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3-PATHS IN GRAPHS WITH BOUNDED AVERAGE DEGREE

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Abstract

In this paper we study the existence of unavoidable paths on three vertices in sparse graphs. A path uvw on three vertices u, v, and w is of type (i,j,k) if the degree of u (respectively v, w) is at most i (respectively j, k). We prove that every graph with minimum degree at least 2 and average degree strictly less than m contains a path of one of the types

- $(2,\infty,2)$, $(2,8,3)$, $(4,3,5)$ if $m = \frac{15}{4}$,
- $(2,\infty,2)$, $(2,5,3)$, $(3,2,4)$, $(3,3,3)$ if $m = \frac{10}{3}$,
- $(2,2,\infty)$, $(2,3,4)$, $(2,5,2)$ if $m = 3$,
- $(2,2,13)$, $(2,3,3)$, $(2,4,2)$ if $m = \frac{14}{5}$,
- $(2,2,i)$, $(2,3,2)$ if $m = \frac{3(i+1)}{i+2}$ for $4 \le i \le 7$,
- $(2,2,3)$ if $m = \frac{12}{5}$, and
- $(2,2,2)$ if $m = \frac{9}{4}$.

Moreover, no parameter of this description can be improved.

Keywords: average degree, structural property, 3-path, degree sequence.

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1. Introduction

In this paper we use a standard graph theory terminology according to the book [3]. However we recall here some notions. We use $V(G)$, $E(G)$, and $\delta(G)$ (or simply V, E, δ) to denote the vertex set, edge set, and the minimum degree of G, respectively. The *degree* of a vertex v, that is, the number of edges incident with v, is denoted by $\deg(v)$. The *average degree* of a graph G, denoted by $\mathrm{ad}(G)$, is defined as $\mathrm{ad}(G) = \frac{2|E(G)|}{|V(G)|}$. A *$k$-vertex* is a vertex v with $\deg(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a *k^+-vertex* v satisfies $\deg(v) \geq k$ and a *k^--vertex* v satisfies $\deg(v) \leq k$. A path on three vertices uvw is a *path of type* (i, j, k) or an (i, j, k)-*path* if $\deg(u) \leq i, \deg(v) \leq j$, and $\deg(w) \leq k$. If a 3-path is of type (i, j, k), then we say that i, j, and k are *parameters* of the type. The *girth* $g(G) = g$ of G is the length of a shortest cycle in G.

The main motivation for our research comes from the paper [2], where the results about the structure of paths on two vertices in graphs with given minimum and average degree are presented, and from the following results.

Theorem 1 [1]**.** *Every 3-polytope*[1] *contains an (i, j, k)-path with $i + j + k \leq 21$, which is tight.*

Theorem 2 [7]**.** *Every 3-polytope has a 3-path of one of the following types:* $(10, 3, 10)$, $(7, 4, 7)$, $(6, 5, 6)$, $(3, 4, 15)$, $(3, 6, 11)$, $(3, 8, 5)$, $(3, 10, 3)$, $(4, 4, 11)$, $(4, 5, 7)$, *and* $(4, 7, 5)$.

Theorem 3 [4]**.** *Every normal plane map*[2] *without two adjacent 3-vertices lying in two common 3-faces has a 3-path of one of the following types:* $(3, 4, 11)$, $(3, 7, 5)$, $(3, 10, 4)$, $(3, 15, 3)$, $(4, 4, 9)$, $(6, 4, 8)$, $(7, 4, 7)$, *and* $(6, 5, 6)$. *Moreover, no parameter of this description can be improved.*

All of the above mentioned theorems deal with graphs having minimum degree at least three. In [8] and [9], the minimum degree condition was relaxed, and planar graphs with minimum degree at least 2 and given girth were studied.

Theorem 4 ([8, 9])**.** *Every planar graph G with minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq g$ has a 3-path of one of the following types:*

(i) $(2, \infty, 2)$, $(2, 7, 3)$, $(3, 5, 3)$, $(4, 2, 5)$, $(4, 3, 4)$ *if* $g = 4$,

(ii) $(2, \infty, 2)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$, $(3, 3, 3)$ *if* $g = 5$,

(iii) $(2, 2, \infty)$, $(2, 3, 5)$, $(2, 4, 3)$, $(2, 5, 2)$ *if* $g = 6$,

(iv) $(2, 2, 6)$, $(2, 3, 3)$, $(2, 4, 2)$ *if* $g = 7$,

[1] *3-polytopes* are precisely 3-connected planar graphs (Steinitz's theorem).

[2] A *normal plane map* is a plane graph in which loops and multiple edges are allowed, and the degree of each vertex and face is at least three.

(v) $(2,2,5)$, $(2,3,3)$ *if* $g \in \{8,9\}$,
(vi) $(2,2,3)$, $(2,3,2)$ *if* $g \geq 10$, *and*
(vii) $(2,2,2)$ *if* $g \geq 16$.

