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SOLUTIONS OF SOME $L(2,1)$-COLORING RELATED OPEN PROBLEMS

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Abstract

An $L(2,1)$-*coloring* (or *labeling*) of a graph G is a vertex coloring $f : V(G) \to Z^+ \cup \{0\}$ such that $|f(u) - f(v)| \geq 2$ for all edges uv of G, and $|f(u)-f(v)| \geq 1$ if $d(u,v) = 2$, where $d(u,v)$ is the distance between vertices u and v in G. The *span of an* $L(2,1)$-*coloring* is the maximum color (or label) assigned by it. The *span of a graph* G is the smallest integer λ such that there exists an $L(2,1)$-coloring of G with span λ. An $L(2,1)$-coloring of a graph with span equal to the span of the graph is called a *span coloring*. For an $L(2,1)$-coloring f of a graph G with span k, an integer h is a *hole* in f if $h \in (0,k)$ and there is no vertex v in G such that $f(v) = h$. A *no-hole coloring* is an $L(2,1)$-coloring with no hole in it. An $L(2,1)$-coloring is *irreducible* if color of none of the vertices in the graph can be decreased to yield another $L(2,1)$-coloring of the same graph. A graph G is *inh-colorable* if there exists an irreducible no-hole coloring of G. Most of the results obtained in this paper are answers to some problems asked by Laskar *et al.* [5]. These problems are mainly about relationship between the span and maximum no-hole span of a graph, lower inh-span and upper inh-span of a graph, and the maximum number of holes and minimum number of holes in a span coloring of a graph. We also give some sufficient conditions for a tree and an unicyclic graph to have inh-span $\Delta + 1$.

Keywords: $L(2,1)$-coloring, span of a graph, no-hole coloring, irreducible coloring, unicyclic graph.

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1. Introduction

The channel assignment problem is the problem of efficiently assigning frequencies to radio transmitters at various places without interference. This problem can be modeled as some kind of vertex coloring problem of the graph in which transmitters are taken as vertices and based on the proximity of the transmitters and the power of the transmissions, edges are placed between them to represent possible interference. The channel assignment problem that of prescribing integer labels for vertices so that neighboring vertices receive labels that differ by at least two while vertices with a common neighbor have different labels is called an *$L(2,1)$-coloring* and has been studied extensively in the literature. More precisely, an *$L(2,1)$-coloring* of a graph G is a vertex coloring (or labeling) $f : V(G) \to Z^+ \cup \{0\}$ such that $|f(u) - f(v)| \geq 2$ for all edges uv of G, and $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$, where $d(u,v)$ is the distance between vertices u and v in G. The *span of an $L(2,1)$-coloring* f of a graph G, denoted by span (f), is equal to $\max\{f(v) : v \in V(G)\}$. The *span of a graph* G, denoted by $\lambda(G)$, is equal to $\min\{\text{span } (f) : f \text{ is an } L(2,1)\text{-coloring of } G\}$. An $L(2,1)$-coloring whose span is equal to the span of the graph is called a *span coloring*.

Throughout the paper we consider simple connected graphs and denote the maximum degree of a graph by Δ. In the introductory paper, Griggs and Yeh [3] studied $L(2,1)$-coloring of a graph and gave the following results. For paths P_n, $\lambda(P_2) = 2$, $\lambda(P_3) = \lambda(P_4) = 3$ and $\lambda(P_n) = 4$ for $n \geq 5$. For any cycle C_n, $\lambda(C_n) = 4$. For the n dimensional hypercube Q_n with $n \geq 5$, $n + 3 \leq \lambda(Q_n) \leq 2n + 1$. For any tree T, $\Delta + 1 \leq \lambda(T) \leq \Delta + 2$ and for any graph G, $\lambda(G) \leq \Delta^2 + 2\Delta$. Further, if G has diameter 2, then $\lambda(G) \leq \Delta^2$. Georges *et al.* [2] studied the relationship between the $L(2,1)$-span of a graph and the path covering number of its complement. Wang [8] proved that if a tree T contains no two vertices of maximum degree at distance 1, 2 or 4, then $\lambda(T) = \Delta + 1$. Zhai *et al.* [10] proved that if a tree T with $\Delta \geq 5$ contains no two vertices of maximum degree at distance 2 or 4, then $\lambda(T) = \Delta + 1$.

If f is an $L(2,1)$-coloring of a graph G with span k, then an integer $h \in (0,k)$ is called a *hole* in f if there is no vertex v in G such that $f(v) = h$. An $L(2,1)$-coloring f of a graph G with no hole is called a *no-hole coloring* of G. The *no-hole span* of a graph G, denoted by $\mu(G)$, is ∞ if G does not have any no-hole coloring; otherwise, it is the smallest integer k such that G has a no-hole coloring with span k. The *hole index* of a graph G, denoted by $\rho(G)$, is the minimum number of colors less than $\lambda(G)$ and not used in a span coloring of G. A no-hole coloring f is called a *full coloring* if span (f) is equal to the span of the graph. The *maximum no hole span* [5] of a graph G, denoted by $\Lambda(G)$, is equal to $\max\{\text{span } (f) : f$ is a no-hole $L(2,1)$-coloring of $G\}$. Fishburn and Roberts [1] introduced no-hole colorings and gave the following results. For each $m \geq 1$, there is a graph G with

$\rho(G) = m$ and $\mu(G) = \lambda(G) + m$. For every $m \geq 2$, there is a connected graph G on $\lambda(G) + 2$ vertices with $\lambda(G) = 2m$ and $\rho(G) = m$.

An $L(2,1)$-coloring of a graph G is *reducible* if there exists another $L(2,1)$-coloring g of G such that $g(u) \leq f(u)$ for all vertices $u \in V(G)$ and there exists a vertex $v \in V(G)$ such that $g(v) < f(v)$. Otherwise, f is said to be *irreducible* [7]. An irreducible no-hole coloring is referred to as an *inh-coloring*. A graph is *inh-colorable* if there exists an inh-coloring of it. For an inh-colorable graph G the *lower inh-span* or simply *inh-span* of G, denoted by $\lambda_{inh}(G)$, and the *upper inh-span* of G, denoted by $\Lambda_{inh}(G)$, are defined as $\lambda_{inh}(G) = \min\{\text{span }(f) : f \text{ is an inh-coloring of } G\}$ and $\Lambda_{inh}(G) = \max\{\text{span }(f) : f \text{ is an inh-coloring of } G\}$. If G is not inh-colorable then $\lambda_{inh}(G) = \Lambda_{inh}(G) = \infty$.

Laskar and Villalpando [7] gave the following results on inh-coloring. For any graph G if $\lambda(G) = \Delta + 1$ and $\lambda_{inh}(G) > \Delta + 1$, then for any span coloring f of G either $f(u) = 0$ for all maximum degree vertices u or $f(u) = \Delta + 1$ for the same vertices u. For any connected unicyclic graph G except C_4, G is inh-colorable if and only if $\Delta < n - 1$, and the inh-span of an inh-colorable unicyclic graph is $\Delta + 1$ or $\Delta + 2$. Any triangular lattice $H_{r,c}$, where $r, c \geq 5$, is inh-colorable and $8 \leq \lambda_{inh}(H_{r,c}) \leq 13$. Laskar *et al.* [6] proved that all trees except stars are inh-colorable and for such a tree T, $\lambda_{inh}(T) = \lambda(T)$. Laskar and Eyabi [4] worked on the maximum number of holes in a span coloring of paths, cycles, trees and complete multipartite graphs.

