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A MAXIMUM RESONANT SET OF POLYOMINO GRAPHS [1]

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Abstract

A polyomino graph P is a connected finite subgraph of the infinite plane grid such that each finite face is surrounded by a regular square of side length one and each edge belongs to at least one square. A dimer covering of P corresponds to a perfect matching. Different dimer coverings can interact via an alternating cycle (or square) with respect to them. A set of disjoint squares of P is a resonant set if P has a perfect matching M so that each one of those squares is M-alternating. In this paper, we show that if K is a maximum resonant set of P, then $P - K$ has a unique perfect matching. We further prove that the maximum forcing number of a polyomino graph is equal to the cardinality of a maximum resonant set. This confirms a conjecture of Xu *et al.* [26]. We also show that if K is a maximal alternating set of P, then $P - K$ has a unique perfect matching.

Keywords: polyomino graph, dimer problem, perfect matching, resonant set, forcing number, alternating set.

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1. Introduction

There are two families of interesting plane bipartite graphs, hexagonal systems [22] and polyomino graphs [30], which often arise in some real-world problems. A hexagonal system with a perfect matching is viewed as the carbon-skeleton of a benzenoid hydrocarbon [4, 8]. The dimer problem in statistical mechanics is to count perfect matchings of polyomino graphs [5, 11, 17, 23, 27].

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A dimer covering of a lattice coincides with a perfect matching of the corresponding graph. Kasteleyn [11] developed a so-called "Pfaffian method" and derived the explicit expression of the number of dimer coverings of $m \times n$ chessboard with even mn. Sachs and Zernitz [23] gave a solution for the dimer problem of another sequence of "almost square shaped" polyominoes. Up to now, dimer statistics on some lattices on other surfaces such as torus, Klein-bottom, Möbius strip, cylinder and 3-dimensional space as well has been also considered; for example, see [14, 16, 27].

In relating with dimer covering, perfect matching existence [30], elementary components [12, 25], matching forcing number [19] and maximal resonance [15] of polyomino graphs have been investigated. In addition, polyomino graphs are also models of many interesting combinatorial subjects, such as hypergraphs [2], domination problem [6, 7], rook polynomials [18], etc.

In space of dimer coverings, different dimer coverings (or dimer patterns) can interact via an alternating cycle with respect to them [28, 29, 31]. Such cycles are resonant cycles or conjugated cycles, which play an important role in Randić's conjugated circuits method [10, 20, 21]. The simple number of isolated alternating squares of polyomino graphs can estimate the number of dimer coverings, since each one of such squares can admit independently two selections for dimer covering. For example, by using this method we know that $2n \times 2n$ chessboard has at least 2^{n^2} dimer coverings.

For convenience, we clearly describe some concepts and notations. Let G be a plane bipartite graph with a *perfect matching* M (or Kekulé structure in chemical literature, a set of edges of G such that each vertex of G is incident with exactly one of those edges). A cycle of G is called an *M-alternating cycle* if its edges appear alternately in M and off M. A face f is said to be *M-resonant* or *M-alternating* if its boundary is an M-alternating cycle. Let Q be a set of finite faces (the intersection is allowed) of G. Q is called an *M-alternating set* if all faces in Q are M-resonant. Further, an M-alternating set Q is called an *M-resonant set* of G if the faces in Q are mutually disjoint. Simply, Q is a *resonant set* and *alternating set* of G if G has a perfect matching M such that Q is an M-resonant set and M-alternating set respectively. The cardinality of a maximum resonant set of G is called the *resonant number* of G, denoted by $res(G)$.

A hexagonal system can be formed by a cycle of an infinite plane hexagonal lattice and its interior. For a subgraph H of a graph G, $G - H$ stands for the subgraph obtained from G by deleting all vertices of H together with their incident edges. In 1985, Zheng and Chen [32] gave an important property for a maximum resonant set of a hexagonal system.

Theorem 1.1 [32]. *Let H be a hexagonal system with a perfect matching and K a maximum resonant set of H. Then $H - K$ has a unique perfect matching.*

A *forcing set* of a perfect matching M of a graph G is a subset $S \subseteq M$ such that S is not contained in any other perfect matching of G. The *forcing number* of a perfect matching M, denoted by $f(G, M)$, is the cardinality of a minimum forcing set of M. The *maximum forcing number* of G is the maximum value of forcing numbers of all perfect matchings of G, denoted by $F(G)$. The concept of forcing number of graphs was originally introduced for benzenoid systems by Harary *et al.* [9]. The same idea appeared in an earlier paper [13] of Klein and Randić by the name "innate degree of freedom". The most known results on forcing number are referred to [3].