In this paper we focus on general graphs with bounded average degree and prove

Theorem 5. *Let G be a graph with minimum degree $\delta(G) \geq 2$ and average degree strictly less than m. Then the graph G contains a 3-path of one of the following types:*

(i) $(2,\infty,2)$, $(2,8,3)$, $(4,3,5)$ *if* $m = \frac{15}{4}$,
(ii) $(2,\infty,2)$, $(2,5,3)$, $(3,2,4)$, $(3,3,3)$ *if* $m = \frac{10}{3}$,
(iii) $(2,2,\infty)$, $(2,3,4)$, $(2,5,2)$ *if* $m = 3$,
(vi) $(2,2,13)$, $(2,3,3)$, $(2,4,2)$ *if* $m = \frac{14}{5}$,
(v) $(2,2,i)$, $(2,3,2)$ *if* $m = \frac{3(i+1)}{i+2}$ *for* $4 \leq i \leq 7$,
(vi) $(2,2,3)$ *if* $m = \frac{12}{5}$, *and*
(vii) $(2,2,2)$ *if* $m = \frac{9}{4}$.

Moreover, all parameters are optimal (*i.e., none of the types of* 3*-paths of the list can be omitted, none of the parameters of any type of* 3*-paths can be decreased, and the value of m cannot be increased without changing the others*).

As every planar graph G with girth at least g satisfies $\mathrm{ad}(G) < \frac{2g}{g-2}$ (see [5]), we deduce

Corollary 6. *Every planar graph G with minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq g$ has a* 3*-path of one of the following types:*

(i) $(2,\infty,2)$, $(2,5,3)$, $(3,2,4)$, $(3,3,3)$ *if* $g = 5$,
(ii) $(2,2,\infty)$, $(2,3,4)$, $(2,5,2)$ *if* $g = 6$,
(iii) $(2,2,7)$, $(2,3,2)$ *if* $g = 8$,
(iv) $(2,2,5)$, $(2,3,2)$ *if* $g = 9$, *and*
(v) $(2,2,3)$ *if* $g = 12$.

In the proof of Theorem 5(ii)–(vi) there are shown tight examples having the requested value of g which are planar graphs. Therefore we can state the following

Observation 7. *None of the parameters of the types of* 3*-paths in Corollary* 6 *can be dropped, except maybe in case* (iii)*. For* (i) *and* (ii) *the value of g cannot be decreased.*

The rest of the paper is organised as follows: Sections 2–7 are dedicated to the proof of Theorem 5. In Section 8 we discuss the quality of our results.

2. Proof of Theorem (i)

We prove Theorem 5(i) by contradiction. Suppose there exists a counterexample $G = (V, E)$ with $\delta(G) \geq 2$ and the average degree $\mathrm{ad}(G) = \frac{2|E(G)|}{|V(G)|} < \frac{15}{4}$ that contains no 3-paths of types $(2, \infty, 2)$, $(2, 8, 3)$, and $(4, 3, 5)$. We will achieve a contradiction by applying a discharging procedure (see [6] for a nice guide on discharging methods).

First we assign a *charge* $\omega(v) = 4\deg(v) - 15$ to each vertex v. Then we redistribute the charges according to the discharging rules R1, R2, R3 and R4 (see below); once the discharging process is finished, a new charge function φ is produced. During this process, no charges are created and no charges disappear; hence, the total sum of charges remains the same. Nevertheless, by the non-existence of 3-paths of types $(2, \infty, 2)$, $(2, 8, 3)$, and $(4, 3, 5)$ in G, we will show that $\varphi(v) \geq 0$ for all $v \in V(G)$. This leads to the following contradiction that completes the proof of the nonexistence of the counterexample:

$$\begin{aligned} 0 &\leq \sum\nolimits_{v\in V(G)} \varphi(v) = \sum\nolimits_{v\in V(G)} \omega(v) = \sum\nolimits_{v\in V(G)} (4\deg(v) - 15) \\ &= 4 \times \sum\nolimits_{v\in V(G)} \deg(v) - 15 \times |V(G)| = 4 \times 2|E(G)| - 15 \times |V(G)| \\ &= |V(G)| \times (4 \times \mathrm{ad}(G) - 15) < 0. \end{aligned}$$

The discharging rules are the following:

R1. Every 6^+-vertex gives 7 to each adjacent 2-vertex.
R2. Every 6^+-vertex gives $\frac{3}{2}$ to each adjacent 3-vertex.
R3. Every 5-vertex gives 5 to each adjacent 2-vertex.
R4. Every 5-vertex gives 1 to each adjacent 3-vertex.

We show now that the new charge $\varphi(v)$ of any vertex v is non-negative. Observe that every vertex is adjacent to at most one 2-vertex (otherwise G would contain a $(2, \infty, 2)$-path). Let v be a k-vertex ($k \geq 2$). The following cases have to be considered.

Case 1. $k = 2$. The initial charge of v is -7. Observe that v is adjacent either to a 6^+-vertex or to two 5-vertices (otherwise G would contain a $(4, 3, 5)$-path). In the former case, v receives at least 7 by R1. In the latter case, v receives 2×5 by R3. Hence $\varphi(v) \geq -7 + \min\{7, 2 \times 5\} = 0$.

Case 2. $k = 3$. The initial charge of v is -3. Observe that now v is adjacent to at least two 6^+-vertices or to three 5^+-vertices (otherwise G would contain a $(4, 3, 5)$-path). In the former case, v receives at least $2 \times \frac{3}{2}$ by R2. In the latter case, v receives at least 3×1 by R2 and R4. It follows that $\varphi(v) \geq -3 + 3 = 0$.

Case 3. $k = 4$. The discharging rules do not involve 4-vertices. Hence $\varphi(v) = \omega(v) = 1 > 0$.