In this paper we prove that for any no-hole colorable graph G, maximum no-hole span is one less than the number of vertices of G. Then we answer the following questions asked by Laskar *et al.* in [5].

Problem 1. Is it true that for all $r \in \mathbb{Z}^+$, there exists an infinite family $\mathcal{F}$ of graphs such that $\Lambda(G) - \lambda(G) = r$ if $G \in \mathcal{F}$?

Problem 2. Is it true that for all $r \in \mathbb{Z}^+$, there exists an infinite family $\mathcal{F}$ of graphs such that $\Lambda_{inh}(G) - \lambda_{inh}(G) = r$ if $G \in \mathcal{F}$?

Problem 3. Is it true that for all $r \in \mathbb{Z}^+ \cup \{0\}$, there exists an infinite family $\mathcal{F}$ of graphs such that the difference between the minimum number of holes and the maximum number of holes in a span coloring of G is r, if $G \in \mathcal{F}$?

Finally, we give partial solutions to the following three problems asked in [5].

Problem 4. For which classes of graphs G, $\lambda(G) = \Lambda(G)$?

Problem 5. Characterize all trees with span $\Delta + 1$.

Problem 6. Characterize all unicyclic graphs with inh-span $\Delta + 1$.

2. Our Results

In this section, $d(u, v)$ stands for the distance between the vertices u and v in a graph and whenever vertices u and v are adjacent we denote $u \sim v$.

We first find the value of maximum no-hole span $\Lambda(G)$ for a no-hole colorable graph G, which will be used in the sequel.

Lemma 1. *For a graph G on n vertices, $\lambda(G) \leq n-1$ if and only if G is no-hole colorable with $\Lambda(G) = n-1$.*

Proof. Let G be a graph on n vertices. The sufficient part is obviously true as a no-hole coloring is also an $L(2,1)$-coloring. Georges *et al.* [2] proved that $\overline{G}$ contains a Hamiltonian path if and only if $\lambda(G) \leq n-1$. Griggs and Yeh [3] proved that there exists an injection $f : V(G) \to [0, n-1]$ such that $|f(x) - f(y)| \geq 2$ for all $xy \in E(G)$, if and only if $\overline{G}$ contains a Hamiltonian path. In other words, f is a no-hole $L(2,1)$-coloring of G with span $n-1$ if and only if $\lambda(G) \leq n-1$. Therefore, whenever $\lambda(G) \leq n-1$, there is a no-hole coloring of G with span $n-1$ and so $\Lambda(G) \geq n-1$. Since G has n vertices, $\Lambda(G) \leq n-1$, and we prove the necessary part. ■

Theorem 2. *For all no-hole colorable graphs G on n vertices, $\Lambda(G) = n-1$.*

Proof. Let f be a no-hole coloring of a graph G on n vertices. Then f uses at most n colors and f is also an $L(2,1)$-coloring. Therefore, $\lambda(G) \leq n-1$ and the result follows from Lemma 1. ■

Theorem 3 below gives a condition on graph G to have $\lambda(G) = \Lambda(G)$. This condition is also sufficient to obtain an irreducible no-hole coloring of a graph. Thus we get a partial solution to Problem 4 stated in the Introduction.

Theorem 3. *If G is a no-hole colorable diameter two graph on n vertices, then G has an irreducible no-hole coloring and $\lambda(G) = \lambda_{inh}(G) = \Lambda_{inh}(G) = \Lambda(G) = n-1$.*

Proof. By Theorem 2, $\Lambda(G) = n-1$. Then by Lemma 1, $\lambda(G) \leq n-1$. Since the graph has diameter two, distinct vertices get distinct colors in an $L(2,1)$-coloring of G. So $\lambda(G) \geq n-1$ and then $\lambda(G) = n-1$. Consider a span coloring f of G. Since f gives different colors to different vertices and span of f is $n-1$, f is an irreducible no-hole coloring of G. Now $\lambda(G) = \Lambda(G) = n-1$ and $\lambda(G) \leq \lambda_{inh}(G) \leq \Lambda_{inh}(G) \leq \Lambda(G)$. So we get the result. ■

The corollary below gives a condition for a diameter two graph to have a no-hole coloring.

Corollary 4. *If G is a diameter two graph on n vertices and $3 \leq \Delta < \frac{n-1}{2}$, then G is inh-colorable and $\lambda(G) = \lambda_{inh}(G) = \Lambda_{inh}(G) = \Lambda(G) = n-1$.*

Proof. Griggs and Yeh [3] proved that if $3 \leq \Delta < \frac{n-1}{2}$, then G has an $L(2,1)$-coloring with span $n-1$. So $\lambda(G) \leq n-1$ and, by Lemma 1, G is no-hole colorable. Now since G has diameter two, by Theorem 3, $\lambda(G) = \lambda_{inh}(G) = \Lambda_{inh}(G) = \Lambda(G) = n-1$. ■

Theorem 5 below gives a solution of Problem 1 stated in the Introduction.

Theorem 5. *For any positive integer r there exists an infinite family $\mathcal{F}$ of no-hole colorable graphs such that $\Lambda(G) - \lambda(G) = r$ if $G \in \mathcal{F}$.*

Proof. For any integer $k \geq 3$, we construct a graph G_{kr} as follows. $V(G_{kr}) = \{u_1, u_2, \ldots, u_{k+1}, v_1, v_2, \ldots, v_{k+r}\}$, $u_i \sim u_j$ for all $i \neq j$, $u_{k+1} \sim v_1, v_j \sim v_{j+1}$ $(1 \leq j \leq k+r-1)$. In other words, $\{u_1, u_2, \ldots, u_{k+1}\}$ induces the complete graph K_{k+1}, $\{v_1, v_2, \ldots, v_{k+r}\}$ induces a path, and u_{k+1} is adjacent to v_1.

Since K_{k+1} is a subgraph of G_{kr}, $\lambda(G_{kr}) \geq 2k$. We give an $L(2,1)$-coloring f to G_{kr} with span $2k$ as follows: $f(u_i) = 2i - 2$ for $1 \leq i \leq k+1$, $f(v_1) = 1$, $f(v_{3j}) = 0$ for $j \geq 1$, $f(v_{3j+1}) = 2$ for $j \geq 1$, $f(v_{3j+2}) = 4$ for $j \geq 0$. The difference between the labels of any two vertices in K_{k+1} is at least two. Whenever two vertices in the path get the same label their distance is at least three from the way of defining f. The difference between the labels of any two adjacent vertices in the path is at least two. All u_i, $1 \leq i \leq k$, have distance at least 3 from all v_j, $2 \leq j \leq k+r$. Moreover, $d(u_i, v_1) = 2$ for $1 \leq i \leq k$. $|f(u_i) - f(v_1)| \geq 1$ for $1 \leq i \leq k$. $|f(u_{k+1}) - f(v_j)| \geq 2$ for $1 \leq j \leq k+r$, since $k \geq 3$. So f is an $L(2,1)$-coloring with $\lambda(G_{kr}) \leq 2k$, and finally we get $\lambda(G_{kr}) = 2k$.