Pachter and Kim revealed a minimax result that connects the forcing number of a perfect matching and its alternating cycles as follows.

Theorem 1.2 [19]**.** *Let G be a plane bipartite graph with a perfect matching. Then for any perfect matching M of G, $f(G, M) = c(M)$, where $c(M)$ denotes the maximum number of disjoint M-alternating cycles in G.*

By combining Theorems 1.1 and 1.2, Xu *et al.* [26] obtained a relation between the forcing number and resonant number of a hexagonal system as follows.

Theorem 1.3 [26]**.** *Let H be a hexagonal system with a perfect matching. Then $F(H) = res(H)$.*

An alternating set of a graph G is called *maximal* if it is not properly contained in another alternating set of G. In 2006, Salem and Abeledo obtained the following result.

Theorem 1.4 [24]**.** *Let H be a hexagonal system and K a maximal alternating set of H. Then $H - K$ has a unique perfect matching.*

Motivated by the above works, we will investigate polyomino graphs. This paper is mainly concerned with a maximum resonant set of a polyomino graph. By applying Zheng and Chen's approach [32], we prove that if K is a maximum resonant set of a polyomino graph G, then $G - K$ has a unique perfect matching. For a maximal alternating set of G, this property still holds. As a corollary, we have that the maximum forcing number of a polyomino graph is equal to its resonant number. Based on these results, it can be shown that the maximum forcing number of a polyomino graph can be computed in a polynomial time, and thus confirms the conjecture proposed by Xu *et al.* [26].

2. Maximum Resonant Set

A *polyomino graph* is a connected finite subgraph of the infinite plane grid such that each interior face is surrounded by a regular square of side length one and

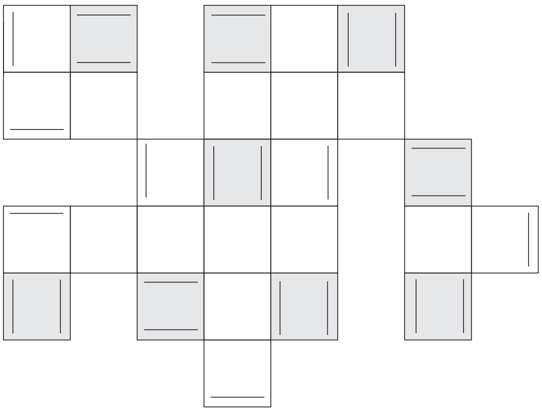

Figure 1. A polyomino graph with a resonant set (gray squares).

each edge belongs to at least one square [30]. An example of polyomino graph with a resonant set is shown in Figure 1.

For a polyomino graph P, the boundary of the infinite face of P is called the *boundary* of P, denoted by $\partial(P)$, and each edge on the boundary is called a *boundary edge* of P. It is well known that polyomino graphs are bipartite. For convenience, we always place a polyomino graph considered on a plane so that one of the two edge directions is horizontal and the other is vertical. Two squares are *adjacent* if they have an edge in common. A vertex of P lying on the boundary of P is called an *external vertex*, and a vertex not being external is called an *internal vertex*. A square of P with external vertices is called an *external square*, and a square with no external vertices is called an *internal square*. In what follows, we always restrict our attention to polyomino graphs with perfect matchings.

Let G be a graph with a perfect matching M and an M-alternating cycle C. Then $M \oplus C(= M \oplus E(C))$ is also a perfect matching of G and C is an $(M \oplus C)$-alternating cycle of G [31]. Let M and N be two perfect matchings of a graph G. The *symmetric difference* of M and N, denoted by $M \oplus N$, is the set of edges contained in either M or N, but not in both, i.e., $M \oplus N = (M \cup N) - (M \cap N)$. An (M, N)-alternating cycle of G is a cycle whose edges are in M and N alternately. It is well known that the symmetric difference of two perfect matchings M and N of G is a disjoint union of (M, N)-alternating cycles.

We now state our main result as follows.