Case 4. $k = 5$. The initial charge of v is 5. If v is adjacent to a 2-vertex, then it is not adjacent to a 3-vertex (otherwise G would contain a $(2, 8, 3)$-path) and $\varphi(v) = 5 - 5 = 0$ by R3. Otherwise, v is adjacent to at most five 3-vertices. Then $\varphi(v) \geq 5 - 5 \times 1 = 0$ by R4.

Case 5. $6 \leq k \leq 8$. If v is adjacent to a 2-vertex, then v is not adjacent to 3-vertices (otherwise G would contain a $(2, 8, 3)$-path) and $\varphi(v) = 4k - 15 - 7 = 4k - 22 > 0$ by R1. If v is not adjacent to 2-vertices, then $\varphi(v) \geq 4k - 15 - k \times \frac{3}{2} = \frac{5k-30}{2} \geq 0$ by R2.

Case 6. $k \geq 9$. The initial charge of v is $4k - 15$. By rules R1 and R2, we have $\varphi(v) \geq 4k - 15 - 7 - (k - 1)\frac{3}{2} = \frac{5k-41}{2} > 0$.

Hence $\varphi(v) \geq 0$ for all $v \in V(G)$ as claimed.

Optimality of Theorem 5(i). We cannot omit the $(2, \infty, 2)$-path in Theorem 5(i) because of the graph depicted in Figure 1 (recall that planar graphs with girth 6 have average degree strictly less than $\frac{2\times 6}{6-2} = 3$). The $(2, 8, 3)$-path cannot be omitted due to the graph depicted in Figure 2 on the left (this graph has $\mathrm{ad}(G) = \frac{41}{11} < \frac{15}{4}$). The $(4, 3, 5)$-path cannot be omitted due to the graph depicted in Figure 2 on the right (this graph has $\mathrm{ad}(G) = \frac{40}{11} < \frac{15}{4}$). Finally, the value of m cannot be increased due to the graph $K_{3,5}$ which has $\mathrm{ad}(G) = \frac{15}{4}$ and contains no 3-paths of types $(2, \infty, 2)$, $(2, 8, 3)$, $(4, 3, 5)$.

3. Proof of Theorem 5(ii)

We proceed by contradiction. Suppose there exists a graph G with $\delta(G) \geq 2$ and $\mathrm{ad}(G) < \frac{10}{3}$ that contains no 3-paths of types $(2, \infty, 2)$, $(2, 5, 3)$, $(3, 2, 4)$, and $(3, 3, 3)$. We achieve a contradiction again by applying a discharging procedure.

First, we assign a charge $\omega(v) = 3\deg(v) - 10$ to each vertex v. Since $\mathrm{ad}(G) < \frac{10}{3}$, the total sum of charges is negative. Next we redistribute the charges according to the following rules:

R1. Every 5^+-vertex gives 4 to each adjacent 2-vertex.

R2. Every 4-vertex gives 2 to each adjacent 2-vertex.

R3. Every 4^+-vertex gives $\frac{1}{2}$ to each adjacent 3-vertex.

Once the discharging process is finished, a new charge $\varphi(v)$ is produced on each vertex v. We show now that $\varphi(v) \geq 0$ for all $v \in V(G)$. Observe that every vertex is adjacent to at most one 2-vertex (otherwise, G would contain a

$(2, \infty, 2)$-path). Let v be a k-vertex ($k \geq 2$). The following cases have to be considered.

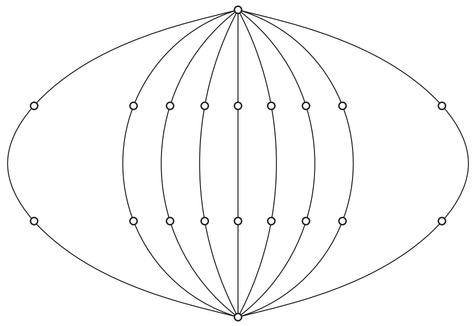


Figure 1. Planar graph with girth 6 and 3-paths of types $(2, 2, \infty)$ and $(2, \infty, 2)$.

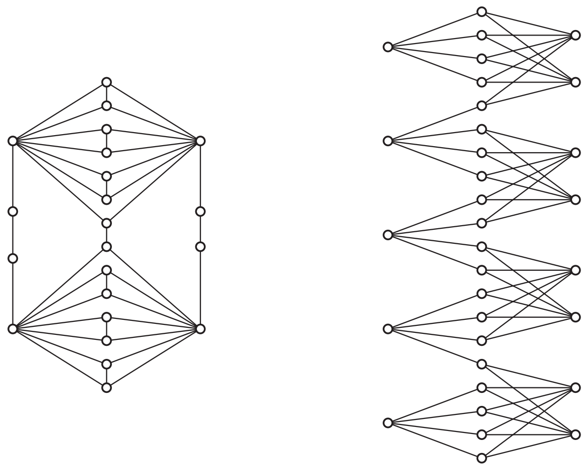


Figure 2. Graphs with 3-paths of types $(2, 8, 3)$ and $(4, 3, 5)$.

Case 1. $k = 2$. The initial charge of v is -4. Observe that v is adjacent either to a 5^+-vertex or to two 4-vertices (otherwise G would contain a $(3, 2, 4)$-path). Hence v receives at least 4 (either by R1 or by R2 applied twice) and $\varphi(v) \geq 0$.