Now since $\lambda(G_{kr}) = 2k < 2k + r + 1 = |V(G_{kr})|$, by Lemma 1, $\Lambda(G_{kr}) = |V(G_{kr})| - 1 = 2k + r = \lambda(G_{kr}) + r$. So $\Lambda(G_{kr}) - \lambda(G_{kr}) = r$. Since k can take any integer value greater than 2, we get an infinite family of graphs, $\mathcal{F} = \{G_{kr} : k > 2\}$, such that $\Lambda(G) - \lambda(G) = r$ if $G \in \mathcal{F}$. ∎

Theorem 6 below gives a solution of Problem 2 stated in the Introduction.

Theorem 6. *For any positive integer r there exists an infinite family $\mathcal{F}$ of irreducible no-hole colorable graphs such that $\Lambda_{inh}(G) - \lambda_{inh}(G) = r$ if $G \in \mathcal{F}$.*

Proof. For any integer $k \geq r + 4$ we construct a graph G_{kr} as below. The vertex set of G_{kr}, $V(G_{kr}) = A \cup B \cup C$, where $A = \{u_1, u_2, \ldots, u_{k+1}\}$, $B = \{v_1, v_2, \ldots, v_{k+1}\}$ and $C = \{w_1, w_2, \ldots, w_r\}$. Adjacency among the vertices in G_{kr} are given as $u_i \sim u_j$, $v_i \sim v_j$ for all $1 \leq i \neq j \leq k+1$, and $u_1 \sim w_i$, $v_1 \sim w_i$ for all $1 \leq i \leq r$.

We give an inh-coloring f to G_{kr} with span $2k+r+1$ as follows: $f(u_i) = 2i-2$, $f(v_i) = 2i-1$ for $1 \leq i \leq k+1$ and $f(w_j) = 2k+j+1$ for $1 \leq j \leq r$, see Figure 1.

We first prove that f is an $L(2,1)$ coloring. Difference between the labels of any two vertices in A (or in B) is at least two. Difference between the labels of any two vertices, one in A and the other in B is at least one, and the distance between them is at least two. For any w_i the neighbors of w_i are u_1 and v_1, $f(w_i) \geq 3$, $f(u_1) = 0$ and $f(v_1) = 1$. For any w_i, label of w_i is different from every other vertex. So the given coloring f is an $L(2,1)$-coloring. One sees that f is a no-hole coloring.

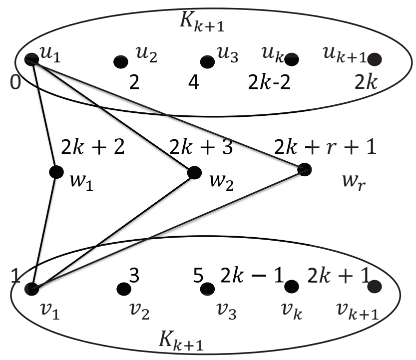

Figure 1. inh-coloring of G_{kr} with span $2k+r+1$.

Next we prove that f is irreducible. Since $f(A)=\{0,2,\dots,2k\}$, for any color p, $0\le p\le 2k$, either $p-1$ or p is used to color a vertex in A and since A induces a complete graph, the color of no u_i can be reduced. Because of the similar reason the color of no v_i, $2\le i\le k+1$, can be reduced. Since v_1 is at distance two from a vertex colored 0, color of v_1 cannot be reduced. Each w_i, $1\le i\le r$, has distance at most two from all the remaining vertices of the graph G_{kr}. So color of each w_i, $1\le i\le r$, have to be different from all the vertices in $G_{kr}-\{w_i\}$. Since $f(A\cup B)=\{0,1,2,\dots,2k+1\}$ and $f(C)=\{2k+2,2k+3,\dots,2k+r+1\}$, the color of no w_i can be reduced. So f is irreducible and $\Lambda_{inh}(G_{kr})\ge 2k+r+1$.

G_{kr} has $2k+r+2$ vertices. According to Lemma 1, $\Lambda(G_{kr})=2k+r+1$. $\Lambda_{inh}(G_{kr})\le\Lambda(G_{kr})\le 2k+r+1$, and so we get $\Lambda_{inh}(G_{kr})=2k+r+1$.

We give an inh-coloring g to G_{kr} with span $2k+1$: $g(u_1)=2k$, $g(u_i)=2i-4$ for $2\le i\le k+1$, $g(v_i)=2r-1+2i$ for $1\le i\le k+1-r$, $g(v_{k+1-j})=2r-2-2j$ for $0\le j\le r-1$, $g(w_i)=2i-1$ for $1\le i\le r$. Note that $g(A)=\{0,2,\dots,2k\}$ with $g(u_1)=2k$, $g(B)=\{0,2,\dots,2r-4,2r-2,2r+1,2r+3,\dots,2k-1,2k+1\}$ with $g(v_1)=2r+1$, and $g(C)=\{1,3,5,\dots,2r-1\}$, see Figure 2.

We first prove that g is an $L(2,1)$-coloring. Difference between labels of any two vertices within A,B or C is at least two. Each w_i is adjacent to both u_1 and v_1, and obviously $|g(w_i)-g(u_1)|\ge 2$ and $|g(w_i)-g(v_1)|\ge 2$, $1\le i\le r$. The distance between each w_i and a vertex in $(A-\{u_1\})\cup(B-\{v_1\})$ is exactly two and difference between their labels is at least one. $d(u_1,v_1)=2$ and obviously $|g(u_1)-g(v_1)|\ge 1$. The distance between a vertex in A (respectively B) and a vertex in $B-\{v_1\}$ (respectively $A-\{u_1\}$) is at least three, and therefore no need of calculating difference between their labels. Since $g(V(G_{kr}))=\{0,1,2,\dots,2k+1\}$, g is a no-hole coloring.

Next we prove that g is irreducible. A induces a complete graph and $g(A)=$

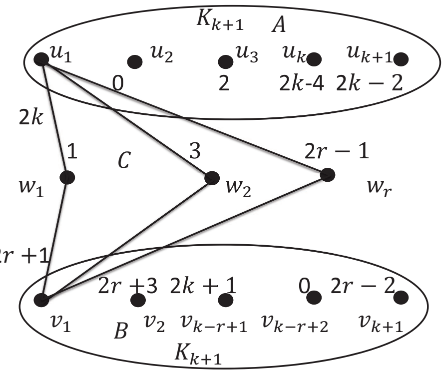

Figure 2. inh-coloring of G_{kr} with span $2k+1$.

$\{0, 2, \ldots, 2k\}$. So color of no vertex in A can be reduced. Every pair of vertices in C are at distance two from each other and each $w_i \in C$ is having distance at most two from all the vertices in A, so color of no vertex in C can be reduced. B can be partitioned into two sets B_1 and B_2 where $g(B_1) = \{0, 2, \ldots, 2r-2\}$ and $g(B_2) = \{2r+1, 2r+3, \ldots, 2k+1\}$. For any $x \in B_2$, color of x cannot be reduced because for any $i \in \{0, 1, 2, \ldots, 2r-1\}$ there exists a vertex y with color i in G_{kr} such that $d(x, y) \leq 2$, and for any $j \in \{2r, 2r+1, \ldots, g(x)-1\}$ there exists a vertex z in G_{kr} with color $j-1$, j or $j+1$ adjacent to x. Because of the similar reason color of no vertex in B_1 can be reduced.