Theorem 2.1. *Let P be a polyomino graph with a perfect matching, and K be a maximum resonant set of P. Then $P - K$ has a unique perfect matching.*

Before proving the main theorem, we will deduce the following crucial lemma.

Let G be a plane bipartite graph, K a set of finite faces and H a subgraph of G. By $K \cap H$ we always mean the intersection of K and the set of faces of H.

Lemma 2.2. *Let P be a 2-connected polyomino graph with a perfect matching, K a resonant set consisting of internal squares of P. If $P - K - \partial(P)$ has a perfect matching or is an empty graph, then K is not a maximum resonant set.*

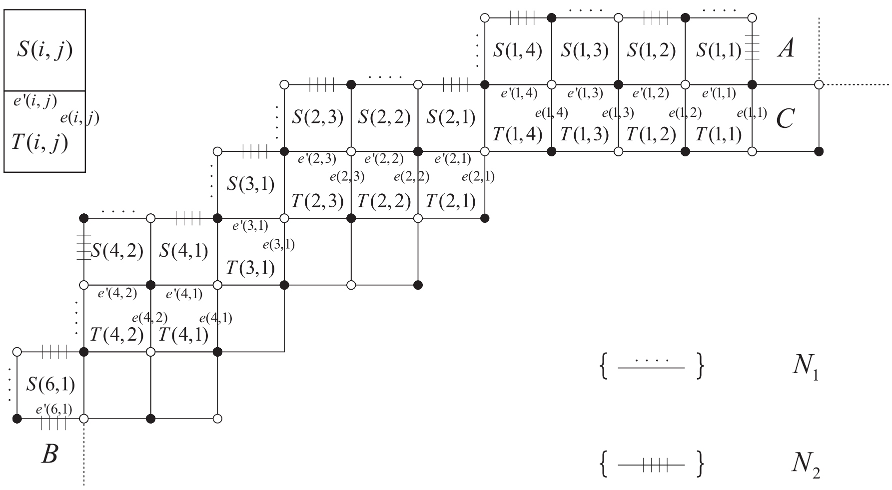

Figure 2. Squares $S(i,j)$ and $T(i,j)$, and edges $e(i,j)$ and $e'(i,j)$ with $m = 6$, $n(1) = 4$, $n(2) = 3$, $n(3) = 1$, $n(4) = 2$, $n(5) = 0$, $n(6) = 1$ and $A, B \notin P$.

Proof. Suppose that $P - K - \partial(P)$ admits a perfect matching M. Decompose the edge set of $\partial(P)$ into two perfect matchings N_1 and N_2 of $\partial(P)$, since $\partial(P)$ is an even cycle. Then it is clear that $M \cup N_1$ and $M \cup N_2$ are two perfect matchings of $P - K$. Let M' be a perfect matching of K such that each edge of M' is vertical. Then $M \cup M'$ is a perfect matching of $P - \partial(P)$. Moreover, $M_1 := N_1 \cup (M \cup M')$ and $M_2 := N_2 \cup (M \cup M')$ are two perfect matchings of P.

Suppose to the contrary that K is a maximum resonant set of P. Adopting the notations of [32], we can take a series of external squares $\{S(i,j) : 1 \leq i \leq m, 1 \leq j \leq n(i)\}$ which satisfy that neither square A nor square B is contained in P as shown in Figure 2. We denote edges, if any, by $e(i,j)$, $e'(i,j)$, $1 \leq i \leq m$ and $1 \leq j \leq n(i)$, and denote the square with edge $e'(i,j)$ which is adjacent to $S(i,j)$, if any, by $T(i,j)$, $1 \leq i \leq m$ and $1 \leq j \leq n(i)$, as shown in Figure 2. We first prove the following claims.

Claim 1. For a pair of parallel edges e_1 and e_2 of a square s of P, they do not lie simultaneously on the boundary of P.

Proof. If e_1 and e_2 lie on the boundary of P, then $\{e_1, e_2\} \subseteq N_1$ or N_2, say N_1. So the square s is N_1-alternating, and $K \cup \{s\}$ is a resonant set of P, which contradicts that K is a maximum resonant set of P. Hence Claim 1 holds. □

Claim 2. $n(1) \geq 2$ is even, $n(m) = 0$, the square $C \in P$, $n(2) \geq 1$ and $m \geq 3$. If $n(i) > 0$, then $T(i,j) \in P$ for all j, $1 \leq j \leq n(i)$.