Case 2. $k = 3$. The initial charge of v is -1. Observe that v is adjacent to two 4^+-vertices (otherwise G would contain a $(3, 3, 3)$-path). Hence v receives $\frac{1}{2}$ from each of them by R3 and $\varphi(v) \geq 0$.

Case 3. $k = 4$. In this case the initial charge of v is 2. If v is adjacent to a 2-vertex, then v is not adjacent to 3-vertices (otherwise G would contain a $(2, 4, 3)$-path, and hence a $(2, 5, 3)$-path) and $\varphi(v) = 2 - 2 = 0$ by R2. If v is not adjacent to any 2-vertex, then $\varphi(v) \geq 2 - 4 \times \frac{1}{2} = 0$ by R3.

Case 4. $k = 5$. Now the initial charge of v is 5. If v is adjacent to a 2-vertex, then v is not adjacent to 3-vertices (otherwise G would contain a $(2,5,3)$-path) and $\varphi(v) = 5 - 4 = 1$ by R1. If v is not adjacent to any 2-vertex, then $\varphi(v) \geq 5 - 5 \times \frac{1}{2} = \frac{5}{2}$ by R3.

Case 5. $k \geq 6$. In this case the initial charge of v is $3k - 10$. By rules R1 and R3, we have $\varphi(v) \geq 3k - 10 - 4 - (k-1) \times \frac{1}{2} = \frac{5k-27}{2} \geq 0$.

Hence $\varphi(v) \geq 0$ for all $v \in V(G)$. This leads to a contradiction with the fact that the sum of original charges is negative.

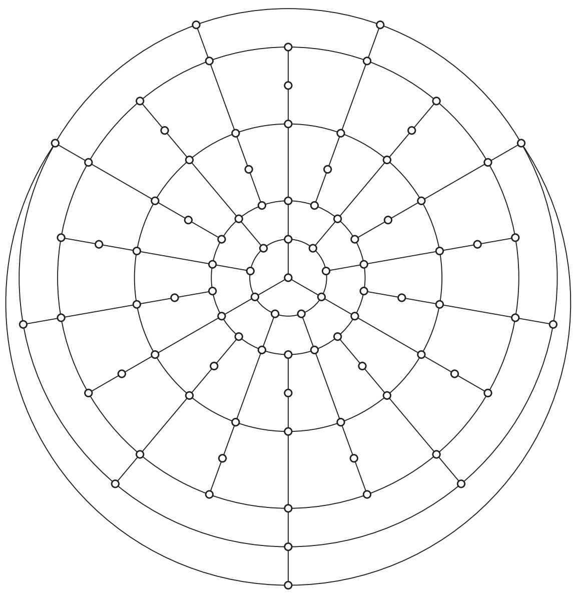


Figure 3. Planar graph with girth 5 and $(3,2,4)$-paths.

Optimality of Theorem 5(ii). We cannot omit the $(2,\infty,2)$-path in Theorem 5(ii) because of the graph depicted in Figure 1 (it is planar and has girth $g(G) = 6 \geq 5$). The $(3,3,3)$-path cannot be omitted, because of the dodecahedron (which has average degree equal to 3, see Figure 4). The $(3,2,4)$-path cannot be omitted due to the graph depicted in Figure 3 (recall that planar graphs with girth 5 have average degree strictly less than $\frac{2\times 5}{5-2} = \frac{10}{3}$). Finally, the $(2,5,3)$-path cannot be omitted either. It suffices to take the dodecahedron and replace all vertices with 6-faces, every edge by two 4-faces, and insert a special configuration into every original 5-face (see Figure 5). The resulting graph G is planar and has girth $g(G) = 5$. Hence, $\mathrm{ad}(G) < \frac{2\times 5}{5-2} = \frac{10}{3}$ (see [5]). Lastly, the value of m cannot be increased due to the graph G_1 depicted in Figure 6. Indeed, G_1 has $\mathrm{ad}(G_1) = \frac{10}{3}$ and no 3-paths of types $(2,\infty,2)$, $(3,2,4)$, $(3,3,3)$, $(2,5,3)$. To prove the optimality of Corollary 6(i), it remains to observe that the value of g cannot be decreased: see graph G_2 depicted in Figure 6.

