpha} - 1 \\ &= n + \alpha - 2 - \frac{\kappa(G)(n + \alpha - \kappa(G))}{\alpha} = d_0, \end{aligned}$$

a contradiction. This completes the proof of Theorem 5. ■

In order to use the induction method, we prove the following stronger theorem instead of Theorem 7.

Theorem 24. *Suppose $\alpha \geq 4$ and $n \geq 3$ are two integers and d is defined as in Theorem 7. Let G be a 2-connected graph with $n(G) \leq n$ and $\alpha(G) \leq \alpha$. Then every longest cycle of G contains all the vertices of degree at least d, unless $G \in \mathcal{L}$.*

Proof. We use induction on $n(G)$. If G has only three or four vertices, then G is Hamiltonian and the result is trivially true. Now we assume that G has at least five vertices and assume that the assertion holds for all graphs with order less than $n(G)$. This implies that $n \geq 5$ and $q \geq 0$.

Suppose that $q = 0$. Then $r = n - 5$. If $n \leq 7$, then $r \leq 2$ and $d = 3$. By Lemma 19, $G \in \mathcal{L}$ or every longest cycle contains all vertices of degree at least d. If $n \geq 8$, then $r \geq 3$ and $d = r + 1 = n - 4$. By Theorem 1, every longest cycle contains all vertices of degree at least d. Thus we are done. So in the following, we assume that $q \geq 1$ (i.e., $n \geq \alpha + 5$).

Let C be a longest cycle of G. We suppose on the contrary that there is a vertex x in $V(G - C)$ with $d(x) \geq d$. Let H be the component of $G - C$ containing x.

Let $b = n - 1 - d$. Then

$$b = \begin{cases} 2q + r + 1, & 0 \leq r \leq 2, \\ 2q + 3, & 3 \leq r < \alpha. \end{cases}$$

By $b(x)$ we denote the number of vertices in $V(G) \backslash N[x]$. Then

$$b(x) \leq b \leq 2q + 3. \tag{7}$$

Suppose first that H has only one vertex x. By Lemma 12, x is nonadjacent to every vertex of $N_C^+(x)$. Thus $b \geq b(x) \geq d(x) \geq d$. By comparing the formulas of b and d, we can see that $r = 2$ and $\alpha = 4$. Since $q \geq 1$, we have $d \geq \alpha + 1 \geq 5$. But in this case $N_C^+(x)$ is an independent set with $d(x) \geq 5$ vertices, a contradiction. This implies that H has at least two vertices.

Note that

$$d-\alpha=\begin{cases}(q-1)(\alpha-2)+1, & 0\leq r\leq 2,\\ (q-1)(\alpha-2)+r-1, & 3\leq r<\alpha.\end{cases}$$

We have $d-\alpha\geq(q-1)(\alpha-2)+1$, and

$$\left\lceil\frac{d-\alpha}{\alpha-2}\right\rceil\geq\left\lceil\frac{(q-1)(\alpha-2)+1}{\alpha-2}\right\rceil=q. \tag{8}$$

Suppose that there is some component of $G-C$ other than H. Let G' be the graph obtained from G by removing all other components of G, i.e., $G'=G[V(C)\cup V(H)]$. Then G' is 2-connected, $n(G')<n(G)$, $\alpha(G')\leq\alpha(G)$, and $d_{G'}(x)=d(x)$. By induction hypothesis, every longest cycle of G' contains x. This implies that there is a cycle in G', and then in G, longer than C, a contradiction. Hence we conclude that there is only one component H of $G-C$, i.e., $G-C=H$.

Claim 25. $N(x)=(V(H)\cup N_C(H))\backslash\{x\}$.

Proof. Suppose that there is a vertex y in H such that $xy\notin E(G)$. We choose a vertex $z\in N(y)$ in such a way that if $G-y$ is 2-connected, then let z be an arbitrary neighbor of y; if $G-y$ is separable, then let z be a neighbor of y which is an inner-vertex of some end-block of $G-y$. In any case, $\{y,z\}$ is not a vertex-cut and thus $G'=G\cdot yz$ is 2-connected. Note that $n(G')<n(G)$, $\alpha(G')\leq\alpha(G)$, and $d_{G'}(x)=d(x)$. By induction hypothesis, every longest cycle of G' contains x. This implies that there is a cycle in G' longer than C. But if G' contains such a cycle, then so is G by Lemma 13, a contradiction. This implies that x is adjacent to all the vertices in $V(H)\backslash\{x\}$.

Note that every vertex in $V(H)\backslash\{x\}$ is not a cut-vertex of H. Suppose that there is a vertex z in $N_C(H)$ such that $xz\notin E(G)$. It is not difficult to see that there is a neighbor y of z in H such that $\{y,z\}$ is not a vertex-cut of G. Thus $G'=G\cdot yz$ is 2-connected. Note that $n(G')<n(G)$, $\alpha(G')\leq\alpha(G)$, and $d_{G'}(x)=d(x)$. By induction hypothesis, every longest cycle of G' contains x. This implies that there is a cycle in G', and then in G, longer than C, a contradiction. Now we conclude that x is adjacent to all the vertices in $(V(H)\cup N_C(H))\backslash\{x\}$. □

By Claim 25, $\alpha(H)=\alpha(H-x)$ and $d_H(x)=n(H-x)$. By Lemma 18, there is a C-path $P=P(u,v)$ such that

$$|V(P)\cap V(H-x)|\geq\left\lceil\frac{n(H-x)}{\alpha(H-x)}\right\rceil=\left\lceil\frac{d_H(x)}{\alpha(H)}\right\rceil.$$