Since A induces K_{k+1}, $\lambda_{inh}(G_{kr}) \geq 2k$. If $\lambda_{inh}(G_{kr}) = 2k$, then there is an inh-coloring h of G_{kr} with span $(h) = \lambda_{inh}(G_{kr}) = 2k$. Then h assigns colors $\{0, 2, \ldots, 2k\}$ to vertices of both A and B, and the remaining vertices $\{w_1, w_2, \ldots, w_r\}$ use at most r colors. So there will be at least $k-r$ (which is at least 4) holes in h and this is a contradiction. Hence $\lambda_{inh}(G_{kr}) \geq 2k+1$ and we have $\lambda_{inh}(G_{kr}) = 2k+1$.

So $\Lambda_{inh}(G_{kr}) - \lambda_{inh}(G_{kr}) = r$. Since k can take any integer value greater than $r+3$, we get an infinite family of graphs, $\mathcal{F} = \{G_{kr} : k > r+3\}$, such that $\Lambda_{inh}(G) - \lambda_{inh}(G) = r$ if $G \in \mathcal{F}$. ■

Theorem 7 below gives a solution of Problem 3 stated in the Introduction.

Theorem 7. *For all non-negative integer r there exists an infinite family $\mathcal{F}$ of graphs such that the difference between the minimum number of holes in a span coloring and the maximum number of holes in a span coloring of G is r, if $G \in \mathcal{F}$.*

Proof. The result is true for the case $r = 0$ because the number of holes in any span coloring of K_n $(n \geq 2)$ is $n-1$.

Next let $r = 1$. For integers $m, n \geq 3$, we construct a graph $G_{1,m,n}$ as below. $V(G_{1,m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, w\}$ where $u_i \sim v_j$ for $i = 1, 2, \ldots, m$,

$j = 1, 2, \ldots, n$, and $w \sim v_n$. In other words, $G_{1,m,n}$ is the complete bipartite graph $K_{m,n}$ with an extra vertex w, which is adjacent to only one vertex of the $K_{m,n}$. We first show that the span of $G_{1,m,n}$ is $m+n$. It is known that $\lambda(K_{m,n}) = m+n$, see [4]. Since $G_{1,m,n}$ contains $K_{m,n}$ as a subgraph, $\lambda(G_{1,m,n}) \geq m+n$.

We give an $L(2,1)$-coloring f to $G_{1,m,n}$ with span $m+n$ and no holes. Namely, let $f(u_i) = i-1$ for $1 \leq i \leq m$, $f(v_j) = m+j$ for $1 \leq j \leq n$, and $f(w) = m$.

We prove that f is an $L(2,1)$-coloring. $G_{1,m,n}$ is a bipartite graph. If two vertices are assigned consecutive colors by f, then they are in the same partite set except the vertices w and v_1. $d(w, v_1) = 3$. So if for two vertices u and v, $|f(u) - f(v)| = 1$ then $d(u,v) \geq 2$. No color is repeated. Hence, the given coloring is an $L(2,1)$-coloring. So $\lambda(G_{1,m,n}) \leq m+n$. Notice that f is a no-hole coloring.

We give a span coloring g of $G_{1,m,n}$ with only one hole: $g(u_i) = i-1$ for $1 \leq i \leq m$, $g(v_j) = m+j$ for $1 \leq j \leq n$, and $g(w) = m+1$. g assigns the same color as f except for the color of w. The vertex w gets the color which is the same as the color of v_1. But $d(v_1, w) = 3$. No vertex has the color $g(w) - 1$. Only the vertex v_2 has the color $g(w)+1$, but $d(v_2, w) = 3$. So one checks that g is an $L(2,1)$-coloring. Since $G_{1,m,n}$ contains $K_{m,n}$ as a subgraph and all the vertices in $K_{m,n}$ have to get distinct colors (as $K_{m,n}$ has diameter two), and $\lambda(G_{1,m,n}) = m+n$, a span coloring of $G_{1,m,n}$ can have at most one hole. Hence the theorem is true for $r = 1$.

Let $r \geq 2$. For any integer $n \geq r+1$, we construct a graph $G_{r,n}$ as follows. $V(G_{r,n}) = \{u_1, u_2, \ldots, u_{r+2}, v_1, v_2, \ldots, v_n\}$ with $u_i \sim u_j$ for all $i \neq j$, $u_{r+2} \sim v_1$, $v_i \sim v_{i+1}$ $(1 \leq i \leq n-1)$. In other words, $\{u_1, u_2, \ldots, u_{r+2}\}$ induces a complete graph, $\{v_1, v_2, \ldots, v_n\}$ induces a path, and u_{r+2} is adjacent to v_1. Since K_{r+2} is a subgraph of $G_{r,n}$, $\lambda(G_{r,n}) \geq 2r+2$.

We show that $\lambda(G_{r,n}) = 2r+2$, and for this we give an $L(2,1)$-coloring f' of $G_{r,n}$ with span $2r+2$ as follows: $f'(u_i) = 2i-2$ for $1 \leq i \leq r+2$ and $f'(v_j) = 2k-1$ for $1 \leq j \leq n$, where $j \equiv k (\text{mod } r+1)$, and $1 \leq k \leq r+1$.

The difference between the labels of any two vertices in K_{r+2} is at least two. Whenever two vertices in the path get the same label their distance is at least $r+1$ from the way of defining f'. The difference between the labels of any two adjacent vertices in the path is at least two. All u_i, $1 \leq i \leq r+1$, have distance at least 3 from all v_j, $2 \leq j \leq n$. Moreover, $d(u_i, v_1) = 2$ for $1 \leq i \leq r+1$. $|f'(u_i) - f'(v_1)| \geq 1$ for $1 \leq i \leq r+1$. $|f'(u_{r+2}) - f'(v_j)| \geq 2$ for $1 \leq j \leq 2$, since $r \geq 2$. $d(u_{r+2}, v_j) > 2$ if $j > 2$. So f' is an $L(2,1)$-coloring and $\lambda(G_{r,n}) \leq 2r+2$. So $\lambda(G_{r,n}) = 2r+2$ and f' is also a no-hole coloring, and therefore the minimum number of holes in a span coloring of $G_{r,n}$ is 0.

Every $L(2,1)$-coloring of $G_{r,n}$ has to use at least $r+3$ different colors because the vertices $u_1, u_2, \ldots, u_{r+2}, v_1$ get distinct colors. So the maximum number of holes in a span coloring of $G_{r,n}$ is at most r. We give a span coloring g' of $G_{r,n}$ with r holes as follows: $g'(u_i) = 2i-2$ for $1 \leq i \leq r+2$, $g'(v_1) = 1$, $g'(v_{3j}) = 0$

for $j \geq 1$, $g'(v_{3j+1}) = 2$ for $j \geq 1$, and $g'(v_{3j+2}) = 4$ for $j \geq 0$.

The difference between the labels of any two vertices in K_{r+2} is at least two. Whenever two vertices in the path get the same label their distance is at least three from the way of defining g'. The difference between the labels of any two adjacent vertices in the path is at least two. All u_i, $1 \leq i \leq r+1$, have distance at least 3 from all v_j, $2 \leq j \leq n$. $d(u_i, v_1) = 2$ for $1 \leq i \leq r+1$. Moreover, $|g'(u_i) - g'(v_1)| \geq 1$ for $1 \leq i \leq r+1$. $|g'(u_{r+2}) - g'(v_j)| \geq 2$ for $1 \leq j \leq n$, since $r \geq 2$. So g' is an $L(2,1)$-coloring. In fact g' is a span coloring of $G_{r,n}$ with r holes and therefore the result is true in this case. ∎

Recall that for any tree T, $\Delta + 1 \leq \lambda(T) \leq \Delta + 2$, see [3]. Also recall that if a tree T contains no two vertices of maximum degree at distance 1, 2 or 4, then $\lambda(T) = \Delta + 1$ [8], and if a tree T with $\Delta \geq 5$ contains no two vertices of maximum degree at distance 2 or 4, then $\lambda(T) = \Delta + 1$ [10]. Here we prove that for a tree T with $\Delta \geq 5$, if the distance between only one pair of maximum degree vertices is 2 or 4, and the distance between every other pair of maximum degree vertices is greater than or equal to 7, then $\lambda(T) = \Delta + 1$. This gives a partial solution to Problem 5 stated in the Introduction. In the rest of the paper we use greedy $L(2,1)$-coloring of a graph which is given below.