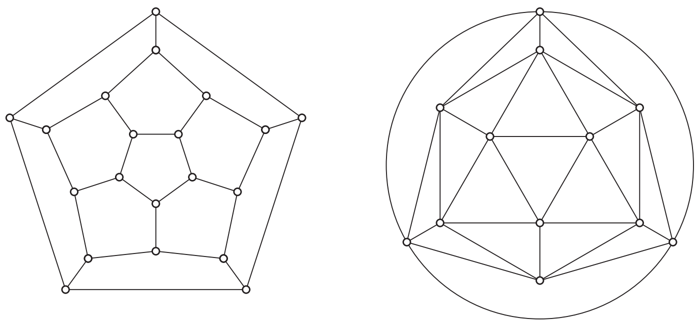
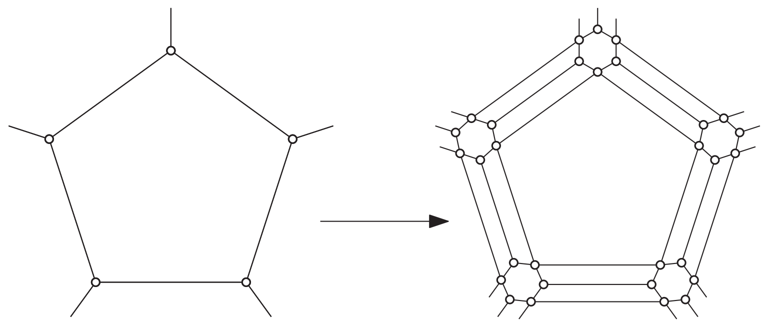
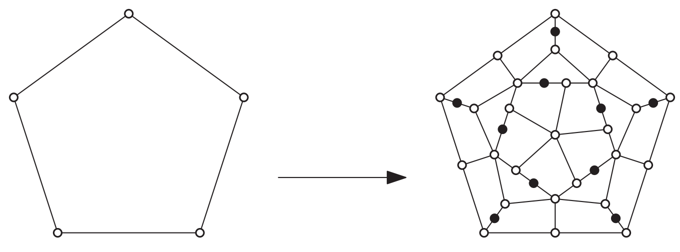


Figure 4. The graphs of the dodecahedron (3-regular) and the icosahedron (5-regular).



(a) Replacement of vertices and edges



(b) Replacement of 5-faces

Figure 5. Construction of a planar graph with a $(2, 5, 3)$-path and with girth 5.

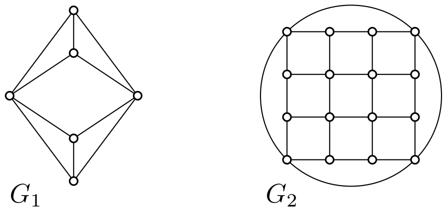


Figure 6. Graph G_1 has $\mathrm{ad}(G_1) = \frac{10}{3}$ and planar graph G_2 has girth 4.

4. Proof of Theorem 5(iii)

Suppose there exists a counterexample $G = (V, E)$ with $\delta(G) \geq 2$ and $\mathrm{ad}(G) < 3$ that contains no 3-paths of types $(2, 2, \infty)$, $(2, 3, 4)$, and $(2, 5, 2)$. We will reach a contradiction by applying a discharging procedure. First we assign a charge $\omega(v) = \deg(v) - 3$ to each vertex v. Since $\mathrm{ad}(G) < 3$, the total sum of the charges is negative. Then we redistribute the charges according to the discharging rules R1 and R2:

R1. Every 3^+-vertex gives $\frac{1}{2}$ to each adjacent 2-vertex.

R2. Every 5^+-vertex gives $\frac{1}{4}$ to each adjacent 3-vertex.

Let $\varphi(v)$ be the charge of a vertex v after the procedure. We are going to show that $\varphi(v) \geq 0$ for all $v \in V(G)$. Observe that there are no adjacent 2-vertices in G (otherwise G would contain a $(2, 2, \infty)$-path) and every 5^--vertex is adjacent to at most one 2-vertex (otherwise G would contain a $(2, 5, 2)$-path). Let v be a k-vertex ($k \geq 2$).

Case 1. $k = 2$. The vertex v is adjacent to 3^+-vertices. It receives charge $\frac{1}{2}$ from each neighbor by R1. Hence $\varphi(v) = -1 + 2 \times \frac{1}{2} = 0$.

Case 2. $k = 3$. If v is adjacent to a 2-vertex, then the two other neighbors of v are 5^+-vertices (otherwise G would contain a $(2, 3, 4)$-path). It follows that v receives twice $\frac{1}{4}$ by R2, gives $\frac{1}{2}$ by R1, and $\varphi(v) = 0 + 2 \times \frac{1}{4} - \frac{1}{2} = 0$. If v is not adjacent to 2-vertices, then it gives nothing by the rules and $\varphi(v) \geq 0$.

Case 3. $k = 4$. In this case we have $\varphi(v) \geq 1 - \frac{1}{2} = \frac{1}{2} > 0$ by R1.

Case 4. $k = 5$. We have $\varphi(v) \geq 2 - \frac{1}{2} - 4 \times \frac{1}{4} = \frac{1}{2} > 0$ by R1 and R2.

Case 5. $k \geq 6$. Now we have $\varphi(v) \geq k - 3 - k \times \frac{1}{2} = \frac{k}{2} - 3 \geq 0$.

Hence $\varphi(v) \geq 0$ for all $v \in V(G)$ and we have a contradiction.

Optimality of Theorem 5(iii). It follows from the graph in Figure 1 that the $(2, 2, \infty)$-path cannot be omitted in Theorem 5(iii) (this graph is planar with girth at least 6 and so has $\mathrm{ad}(G) < 3$). The $(2, 3, 4)$-path cannot be omitted as one can see in Figure 7 (observe that this graph is planar, has girth 6 and hence it has $\mathrm{ad}(G) < \frac{2\times 6}{6-2} = 3$; moreover, it contains neither two adjacent 2-vertices nor vertices adjacent to two 2-vertices). The $(2,5,2)$-path cannot be omitted because of the subdivided icosahedron, i.e., the graph obtained from the icosahedron (see Figure 4) by adding a 2-vertex on each edge (the resulting graph has girth 6). Finally, the value of m cannot be increased: a 3-regular graph G has $\mathrm{ad}(G) = 3$ and contains no 3-paths of types $(2,2,\infty)$, $(2,3,4)$, and $(2,5,2)$. Similarly, in Corollary 6(ii), the value of g cannot be decreased as shown by the dodecahedron.