Proof. Claim 1 implies that $n(1) \geq 2$, $n(m) = 0$, and $T(i,j) \in P$ for all $1 \leq j \leq n(i)$. It remains to show that $n(1)$ is even, $C \in P$, $n(2) \geq 1$ and $m \geq 3$.

Since K is a maximum resonant set of P, $e'(i,j) \notin M$ for all $1 \leq j \leq n(i)$, $1 \leq i \leq m$. Otherwise, $K \cup \{S(i,j)\}$ is a resonant set of P, since the square $S(i,j)$ is either M_1-alternating or M_2-alternating, a contradiction. So $e(1,j) \in M \cup M'$ for all $2 \leq j \leq n(1)$.

First, we show that $n(1)$ is even. Suppose to the contrary that $n(1)$ is odd with $n(1) \geq 3$ (see Figure 3). We use P_0 to denote the subgraph of P formed by squares $S(1,1), S(1,2), \ldots, S(1,n(1))$, $T(1,2), T(1,3), \ldots, T(1,n(1)-1)$. Then we can see that the restriction of M_2 on P_0 is a perfect matching of P_0. Let $M_2' = M_2 \oplus T(1,2) \oplus T(1,4) \oplus \cdots \oplus T(1,n(1)-1) \oplus S(1,2) \oplus S(1,4) \oplus \cdots \oplus S(1,n(1)-1)$. Then M_2' is a perfect matching of P such that each member in the set

$$S_0 := \Big(K \cup \{S(1,1), S(1,3), \ldots, S(1,n(1))\}\Big) \Big\backslash \Big(K \cap P_0\Big)$$

is an M_2'-alternating square. Note that the set $\{S(1,1), S(1,3), \ldots, S(1,n(1))\}$ is of cardinality $\frac{n(1)+1}{2}$, whereas $|K \cap P_0| \leq \frac{n(1)-1}{2}$. Hence, S_0 is a resonant set of P larger than K. This contradicts that K is a maximum resonant set of P.

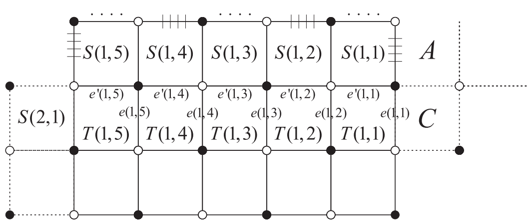

Figure 3. Illustration for Claim 2 in the proof of Lemma 2.2: $n(1)$ is odd.

Next, we show that $C \in P$. Suppose to the contrary that $C \notin P$. Then $e(1,1) \in N_1$. We use P_1 to denote the subgraph of P formed by squares $T(1,1), T(1,2), \ldots, T(1,n(1)-1)$. Then the restriction of M_1 on P_1 is a perfect matching of P_1. Note that $T(1,1), T(1,3), \ldots, T(1,n(1)-1)$ are M_1-alternating squares and $|K \cap P_1| \leq \frac{n(1)}{2} - 1$. Hence, we can see that

$$\Big(K \cup \{T(1,1), T(1,3), \ldots, T(1,n(1)-1)\}\Big) \Big\backslash \Big(K \cap P_1\Big)$$

is a resonant set of P larger than K, a contradiction. Similarly, $S(2,1) \in P$ and $n(2) > 0$. Moreover, $m \geq 3$. So we complete the proof of Claim 2. □

Let ℓ be an integer with $2 \leq \ell \leq m$ such that $n(\ell)$ is even, and $n(t)$ is odd for all $2 \leq t \leq \ell - 1$. It follows from $e(1, n(1)) \in M \cup M'$, $e'(i,j) \notin M$ that $e(i,j) \in M \cup M'$ for all i and j, $2 \leq i \leq \ell - 1$, $1 \leq j \leq n(i)$. We now need to distinguish the following two cases.

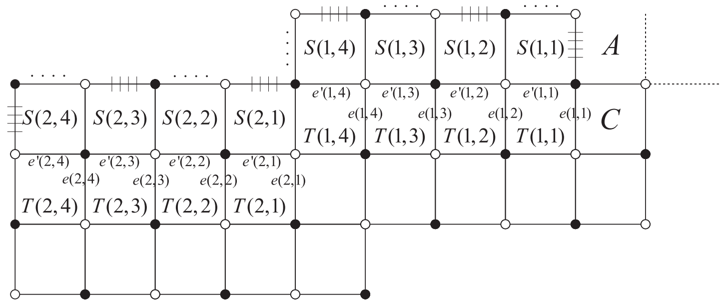

Figure 4. Illustration for Subcase 1.1 in the proof of Lemma 2.2: $\ell = 2$ and $n(\ell) = 4$.