Algorithm 8 (Greedy coloring)**.** Let G be a graph whose few vertices might have been colored before.

1. Order the vertices of the given graph as $u_1, u_2, \ldots, u_n$ such that all colored vertices (if any) appear at the beginning of the list.
2. Let u_i be the first uncolored vertex that appears in the list.
3. Color u_i with the smallest possible color k such that no lower indexed neighbor of u_i in the list is colored with $k-1$, k or $k+1$ and no lower indexed vertex at distance two from u_i is colored with k.
4. If all the vertices of the graph have received color then stop; otherwise set $i = i + 1$ and go to 3.

Theorem 9. *Algorithm 8 gives an $L(2,1)$-coloring of G if and only if the pre-colored vertices of G satisfy constraints of an $L(2,1)$-coloring in the graph G.*

Definition. Here we define the *distance between a vertex u and a subgraph G_1 of a graph G* as $d(u, G_1) = \min\{d(u,v) : v \in V(G_1)\}$.

Notation 10. For any rooted tree and a vertex x in it, $p(x)$ denotes the parent of x in the tree.

Theorem 11. *Let T be a tree with $\Delta \geq 5$. If the distance between only one pair of maximum degree vertices in T is 2 or 4, and the distance between every other pair of maximum degree vertices in T is greater than or equal to 7, then $\lambda(T) = \Delta + 1$.*

Proof. Here we give an $L(2,1)$-coloring f of T with span $\Delta + 1$ following the steps of the algorithm below.

Algorithm: Step 1. We construct an induced subgraph T_1 of T such that T_1 consists of all the maximum degree vertices of T and the paths between them.

Step 2. Let u and v be the two maximum degree vertices in T at distance 2 or 4. We make T and T_1 rooted trees with u as the root vertex. We assign $f(u) = 0$ and $f(v) = \Delta + 1$. If $d(u, v) = 2$, we color the vertex between u and v with color 2. If $d(u, v) = 4$, we color the vertices between u and v in order with colors 2, 4 and 0. We color all other vertices of T_1 which are maximum degree vertices in T with the color 0.

Step 3. We partition $V(T_1)$ into three sets S_1, S_2 and S_3, where S_1 consists of all the vertices on the $u - v$ path, $S_2 = \{x : u - x$ path does not contain the vertex $v\} - S_1$ and $S_3 = V(T_1) - S_1 - S_2$. Notice that all the vertices of S_1 have received colors.

Step 4. For every uncolored vertex x in S_2 (or S_3), let $h(x) = \min\{d(x, w) : w$ is a descendant of x and is a vertex of maximum degree in $T\}$.

Step 5. The uncolored vertices of S_2 (respectively S_3) are listed in nondecreasing order of their distances from u. When two vertices are at the same distance from u we give priority to the vertex x with minimum value of $h(x)$. When two vertices have the same distance from u and have the same value of $h(x)$, any vertex can be given priority. We color the vertices in S_2 (respectively S_3) greedily (Algorithm 8) according to this ordering.

Step 6. We extend the coloring of T_1 to T. We list the vertices of T which are not in T_1 in nondecreasing order of their distances from T_1. We color T following Algorithm 8.

We observe that the vertices on the $u - v$ path and the other maximum degree vertices satisfy the constraints of an $L(2,1)$-coloring on T. Since all other vertices of T_1 are colored greedily, f is an $L(2,1)$-coloring of T_1 by Theorem 9. Since vertices in $V(T) - V(T_1)$ are colored following greedy algorithm and since the distance between two vertices in T_1 is the same as the distance in T, the coloring obtained for T is an $L(2,1)$-coloring again by Theorem 9.

We prove that span $(f) = \Delta+1$. We first prove that $f(z) \leq \Delta+1$ if $z \in V(T_1)$. Since the distance between any two maximum degree vertices (except between u and v) of T is greater than 6, and the vertices are colored in nondecreasing order of their distances from the root vertex, when a vertex in S_2 or S_3 is colored at most one of its children or grandchildren (which is a maximum degree vertex) has already received color. Let x_1 be a vertex of T_1 such that when it is colored no children or grandchildren of it is already colored. Then the parent $p(x_1)$ of

x_1 is a maximum degree vertex of T or not. If $p(x_1)$ is a vertex of maximum degree then $p(x_1)$ is colored with 0 or $\Delta+1$. $p(x_1)$ has at most $\Delta-1$ numbers of colored neighbors. If $p(x_1)$ gets the color 0 (respectively $\Delta+1$) then there is a color available for x_1 in $[2,\Delta+1]$ (respectively $[0,\Delta-1]$). If $p(x_1)$ is not a vertex of maximum degree, then $p(x_1)$ can have at most $\Delta-2$ colored neighbors, so at least one color is available for x_1 in $[0,\Delta+1]$. Let x_2 be a vertex of T_1 such that when it is colored only one of its children is already colored and no grandchildren is colored. Therefore $h(x_2)=1$. If $p(x_2)\in S_1$ then $p(x_2)$ is at distance less than 3 from u or v, and so we get a pair of maximum degree vertices of T except the pair u,v at distance less than 7 from each other which is not possible. So $p(x_2)\notin S_1$ and $p(x_2)\in S_2$ or S_3 according as $x_2\in S_2$ or S_3. Let y_2 be any sibling of x_2 if one exists. Since the distance between the colored child of x_2 and a maximum degree vertex in T which is a descendant of y_2 is at least 7, $h(y_2)\geq 4>h(x_2)$. So y_2 is not colored before and the number of vertices already colored at distance 2 from x_2 is 1 (the grandparent of x_2). Thus x_2 is adjacent to only two vertices already colored and one of which gets the color 0. Since we are following greedy algorithm and x_2 can use a color different from $0,1,f(p(x_2)),f(p(x_2))\pm 1,f(p(p(x_2)))$, there is at least one color less than 7 is available for x_2. Thus $f(x_2)\leq\Delta+1$ since $\Delta\geq 5$. Let x_3 be a vertex of T_1 such that when it is colored only one of its grandchildren is already colored and no children is colored. Then $h(x_3)=2$. $p(x_3)\notin S_1$ as in the earlier case. No sibling y_3 of x_3 is colored before as $h(y_3)\geq 3>h(x_3)$. So number of vertices already colored at distance 2 from x_3 is 2 (the grandparent and the maximum degree grandchild of x_3). Thus x_3 is adjacent to only one vertex (parent of x_3) colored before. Since we are following greedy algorithm and x_3 can use a color different from $0,f(p(x_3)),f(p(x_3))\pm 1,f(p(p(x_3)))$, there is at least one color less than 6 available for x_3. Thus $f(x_3)\leq\Delta+1$ since $\Delta\geq 5$. So we get span $(f)=\Delta+1$ for T_1.