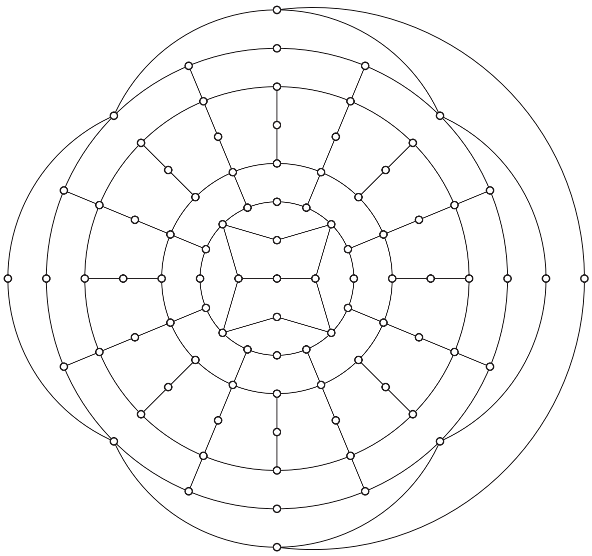


Figure 7. Planar graph with girth 6 and $(2,3,4)$-paths.

5. Proof of Theorem 5(iv)

Let $G = (V,E)$ be a counterexample to Theorem 5(iv) with $\delta(G) \geq 2$ and $\mathrm{ad}(G) < \frac{14}{5}$ that contains no 3-paths of any of the types $(2,2,13)$, $(2,3,3)$, and $(2,4,2)$. We assign a charge $\omega(v) = 5\deg(v) - 14$ to each vertex v. From the assumption $\mathrm{ad}(G) < \frac{14}{5}$ it follows that the total sum of the charges is negative.

We now redistribute the charges according to rules R1 to R3:

R1. Every 14^+-vertex gives 4 to each adjacent 2-vertex and $\frac{1}{2}$ to each adjacent 3-vertex.

R2. Every d-vertex, with $4 \leq d \leq 13$, gives 2 to each adjacent 2-vertex and $\frac{1}{2}$ to each adjacent 3-vertex.

R3. Every 3-vertex gives 2 to each adjacent 2-vertex.

Let $\varphi(v)$ be the charge of v after the procedure. Let v be a k-vertex ($k \geq 2$). Consider the following cases.

Case 1. $k = 2$. The initial charge of v is -4. If v is adjacent to a 2-vertex, then its other neighbor has degree at least 14 (as G does not contain any $(2,2,13)$-path). By R1, we have $\varphi(v) = -4+4 = 0$. Otherwise, v is adjacent to two 3^+-vertices. By rules R1 to R3, $\varphi(v) \geq -4+2\times 2 = 0$.

Case 2. $k = 3$. The initial charge of v is 1. If v is adjacent to a 2-vertex, then its two other neighbors have degree at least 4 (as G does not contain any $(2,3,3)$-path). It follows by rules R1 to R3 that $\varphi(v) = 1+2\times\frac{1}{2}-2 = 0$. If v is not adjacent to 2-vertices, then $\varphi(v) \geq \omega(v) = 1 > 0$.

Case 3. $k = 4$. The initial charge of v is 6. As G does not contain any $(2,4,2)$-path, the vertex v is adjacent to at most one 2-vertex. By R2, $\varphi(v) \geq 6-2-3\times\frac{1}{2} = \frac{5}{2} > 0$.

Case 4. $5 \leq k \leq 13$. The initial charge of v is $5k-14$. By R2, $\varphi(v) \geq 5k-14-k\times 2 = 3k-14 > 0$.

Case 5. $k \geq 14$. The initial charge of v is $5k-14$. By R1, $\varphi(v) \geq 5k-14-k\times 4 = k-14 \geq 0$.

Therefore, for all $v \in V(G)$, $\varphi(v) \geq 0$ and we have a contradiction.

Optimality of Theorem 5(iv). See Figure 8. Graph G_1 has $\delta(G_1) \geq 2$ and $\mathrm{ad}(G_1) = \frac{39}{14}$, and contains only $(2,2,13)$-paths and $(2,13,2)$-paths. Graph G_2 has $\delta(G_2) \geq 2$ and $\mathrm{ad}(G_2) = \frac{8}{3}$, and contains only $(4,2,4)$-paths and $(2,4,2)$-paths. Graph G_3 has $\delta(G_3) \geq 2$ and $\mathrm{ad}(G_3) = \frac{8}{3}$, and contains only $(2,3,3)$-paths, $(3,2,3)$-paths, and $(3,3,3)$-paths. Finally, G_4 is a graph with $\delta(G_4) \geq 2$ and $\mathrm{ad}(G_4) = \frac{14}{5}$, and G_4 does not contain any 3-paths of types $(2,2,13)$, $(2,3,3)$, and $(2,4,2)$.