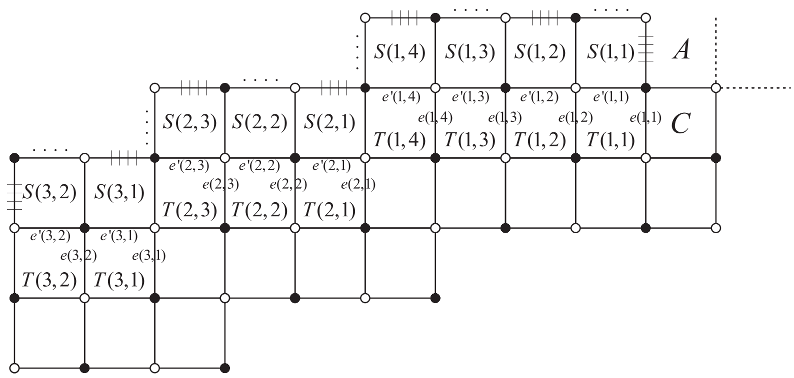

Figure 5. Illustration for Subcase 1.2 in the proof of Lemma 2.2: $\ell = 3$, $n(\ell) = 2$.

Case 1. $n(\ell) > 0$. In this case we have $e(\ell, j) \in M \cup M'$ for all $1 \leq j \leq n(\ell)$.

Subcase 1.1. $\ell = 2$ (see Figure 4). Let P_2 denote the subgraph of P formed by squares in the set

$$\{T(1,j) : 2 \leq j \leq n(1)\} \cup \{S(2,j) : 1 \leq j \leq n(2)\} \cup \{T(2,j) : 1 \leq j \leq n(2) - 1\}.$$

Then the restriction of M_2 on P_2 is a perfect matching of P_2. Let

$$M_2'' = M_2 \oplus T(2,1) \oplus T(2,3) \oplus \cdots \oplus T(2,n(2)-1) \oplus S(2,1) \oplus S(2,3) \oplus \cdots \oplus S(2,n(2)-1),$$

$$S_1 := \{T(1,2), T(1,4), \ldots, T(1,n(1))\} \cup \{S(2,2), S(2,4), \ldots, S(2,n(2))\}.$$

Then M_2'' is a perfect matching of P such that each member of $(K \cup S_1)\backslash(K \cap P_2)$ is an M_2''-alternating square. Note that $|S_1| = \frac{n(1)+n(2)}{2}$, whereas

$$|K \cap \{T(1,j) : 2 \le j \le n(1)\}| \le \frac{n(1)}{2} - 1,$$
$$|K \cap \{T(2,j) : 1 \le j \le n(2)-1\}| \le \frac{n(2)}{2}.$$

Hence, $(K \cup S_1)\backslash(K \cap P_2)$ is a resonant set of P larger than K, a contradiction.

Subcase 1.2. $\ell \ge 3$ (see Figure 5). Let P_3 denote the subgraph of P formed by squares in

$$\{T(\ell-1,j) : 1 \le j \le n(\ell-1)\} \cup \{S(\ell,j) : 1 \le j \le n(\ell)\} \cup \{T(\ell,j) : 1 \le j \le n(\ell)-1\}.$$

Then the restriction of M_2 on P_3 is a perfect matching of P_3. Let

$$M_2''' = M_2 \oplus T(\ell,1) \oplus T(\ell,3) \oplus \cdots \oplus T(\ell,n(\ell)-1) \oplus S(\ell,1) \oplus S(\ell,3) \oplus \cdots \oplus S(\ell,n(\ell)-1),$$

$$S_2 := \{S(\ell,2), S(\ell,4), \ldots, S(\ell,n(\ell))\} \cup \{T(\ell-1,1), T(\ell-1,3), \ldots, T(\ell-1,n(\ell-1))\}.$$

Then M_2''' is a perfect matching of P such that each member of $(K \cup S_2)\backslash(K \cap P_3)$ is an M_2'''-alternating square. Note that $|S_2| = \frac{n(\ell)+n(\ell-1)+1}{2}$, whereas

$$|K \cap \{T(\ell-1,j) : 1 \le j \le n(\ell-1)\}| \le \frac{n(\ell-1)-1}{2},$$
$$|K \cap \{T(\ell,j) : 1 \le j \le n(\ell)-1\}| \le \frac{n(\ell)}{2}.$$

Hence, $(K \cup S_2)\backslash(K \cap P_3)$ is a resonant set of P larger than K, a contradiction.