Now we prove that $f(z)\leq\Delta+1$ if $z\in V(T)-V(T_1)$. When a vertex in $V(T)-V(T_1)$ is colored greedily we only have to consider the color of its parent, grandparent and siblings. When $p(z)$ has degree Δ then it is colored with 0 or $\Delta+1$. z can use any color other than 0 ($\Delta+1$ respectively), 1 (Δ respectively), color of grandparent of z and color of at most $\Delta-2$ siblings, which are total at most $\Delta+1$ in number. When $p(z)$ has degree less than Δ, z can use any color other than $f(p(z)),f(p(z))\pm 1$, color of grandparent of z and color of at most $\Delta-2$ siblings, which are total at most $\Delta+1$ in number. So in any case, there is at least one color less than or equal to $\Delta+1$ available for z and so $f(z)\leq\Delta+1$. So span $(f)=\Delta+1$ for T and $\lambda(T)=\Delta+1$. ■

Example 12. We explain the algorithm given in the proof of Theorem 11 through an example below. Here we consider a tree T (Figure 3) with $\Delta=5$, the distance between a pair of maximum degree vertices is 2 and the distance between every other pair of maximum degree vertices is greater than or equal to 7.

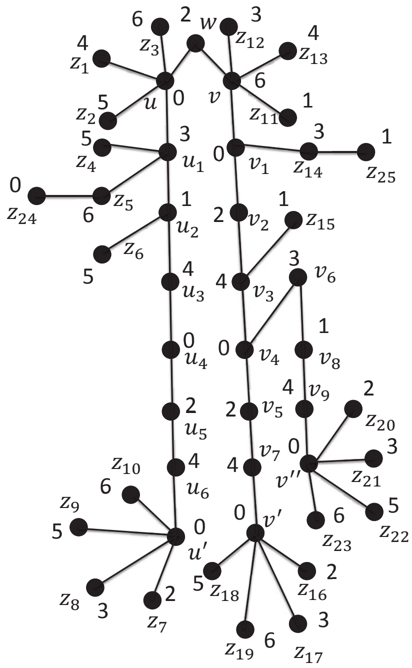

Figure 3. A tree T with $\Delta = 5$ and $\lambda(T) = 6$.

Step 1. Here $V(T_1) = \{u, v, w, u', v', v'', u_i, v_j : 1 \le i \le 6, 1 \le j \le 9\}$. T_1 is the induced subgraph on these vertices.

Step 2. Maximum degree vertices of T are u, v, u', v', v''. We assign $f(u) = f(u') = f(v') = f(v'') = 0, f(v) = 6, f(w) = 2$.

Step 3. Here $S_1 = \{u, w, v\}$, $S_2 = \{u', u_i : 1 \le i \le 6\}$, $S_3 = \{v', v'', v_j : 1 \le j \le 9\}$.

Step 4. $h(u_i) = 7 - i$ for $1 \le i \le 6$. $h(v_1) = 6, h(v_2) = 5, h(v_3) = 4, h(v_4) = 3, h(v_5) = 2, h(v_6) = 3, h(v_7) = 1, h(v_8) = 2, h(v_9) = 1$.

Step 5. The uncolored vertices of S_2 are ordered as $u_1, u_2, u_3, u_4, u_5, u_6$. These vertices are colored greedily according to this ordering. The uncolored vertices of S_3 are ordered as $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9$. Here $d(v, v_5) = d(v, v_6)$ but $h(v_5) < h(v_6)$ so v_5 appears before v_6. Similar reasoning is used for v_7 and v_8. These vertices are colored greedily according to this ordering.

Step 6. We order the vertices of $V(T) - V(T_1)$ as $z_1, z_2, \ldots, z_{24}, z_{25}$. These vertices are colored applying Algorithm 8. Thus one gets that $\lambda(T) = 6$.

It is known, see [7], that every unicyclic graph on n vertices with $\Delta < n-1$ is inh-colorable with inh-span $\Delta+1$ or $\Delta+2$. Now we give some sufficient conditions for unicyclic graphs to have inh-span $\Delta + 1$. This is a partial answer to Problem 6 stated in the Introduction.

Theorem 13. *Let G be an unicyclic graph with $7 \leq \Delta \leq n-2$. If G has four consecutive vertices u, v, w, x on the cycle such that $deg(u) \leq \Delta - 2, deg(v) = deg(w) = 2$ and $deg(x) \leq \Delta - 5$, and the distance between any two maximum degree vertices is not equal to 2 or 4, then $\lambda_{inh}(G) = \Delta + 1$.*

Proof. Let G_1 be the subgraph of G induced on the set $V(G) - \{v, w\}$. Then G_1 is a tree with maximum degree Δ and the distance between any two maximum degree vertices is not equal to 2 or 4. Hence from [10], $\lambda(G_1) = \Delta + 1$. Since G_1 is a tree different from a star, by [6], $\lambda_{inh}(G_1) = \lambda(G_1) = \Delta + 1$. Let f be an inh-span coloring of G_1. We define an inh-coloring g of G as follows: $g(x) = f(x)$ for all $x \in V(G_1)$, and then g assigns colors greedily to v, and then to w. Next, we check that g is an inh-coloring of G with span $\Delta + 1$. When v is colored there is only one colored vertex (that is u) adjacent to it and there are at most $\Delta - 2$ colored vertices (x and neighbors of u other than v) at distance two from it in G. So there is at least one color in $[0, \Delta + 1]$ available for v. So $f(v) \leq \Delta + 1$. When w is colored there are two colored vertices (that is v and x) adjacent to it and $\Delta - 5$ colored vertices (u and neighbors of x other than w) at distance two from it in G. So there is at least one color in $[0, \Delta + 1]$ available for w. So $g(w) \leq \Delta + 1$. Since $deg(v) = deg(w) = 2$, if the distance between two vertices in G_1 is 2, then it remains so in G also. Further, one checks that if the distance between any two vertices in G_1 is greater than 2 or more, then their distance will not be reduced to 2 or less in G. So constraints of an $L(2,1)$-coloring are satisfied and span $(g) = \Delta + 1$. Since f is an inh-coloring and v and w are colored greedily, g is an irreducible coloring. Since f and g have the same span and f is a no-hole coloring, so is g. Hence g is an inh-coloring and $\lambda_{inh}(G) = \Delta + 1$. ■

Theorem 14. *If G is an unicyclic graph with $9 \leq \Delta \leq n-2$ having exactly one maximum degree vertex, then $\lambda_{inh}(G) = \Delta + 1$.*

Proof. We give an $L(2,1)$-coloring f to G following the algorithm below.

Algorithm: Step 1. Let u be the maximum degree vertex and C be the cycle in G. f assigns color 0 to u. Since $\Delta \leq n-2$, there is at least one vertex at distance 2 from u, say v. Assign $f(v) = 1$. Then we color a vertex w between u and v greedily, that is $f(w) = 3$.

Step 2. Let $V(C) = \{u_1, u_2, \ldots, u_k\}$ and T_i be the largest subtree of G such that $V(T_i) \cap V(C) = \{u_i\}$. Make each T_i a rooted tree with u_i as the root.

Step 3. Order the uncolored vertices of C starting at a vertex nearest to u and other vertices in any order. For each i order the uncolored vertices of T_i in non-decreasing distances from the root vertex and breaking tie if the vertex is either adjacent or at distance two from u, v or w. Then order the vertices of the graph G as u, v, w, ordered uncolored vertices of C, ordered uncolored vertices of $T_1 - u_1, \ldots$, ordered uncolored vertices of $T_k - u_k$.