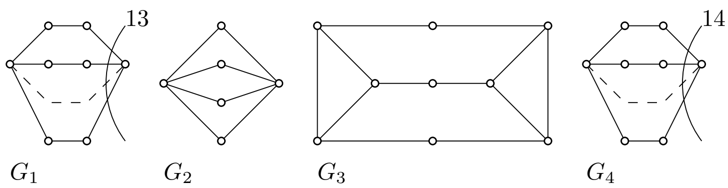


Figure 8. Graphs G_1, G_2, G_3, and G_4.

6. Proof of Theorem 5(v)

Let $G = (V,E)$ be a counterexample to Theorem 5(v) with $\delta(G) \geq 2$ and $\mathrm{ad}(G) < \frac{3(i+1)}{i+2}$ $(4 \leq i \leq 7)$ that does not contain $(2,2,i)$-paths and $(2,3,2)$-paths. First we assign a charge $\omega(v) = (i+2)\deg(v) - 3(i+1)$ to each vertex v. From the assumption on the average degree it follows that the total sum of the charges is negative.

We now redistribute the charges according to the following rules:

R1. Every $(i+1)^+$-vertex gives $(i-1)$ to each adjacent 2-vertex.

R2. Every vertex with degree d, $3 \le d \le i$, gives $\frac{i-1}{2}$ to each adjacent 2-vertex.

Let $\varphi(v)$ denote the charge of vertex v after the discharging process. Let v be a k-vertex ($k \ge 2$). Consider the following cases.

Case 1. $k = 2$. The initial charge of v is $-i+1$. Note that G does not contain any $(2,2,i)$-path. If v is adjacent to a 2-vertex, then it is also adjacent to an $(i+1)^+$-vertex. If v is adjacent to an $(i+1)^+$-vertex, then v receives at least $i-1$ by R1 and $\varphi(v) \ge -i+1+i-1 = 0$. Otherwise, v twice receives at least $\frac{i-1}{2}$ by R2. It follows that $\varphi(v) \ge -i+1+2\times\frac{i-1}{2} = 0$.

Case 2. $k = 3$. In this case the initial charge of v is 3. As G does not contain any $(2,3,2)$-path, v is adjacent to at most one 2-vertex and, by R2, we have $\varphi(v) \ge 3 - \frac{i-1}{2} = \frac{7-i}{2} \ge 0$, because $i \le 7$.

Case 3. $4 \le k \le i$. By R2, we have $\varphi(v) \ge (i+2)k - 3(i+1) - k \times \frac{i-1}{2} = \frac{(k-4)(i+5)}{2} + 7 - i \ge 0$ as $i \le 7$.

Case 4. $k \ge i+1$. By R1, we have $\varphi(v) \ge (i+2)k - 3(i+1) - k\times(i-1) = 3k - 3(i+1) \ge 0$.

Hence for all $v \in V(G), \varphi(v) \ge 0$ and we have a contradiction.

Optimality of Theorem 5(v). See Figure 9. The graph G_1 has $\delta(G_1) \ge 2$ and $\mathrm{ad}(G_1) = \frac{3i}{i+1}$, and contains only $(2,2,i)$-paths and $(2,i,2)$-paths. The graph G_2 has $\delta(G_2) \ge 2$ and $\mathrm{ad}(G_2) = \frac{12}{5}$, and contains only $(2,3,2)$-paths and $(3,2,3)$-paths. The graph G_3 is a graph with $\delta(G_3) \ge 2$ and $\mathrm{ad}(G_3) = \frac{3(i+1)}{i+2}$, and G_3 does not contain 3-paths of types $(2,2,i)$ and $(2,3,2)$. Finally, to prove the optimality of Corollary 6(iv) (girth 9 corresponds to $\mathrm{ad}(G) < \frac{2\times 9}{9-2} = \frac{3\times 6}{7}$), one can take the icosahedron where each edge is subdivided by two 2-vertices (the resulting graph has girth $g(G) = 9$) and the subdivided dodecahedron (the resulting graph has girth $g(G) = 10 \ge 9$).

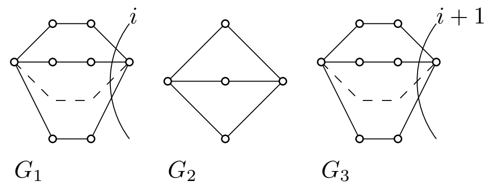


Figure 9. Graphs G_1, G_2, and G_3.

7. Proof of Theorem 5(vi)

Let $G = (V, E)$ be a counterexample to Theorem 5(vi) with $\delta(G) \geq 2$ and $\text{ad}(G) < \frac{12}{5}$ that does not contain $(2, 2, 3)$-paths. First we assign a charge $\omega(v) = 5\text{deg}(v) - 12$ to each vertex v. From the assumption on average degree it follows that the total sum of the charges is negative.

Then we redistribute the charges according to the rules R1 and R2.

R1. Every 4^+-vertex gives 2 to each adjacent 2-vertex.

R2. Every 3-vertex gives 1 to each adjacent 2-vertex.

Let $\varphi(v)$ be the new charge of vertex v. Let v be a k-vertex ($k \geq 2$).

Case 1. $k = 2$. The initial charge of v is -2. If v is adjacent to a 2-vertex, then its other neighbor is a 4^+-vertex (as G does not contain any $(2, 2, 3)$-path), and so it receives 2 by R1 and $\varphi(v) = -2 + 2 = 0$. Otherwise, v is adjacent to two 3^+-vertices and $\varphi(v) \geq -2 + 2 \times 1 = 0$ by R1 and R2.