Case 2. $n(\ell) = 0$ (see Figure 6). Let P_4 denote the subgraph of P formed by squares $T(\ell-1,1), T(\ell-1,2), \ldots, T(\ell-1,n(\ell-1))$. Note that the left vertical edge of the square $T(\ell-1,n(\ell-1))$ belongs to N_2, and moreover each square in P_4 is M_2-alternating. Thus we can see that

$$\Big(K \cup \{T(\ell-1,1), T(\ell-1,3), \ldots, T(\ell-1,n(\ell-1))\}\Big) \Big\backslash \Big(K \cap P_4\Big)$$

is a resonant set of P larger than K, a contradiction.

Now the entire proof of the lemma is complete. ■

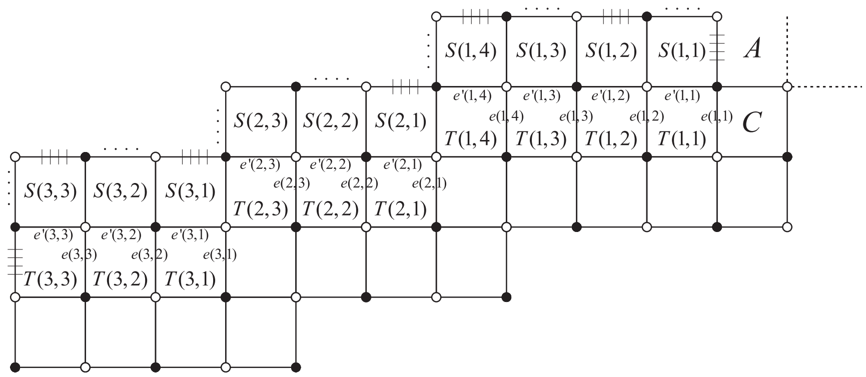

Figure 6. Illustration for Case 2 in the proof of Lemma 2.2: $\ell = 4$, $n(\ell) = 0$.

***Proof of Theorem* 2.1.** Suppose to the contrary that $P - K$ has two perfect matchings M and M'. Then $M \oplus M'$ contains an (M, M')-alternating cycle C. Let $I[C]$ denote the subgraph of P consisting of C together with its interior. Put $K^* = K \cap I[C]$. Then K^* is not a maximum resonant set of $I[C]$, since $I[C]$ and K^* satisfy the condition of Lemma 2.2. Moreover K is also not a maximum resonant set of P, which contradicts the assumption that K is a maximum resonant set of P. ■

3. Maximal Alternating Set

For a maximal alternating set of polyomino graphs, we can obtain the following results.

Lemma 3.1. *Let P be a 2-connected polyomino graph with a perfect matching, K an alternating set consisting of internal squares and $\partial(P)$ the boundary of P. If $P - K - \partial(P)$ has a perfect matching or is an empty graph, then K is not a maximal alternating set.*

Proof. We take the substructure and notations used in the proof of Lemma 2.2 (see Figure 2). Let M be a perfect matching of $P - \partial(P)$ such that all squares in K are M-alternating, and let N_1 and N_2 be two perfect matchings of $\partial(P)$. Then $M_1 := M \cup N_1$ and $M_2 := M \cup N_2$ are two perfect matchings of P.

Suppose to the contrary that K is a maximal alternating set of P. The following Claim 1 and its proof are the same as Claim 1 of Lemma 2.2.

Claim 1. For a pair of parallel edges e_1 and e_2 of a square s of P, they do not lie simultaneously on the boundary of P.

Claim 2. $n(1) \geq 2$, $n(m) = 0$, the square $C \in P$, $n(2) \geq 1$ and $m \geq 3$. $e'(i,j) \notin M$ for all $1 \leq j \leq n(i)$, $1 \leq i \leq m$. Moreover, $e(1,j) \in M$ for all $2 \leq j \leq n(1)$.