Step 4. Color graph G following Algorithm 8. Since vertices u, v and w satisfy the constraints of the $L(2,1)$-coloring in G, by Theorem 9 f is an $L(2,1)$-coloring. Now we prove that span $(f) = \Delta + 1$. When a vertex x in C is colored it has at most two (this will not be equal to 3 as the starting vertex in the ordering of vertices of C is nearest to u) colored neighbors, say x_1, x_2, and at most 4 colored vertices at distance two from x, two of which, say x_3 and x_4, may lie on C and the vertices u and v (when w lies on C). So x can use any color other than $f(x_1), f(x_2), f(x_3), f(x_4), f(x_1) \pm 1, f(x_2) \pm 1, f(u)$ and $f(v)$. Therefore, x can use a color less than or equal to 10. But 10 is less than or equal to $\Delta + 1$ and so $f(x) \leq \Delta + 1$. Now we consider vertices not on $C \cup \{u, v, w\}$. Let y be such a vertex. Suppose none of u, v, w is either a child or grandchild of y. If $p(y) = u$ then y can use any color other than $0, 1$ and color of $\Delta - 1$ neighbors of u. If $p(y) \neq u$ then y can use any color other than $f(p(y)), f(p(y)) \pm 1$, and colors of at most $\Delta - 2$ neighbors of $p(y)$. Therefore, at least one color is available for y in $[0, \Delta + 1]$. Next, let vertices u, v, w be children or grandchildren of y. One checks easily that the worst case is that w is a child and both u and v are grandchildren of y. Therefore, y can use any color other than $f(w), f(w) \pm 1, f(u), f(v), f(p(y)), f(p(y)) \pm 1$ and colors of two neighbors of $p(y)$ lying on C. Notice that in this case no sibling of y is colored so far because priority has been given for vertices nearer to u (see Step 3). So there is at least one color less than or equal to 10 available for y. But 10 is less than or equal to $\Delta + 1$, and so $f(y) \leq \Delta + 1$. Thus span $(f) \leq \Delta + 1$. This is an irreducible coloring since every vertex except u and v is colored greedily. Since u is colored zero, neighbors of u are colored with $2, 3, \ldots, \Delta + 1$ and v is colored with 1, this is a no hole coloring. Thus, $\lambda_{inh}(G) = \Delta + 1$. ∎

Example 15. We give an example of a unicyclic graph G where $\Delta = 9$ and G has only one maximum degree vertex, see Figure 4. This graph is given a coloring f following the algorithm used in the proof of Theorem 14.

Step 1. We assign $f(u) = 0, f(v) = 1$ and $f(w) = 3$.

Step 2. Here $V(C) = \{u_1, u_2, u_3, u_4\}$. $V(T_1) = \{u_1, u, v, w, x_{1j} : 1 \leq j \leq 13\}$, $V(T_2) = \{u_2\}, V(T_3) = \{u_3, x_{3j} : 1 \leq j \leq 4\}$ and $V(T_4) = \{u_4\}$.

Step 3. Vertices of C are ordered u_1, u_2, u_3, u_4. Uncolored vertices of T_1 are ordered $x_{11}, x_{12}, \ldots, x_{113}$. Uncolored vertices of T_3 are ordered $x_{31}, x_{32}, x_{33}, x_{34}$. Finally, all the vertices of $V(G)$ are ordered $u, v, w, u_1, u_2, u_3, u_4, x_{11}, x_{12}, \ldots, x_{113}$, $x_{31}, x_{32}, x_{33}, x_{34}$.

Step 4. Vertices of G are colored following Algorithm 8.

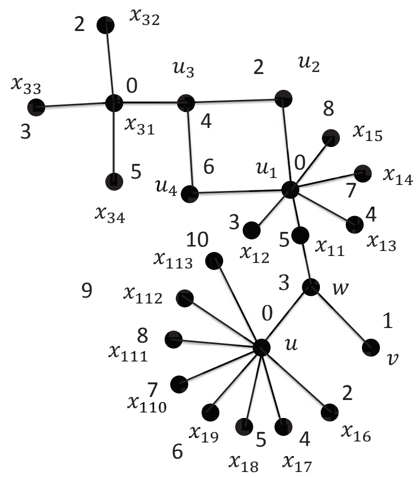

Figure 4. An unicyclic graph G with $\Delta = 9$, one maximum degree vertex and $\lambda_{inh}(G) = 10$.

Theorem 16. *Let G be an unicyclic graph with $9 \leq \Delta \leq n-2$ and suppose G has at least two vertices of maximum degree. If the distance between any two maximum degree vertices in G is at least 7, then $\lambda_{inh}(G) = \Delta + 1$.*

Proof. Here we give an $L(2,1)$-coloring f of G with span $\Delta + 1$ following the steps of the algorithm below.

Algorithm: Step 1. Let C be the cycle of G. Let G_1 be the connected subgraph of G consisting of the cycle C, the maximum degree vertices of G and paths either joining C and a maximum degree vertex or any two maximum degree vertices.

Step 2. We color an arbitrary maximum degree vertex u of G with color $\Delta + 1$ and all other maximum degree vertices of G with color 0.

Step 3. Let the vertices on the cycle be $u_1, u_2, \ldots, u_k$.

Step 4. For each i let T_i be the largest subtree of G_1 such that $V(T_i) \cap C = \{u_i\}$ for $1 \leq i \leq k$. We make T_i a rooted tree with u_i as the root vertex.

Step 5. Let $S_i = V(T_i)$. Let for any vertex x_i in S_i, $p(x_i)$ denote the parent of x_i. For every uncolored vertex x_i in S_i, $h(x_i) = \min\{d(x_i, w_i) : w_i$ is a descendant of x_i and is a vertex of maximum degree in $G\}$.

Step 6. The uncolored vertices of S_i are ordered according to their non-decreasing distances from u_i. When two vertices are at the same distance from u_i we give priority to the vertex x_i with minimum value of $h(x_i)$. When two vertices have the same distance from u_i and have the same value of $h(x_i)$ any vertex can be given priority.

Step 7. Order the vertices of the graph G_1 as the colored vertices of G_1, uncolored vertices of C, ordered uncolored vertices of $T_1 - u_1, \ldots,$ ordered uncolored vertices of $T_k - u_k$.

Step 8. Give a coloring, say f, to G_1 according to this ordering following Algorithm 8.

Step 9. We extend f to G. We order the vertices of G which are not in G_1 according to their non-decreasing distances from the cycle and then apply Algorithm 8 in that order.

Since the distance between any two maximum degree vertices of G is greater than 6, these vertices satisfy the constraints of an $L(2,1)$-coloring in G_1. So by Theorem 9, f is an $L(2,1)$-coloring of G_1. Since for $x, y \in V(G_1)$, the distance between x and y is the same in both G_1 and G, Theorem 9 implies that f is an $L(2,1)$-coloring of G.