Case 2. $k = 3$. Now the initial charge of v is 3. By R2, $\varphi(v) \geq 3 - 3 \times 1 = 0$.

Case 3. $k \geq 4$. In this case the initial charge of v is $5k - 12$. By R1, $\varphi(v) \geq 5k - 12 - k \times 2 = 3k - 12 \geq 0$.

Hence we have again $\varphi(v) \geq 0$ for all $v \in V(G)$, a contradiction.

Optimality of Theorem 5(vi). See graph G_1 of Figure 9. For $i = 3$ it has $\text{ad}(G_1) = \frac{9}{4} < \frac{12}{5}$ and contains only $(2, 2, 3)$ and $(2, 3, 2)$-paths. For $i = 4$ it has $\text{ad}(G_1) = \frac{12}{5}$ and does not contain any $(2, 2, 3)$-path. The optimality of Corollary 6(v) is given by the dodecahedron (see Figure 4) where each edge is subdivided by two 2-vertices (the resulting graph has girth $g(G) = 15 \geq 12$ and contains only $(2, 2, 3)$ and $(3, 2, 3)$-paths).

8. Proof of Theorem 5(vii)

The proof is left to the reader. There is again applied a discharging procedure with initial charge $\omega(v) = 4\deg(v) - 9$ and the following discharging rule R: every 3^+-vertex gives 1 to each adjacent 2-vertex. The optimality is given by the graph G_1 (for $i = 3$) presented in Figure 9 (it has $\text{ad}(G_1) = \frac{9}{4}$ and does not contain any $(2, 2, 2)$-path).

9. Remarks

Consider now the relationship between the girth (when G is planar) and the average degree (recall that if G is planar with girth at least g, then $\text{ad}(G) < \frac{2g}{g-2}$). Observe that Corollaries 6(ii), 6(iv), and 6(v) improve Theorems 4(iii), 4(v) (when

$g = 9$), and 4(vi) (when $g \geq 12$), respectively. It seems natural to ask whether the conclusion of Theorem 4 can be proved by requiring conditions on the average degree and $\delta(G) \geq 2$.

One can observe that the conclusion of Theorem 4(i) cannot be obtained if the $g(G) \geq 4$ condition is replaced by the $\text{ad}(G) < 4$ condition. To see this, consider any arbitrary 3-regular graph and replace every edge with a diamond (i.e., a cycle of length 4 with a chord); the resulting graph G' has $\delta(G') = 3$ and $\text{ad}(G') = \frac{15}{4} < 4$, but contains only 3-paths of types $(3, 3, 6)$, $(3, 6, 3)$ and $(6, 3, 6)$.

Also observe that Corollaries 6(i) and 6(iii) give new descriptions for planar graphs with girth 5 and 8, respectively (they do not imply Theorems 4(ii) and 4(v) (when $g = 8$)); the question how replace the $g(G) \geq 5$ condition by $\text{ad}(G) < \frac{10}{3}$ condition is open.

For $g(G) \geq 8$, we cannot replace the girth condition by the $\text{ad}(G) < \frac{8}{3}$ condition: there exists a graph with $\text{ad}(G) = \frac{18}{7} < \frac{8}{3}$ which contains neither a $(2, 2, 5)$ nor a $(2, 3, 3)$-path (see the graph G_1 on Figure 9 for $i = 6$).

Finally, the conclusions of Theorems 4(iv) and 4(vii) cannot be obtained with the help of the conditions $\text{ad}(G) < \frac{14}{5}$ for $g = 7$ and $\text{ad}(G) < \frac{16}{7}$ for $g = 16$, respectively, as Theorems 5(iv) and 5(vi) are tight.

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References

[1] K. Ando, S. Iwasaki and A. Kaneko, *Every 3-connected planar graph has a connected subgraph with small degree sum*, in: Annual Meeting of Mathematical Society of Japan, (1993), in Japanese.

[2] P. Bose, M. Smid and D.R. Wood, *Light edges in degree-constrained graphs*, Discrete Math. **28** (2004) 35–41.
doi:10.1016/j.disc.2003.12.003

[3] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer, 2008).

[4] O.V. Borodin, A.O. Ivanova, T.R. Jensen, A.V. Kostochka and M. Yancey, *Describing 3-paths in normal plane maps*, Discrete Math. **313** (2013) 2702–2711.
doi:10.1016/j.disc.2013.08.018

[5] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud and E. Sopena, *On the maximum average degree and the oriented chromatic number of a graph*, Discrete Math. **206** (1999) 77–89.
doi:10.1016/S0012-365X(98)00393-8

[6] D.W. Cranston and D.B. West, *A guide to the discharging metho*d, arXiv: 1306.4434 [math.CO] 19 Jun 2013.

[7] S. Jendrol', *A structural property of convex* 3*-polytopes*, Geom. Dedicata **68** (1997) 91–99.
doi:10.1023/A:1004993723280

[8] S. Jendrol' and M. Maceková, *Describing short paths in plane graphs of girth at least* 5, Discrete Math. **338** (2015) 149–158.
doi:10.1016/j.disc.2014.09.014

[9] S. Jendrol', M. Maceková and R. Soták, *Note on* 3*-paths in plane graphs of girth* 4, Discrete Math. **338** (2015) 1643–1648.
doi:10.1016/j.disc.2015.04.011

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