Proof. Claim 1 implies that $n(1) \geq 2$, $n(m) = 0$, and $T(i,j) \in P$ for all $1 \leq j \leq n(i)$, $1 \leq i \leq m$.

Since K is a maximal alternating set of P, $e'(i,j) \notin M$ for all $1 \leq j \leq n(i)$, $1 \leq i \leq m$. Otherwise, $K \cup \{S(i,j)\}$ is an alternating set of P, since the square $S(i,j)$ is either M_1-alternating or M_2-alternating, a contradiction. So $e(1,j) \in M$ for all $2 \leq j \leq n(1)$.

Now we show that $C \in P$. Suppose to the contrary that $C \notin P$. Then $e(1,1) \in N_1$ and $S(1,1)$ is M_1-alternating. So $K \cup \{S(1,1)\}$ is an alternating set of P, a contradiction. Symmetrically, $S(2,1) \in P$ and $n(2) \geq 1$. So $m \geq 3$. Hence Claim 2 is proved. □

Let ℓ be an integer with $3 \leq \ell \leq m$ such that $n(\ell) = 0$, and $n(t) > 0$ for all $2 \leq t \leq \ell - 1$. It follows from $e(1,n(1)) \in M$, $e'(i,j) \notin M$ that $e(i,j) \in M$ for all i and j, $2 \leq i \leq \ell - 1$, $1 \leq j \leq n(i)$. Note that the left vertical edge of the square $T(\ell-1, n(\ell-1))$ belongs to N_1 or N_2, say N_1. So $T(\ell-1, n(\ell-1))$ is M_1-alternating and $K \cup \{T(\ell-1, n(\ell-1))\}$ is an alternating set of P, which contradicts the assumption that K is a maximal alternating set of P. The lemma is proved. ■

Theorem 3.2. *Let P be a polyomino graph with a perfect matching, and K be a maximal alternating set of P. Then $P - K$ has a unique perfect matching.*

Proof. Suppose to the contrary that $P - K$ has two perfect matchings M and M'. Then $M \oplus M'$ contains an (M, M')-alternating cycle C. Put $K^* = K \cap I[C]$. Then K^* is not a maximal alternating set of $I[C]$, since $I[C]$ and K^* satisfy the condition of Lemma 3.1. So K is also not a maximal alternating set of P, a contradiction. ■

4. MAXIMUM FORCING NUMBER

Motivated by Theorem 1.3, it is natural to ask the following question: when is the maximum forcing number of a plane bipartite graph equal to its resonant number? In the following, we shall give a sufficient condition.

Let G be a plane graph with a perfect matching. A cycle C of G is said to be *nice* if G has a perfect matching M such that C is an M-alternating cycle. Denote by $I[C]$ the subgraph of G consisting of C together with its interior. A cycle C of G is called a *face cycle* if it is the boundary of some finite face of G. For convenience, we do not distinguish a face cycle with its finite face.

Theorem 4.1. *Let G be a connected plane bipartite graph with perfect matchings. If for each nice cycle C of G and any maximum resonant set K of $I[C]$, $I[C]-K$ has a unique perfect matching, then $res(G) = F(G)$.*

Proof. Let $F(G) = n$. By the definition of resonant number and Theorem 1.2, we can see that $res(G) \leq n$. In the following we show $res(G) \geq n$. Define $\mathcal{M}(G)$ as the set of perfect matchings of G whose forcing numbers equal n. By Theorem 1.2, for any $M \in \mathcal{M}(G)$, there exist n pairwise disjoint M-alternating cycles in G. We choose a perfect matching M_1 in $\mathcal{M}(G)$ such that n disjoint M_1-alternating cycles $C_1, C_2, \ldots, C_n$ of G have face cycles as many as possible.