By Claim 25, we can choose P such that it satisfies the above inequality and $x\in V(P)$. Thus

$$|V(P)|\geq|V(P)\cap V(H-x)|+|\{u,v,x\}|\geq\left\lceil\frac{d_H(x)}{\alpha(H)}\right\rceil+3.$$

By Claim 25, $d_H(x) = d(x) - |N_C(H)| \geq d - |N_C(H)|$. Note that the union of $N_C^+(H)$ and an independent set of H form an independent set of G. This implies that $\alpha(H) \leq \alpha(G) - |N_C(H)| \leq \alpha - |N_C(H)|$. Together with the above inequality, we have

$$|V(P)| \geq \left\lceil \frac{d - |N_C(H)|}{\alpha - |N_C(H)|} \right\rceil + 3 = \left\lceil \frac{d - \alpha}{\alpha - |N_C(H)|} \right\rceil + 4 \geq \left\lceil \frac{d-\alpha}{\alpha - 2} \right\rceil + 4.$$

By (8), $l(P) = |V(P)| - 1 \geq q + 3$. By Lemma 11,

$$l(C) = l(\overrightarrow{C}[u,v]) + l(\overrightarrow{C}[v,u]) \geq 2l(P) + 2(|N_C(H)| - 2) \geq 2q + 2|N_C(H)| + 2.$$

Thus

$$b(x) = |V(C) \backslash N_C(H)| \geq 2q + 2|N_C(H)| + 2 - |N_C(H)| \geq 2q + 4,$$

contradicting (7).

The proof is complete. ■

***Proof of Theorem* 8.** The case $n = \alpha + 2$ is trivial. The only 2-connected graphs with independent number α and order $\alpha + 2$ are $K_{2,\alpha}$ and $K_{1,1,\alpha}$. Note that every longest cycle of them contains all (the two) vertices with degree at least 3. For the case $n = \alpha + 4$, the bound on d in Theorems 7 and 8 are equal. So the result can be deduced by Theorem 7 immediately.

Now we consider the case $n = \alpha + 3$. Let G be a 2-connected graph with independent number α and order $\alpha + 3$, let C be an arbitrary longest cycle of G, and let x be a vertex of G of degree at least 4. If C contains x, then we have nothing to prove. So we assume that $x \in V(G - C)$. If x is an isolated vertex of $G - C$, then $d_C(x) = d(x) \geq 4$. By Lemma 12, $l(C) \geq 8$. Thus

$$\begin{aligned}\alpha(G) &\leq \alpha(G[V(C)]) + \alpha(G - C) \leq \alpha(C) + |V(G - C)| = \left\lfloor \frac{l(C)}{2} \right\rfloor + n - l(C) \\ &= n - \left\lceil \frac{l(C)}{2} \right\rceil \leq n - 4,\end{aligned}$$

a contradiction. Thus we conclude that x has a neighbor x' in $G - C$. Since G is 2-connected, G has a C-path P passing through the edge xx'. Note that $l(P) \geq 3$, and by Lemma 10, $l(C) \geq 6$. Thus

$$\begin{aligned}\alpha(G) &\leq \alpha(G[V(C)]) + \alpha(G - C) \leq \left\lfloor \frac{l(C)}{2} \right\rfloor + n - l(C) - 1 \\ &= n - \left\lceil \frac{l(C)}{2} \right\rceil - 1 \leq n - 4,\end{aligned}$$

a contradiction. ■

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References

[1] B. Bollobás and G. Brightwell, *Cycles through specified vertices*, Combinatorica **13** (1993) 147–155.
doi:10.1007/BF01303200

[2] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer-Verlag, London, 2008).

[3] V. Chvátal and P. Erdős, *A note on Hamiltonian circuits*, Discrete Math. **2** (1972) 111–113.
doi:10.1016/0012-365X(72)90079-9

[4] G.A. Dirac, *Some theorems on abstract graphs*, Proc. Lond. Math. Soc. **(3)2** (1952) 69–81.
doi:10.1112/plms/s3-2.1.69

[5] M. Kouider, *Cycles in graphs with prescribed stability number and connectivity*, J. Combin. Theory Ser. B **60** (1994) 315–318.
doi:10.1006/jctb.1994.1023

[6] B. Li and S. Zhang, *Forbidden subgraphs for longest cycles to contain vertices with large degree*, Discrete Math. **338** (2015) 1681–1689.
doi:10.1016/j.disc.2014.07.003

[7] D. Paulusma and K. Yoshimoto, *Cycles through specified vertices in triangle-free graphs*, Discuss. Math. Graph Theory **27** (2007) 179–191.
doi:10.7151/dmgt.1354

[8] A. Saito, *Long cycles through specified vertices in a graph*, J. Combin. Theory Ser. B **47** (1989) 220–230.
doi:10.1016/0095-8956(89)90021-X

[9] R. Shi, 2-*neighborhoods and Hamiltonian conditions*, J. Graph Theory **16** (1992) 267–271.
doi:10.1002/jgt.3190160310

[10] S. O, D.B. West and H. Wu, *Longest cycles in k-connected graphs with given independence number*, J. Combin. Theory Ser. B **101** (2011) 480–485. doi:10.1016/j.jctb.2011.02.005

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