Now we prove that span $(f) = \Delta+1$. When a vertex u_i on the cycle is colored it has at most two colored neighbors, say y_1, y_2, on the cycle, at most two colored vertices, say y_3, y_4, at distance 2 from it on the cycle and at most one maximum degree vertex, say v, in G which does not lie on C and is at distance one or two from u_i. The vertex v is colored with 0 or $\Delta + 1$. Therefore, u_i can use any color other than $f(y_1), f(y_1) \pm 1, f(y_2), f(y_2) \pm 1, f(y_3), f(y_4), 0$ (respectively $\Delta + 1$), 1 (respectively Δ). Therefore, u_i can use a color less than or equal to 10. But 10 is less than or equal to $\Delta + 1$, and so $f(u_i) \leq \Delta + 1$. Since the distance between any two maximum degree vertices of G is greater than 6, when a vertex in S_i is colored at most one of its children or grandchildren is already colored. Let y_{i1} be a vertex of T_i such that when it is colored no children or grandchildren of it is already colored. Then $p(y_{i1})$ is a maximum degree vertex of G or not. If $p(y_{i1})$ is a vertex of maximum degree, then $p(y_{i1})$ is colored with 0 or $\Delta+1$. Clearly, $p(y_{i1})$ has at most $\Delta-1$ colored neighbors. If $p(y_{i1})$ gets the color 0 (respectively $\Delta+1$) then there is a color available for y_{i1} in $[2, \Delta+1]$ (respectively $[0, \Delta-1]$). If $p(y_{i1})$ is not a vertex of maximum degree, then $p(y_{i1})$ can have at most $\Delta - 2$ colored neighbors, so at least one color is available for y_{i1} in $[0, \Delta + 1]$. Since we are following a greedy algorithm, $f(y_{i1}) \leq \Delta + 1$. Let y_{i2} be a vertex of T_i such that when it is colored only one of its children is already colored and no grandchildren is colored before. Observe that $h(y_{i2}) = 1$. Let z_{i2} be a sibling of y_{i2}. Since the distance between the colored child of y_{i2} and a descendant of z_{i2} having maximum degree in G is at least 7, $h(z_{i2}) \geq 4 > h(y_{i2})$. So z_{i2} is not colored before and the number of vertices already colored at distance 2 from y_{i2} is at most 2 (if $p(y_{i2})$ lies on the cycle). Let t_1, t_2 be the two colored neighbors of $p(y_{i2})$. Thus y_{i2} is adjacent to only two vertices already colored, one of which is colored with 0 or $\Delta+1$. So y_{i2} can use a color different from $f(p(y_{i2})), f(p(y_{i2})) \pm 1, 0$ (respectively $\Delta + 1$), 1 (respectively Δ), $f(t_1), f(t_2)$. Since we are using a greedy algorithm,

$f(y_{i2}) \le 7$. Thus $f(y_{i2}) \le \Delta + 1$ since $\Delta \ge 9$. Let y_{i3} be a vertex of T_i such that when it is colored only one of its grandchildren is already colored and no children is colored before. Clearly, $h(y_{i3}) = 2$. No sibling z_{i3} of y_{i3} is colored before since $h(z_{i3}) \ge 3 > h(y_{i3})$. So the number of vertices already colored at distance 2 from y_{i3} is at most 3 (if $p(y_{i3})$ lies on the cycle). Let t_3, t_4 be the two colored neighbors of $p(y_{i3})$ and t_5 be the colored grandchild of y_{i3}. Thus y_{i3} is adjacent to only one vertex already colored. So y_{i3} can use a color different from $f(p(y_{i3})), f(p(y_{i3})) \pm 1, f(t_3), f(t_4), f(t_5)$. Since we are using a greedy algorithm, $f(y_{i3}) \le 6$. Thus $f(y_{i3}) \le \Delta + 1$ since $\Delta \ge 9$, and we get span $(f) = \Delta + 1$ for G_1.

When a vertex x in $V(G) - V(G_1)$ is colored, we only have to consider one vertex x' adjacent to x and at most $\Delta - 1$ neighbors of x'. If x' has degree Δ, then it is colored 0 or $\Delta + 1$. Then there is a color other than 0 ($\Delta + 1$ respectively), 1 (Δ respectively) and colors of at most $\Delta - 1$ neighbors of x' available for x. If x' has degree less than Δ, then there is a color other than $f(x'), f(x') \pm 1$ and colors of at most $\Delta - 2$ neighbors of x', available for x. Since we are using a greedy coloring, $f(x) \le \Delta + 1$. So span $(f) = \Delta + 1$ for G. There is a maximum degree vertex colored with 0 and its neighbors are colored with $2, 3, \ldots, \Delta + 1$. There is a maximum degree vertex colored with $\Delta + 1$ and one of its neighbors is colored with 1. So f is a no-hole coloring. Since all the vertices other than the maximum degree vertices are colored following a greedy algorithm and span $(f) = \Delta + 1$, f is an irreducible coloring. Hence $\lambda_{inh}(G) = \Delta + 1$. ■

Example 17. We give below example of an unicyclic graph G where $\Delta = 9$ and the distance between any two maximum degree vertices is greater than or equal to seven, see Figure 5. This graph is given a coloring f following the algorithm used in Theorem 16.

Step 1. Here $V(G_1) = \{u_i, v, v', v'', x_j : 1 \le i \le 6, 1 \le j \le 7\}$.

Step 2. Maximum degree vertices of G are v, v' and v''. $f(v) = 10, f(v') = f(v'') = 0$.

Step 3. Vertices on the cycle are $u_1, u_2, u_3, u_4, u_5, u_6$.

Step 4. Here $V(T_1) = \{u_1, v\}, V(T_5) = \{u_5, v', v'', x_i : 1 \le i \le 7\}$.

Step 5. $h(x_1) = 3, h(x_2) = 2, h(x_3) = 1, h(x_4) = 4, h(x_5) = 3, h(x_6) = 2, h(x_7) = 1$.

Step 6. The uncolored vertices of S_5 are ordered $x_1, x_2, x_3, x_4, x_5, x_6, x_7$. Here $d(u_5, x_3) = d(u_5, x_4)$ but $h(x_3) < h(x_4)$, so x_3 appears before x_4.

Step 7. The vertices of G_1 are ordered $v, v', v'', u_1, u_6, u_5, u_4, u_3, u_2, x_1, x_2, x_3, x_4, x_5, x_6, x_7$.

Step 8. These vertices are colored following Algorithm 8 in this order.

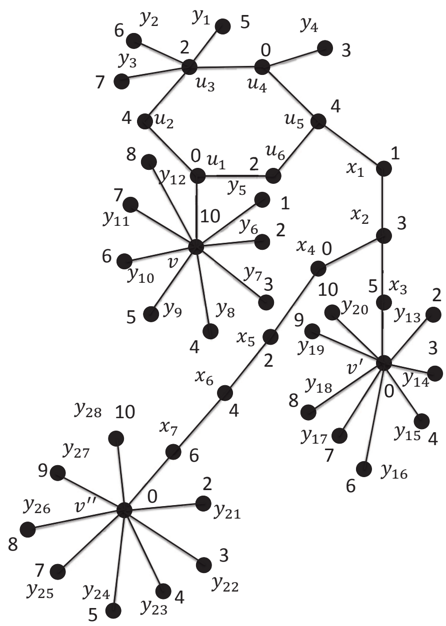

Figure 5. An unicyclic graph G with $\Delta = 9$, more than one maximum degree vertices and $\lambda_{inh}(G) = 10$.

Step 9. We order the vertices of $V(G) - V(G_1)$ as $y_1, y_2, \ldots, y_{27}, y_{28}$. These vertices are colored following Algorithm 8 according to this order. Thus one gets that $\lambda_{inh}(G) = 10$.

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