Put $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$. It suffices to show that all cycles in $\mathcal{C}$ are face cycles. Otherwise, $\mathcal{C}$ has a non-face cycle member and its interior contains only face cycle members of $\mathcal{C}$. Without loss of generality, let C_i denote such a non-face cycle member of $\mathcal{C}$ and $C_1, C_2, \ldots, C_{i-1}$ are all the face cycles in $\mathcal{C}$ contained in the interior of C_i for some i, $1 \leq i \leq n$. Then the restriction of M_1 on $I[C_i]$ is also a perfect matching of $I[C_i]$, denoted by M_c. By the assumption, $\{C_1, C_2, \ldots, C_{i-1}\}$ is a non-maximum resonant set of $I[C_i]$. Let S be a maximum resonant set of $I[C_i]$. Then $|S| \geq i$. Let M_0 be a perfect matching of $I[C_i]$ such that all faces in S are M_0-resonant. Let $M_2 = (M_1 \backslash M_c) \cup M_0$ and $\mathcal{C}' = S \cup \{C_{i+1}, C_{i+2}, \ldots, C_n\}$. Then M_2 is a perfect matching of G and each member of $\mathcal{C}'$ is an M_2-alternating cycle. Note that $M_2 \in \mathcal{M}(G)$ and $\mathcal{C}'$ contains more face cycles than $\mathcal{C}$. This contradicts the choices of M_1 and $\{C_1, C_2, \ldots, C_n\}$. ∎

Combining Theorems 4.1 with 2.1 and 1.1, we immediately obtain the following results.

Corollary 4.2 [26]**.** *Let P be a hexagonal system with a perfect matching. Then $F(P) = res(P)$.*

Corollary 4.3. *Let P be a polyomino graph with a perfect matching. Then $res(P) = F(P)$.*

We now give a weakly elementary property of such graphs that satisfy the conditions of Theorem 4.1. Let G be a connected plane bipartite graph with a perfect matching. An edge of G is called *allowed* if it lies in some perfect matching of G and *forbidden* otherwise. G is called *elementary* if each edge of G is allowed. G is said to be *weakly elementary* if for each nice cycle C of G the interior of C has at least one allowed edge of G that is incident with a vertex of C whenever the interior of C contains an edge of G [31]. A face f of G is said to be a *boundary face* if the boundaries of f and $\partial(G)$ have a vertex in common.

Theorem 4.4. *Let G be a connected plane bipartite graph with perfect matchings. If for each nice cycle C of G and any maximum resonant set K of $I[C]$, $I[C]-K$ has a unique perfect matching, then G is weakly elementary.*

Proof. Suppose to the contrary that G is not weakly elementary. Then there exists a nice non-face cycle C of G such that the interior of C has no allowed edges of G incident with vertices of C. It follows that the interior of C has no allowed edges of $I[C]$ that are incident with vertices of C. So, for every perfect matching M of $I[C]$, C is M-alternating and any maximum resonant set K of $I[C]$ contains no boundary faces of $I[C]$. So $I[C]-K$ has at least two perfect matchings, a contradiction. ■

Xu *et al.* ever gave a conjecture as follows, which can be now confirmed.

Conjecture 4.5 [26]**.** *Let G be an elementary polyomino graph. Then the maximum forcing number of G can be computed in polynomial time.*

Abeledo and Atkinson obtained the following result.

Theorem 4.6 [1]**.** *Let G be a 2-connected plane bipartite graph. Then the resonant number of G can be computed in polynomial time using linear programming methods.*

A polyomino graph P with forbidden edges can be decomposed into some elementary components [25]. There exists a polynomial time algorithm to accomplish this decomposition. Such elementary components are elementary polyominoes and thus 2-connected. Note that the forcing number of any perfect matching M of P equals the sum of forcing numbers of the restrictions of M on its elementary components. So Theorem 4.6 and Corollary 4.3 imply the following result, which confirms Conjecture 4.5.

Theorem 4.7. *Let G be a polyomino graph with a perfect matching. Then the maximum forcing number of G can be computed in polynomial time.*

5. Concluding Remarks

Theorem 2.1 does not hold for general plane bipartite graphs even for plane elementary bipartite graphs. Let us see two elementary bipartite graphs G and G' as shown in Figure 7, where G' is a subgraph of G bounded by a nice cycle of G. Since there exist at most 12 (respectively 6) pairwise disjoint finite faces in G (respectively in G') and the faces with labels 1 (respectively 2) form a resonant set of G (respectively in G'), we have that $res(G)=12$ and $res(G')=6$. There is a maximum resonant set S' (the faces with labels 2) of G' such that $G'-S'$ has two perfect matchings. This shows that Theorem 2.1 does not hold for a graph G'.

For any maximum resonant set S of G, $G-S$ is empty; the empty set can be viewed as a unique perfect matching of an empty graph. This also shows that even if Theorem 2.1 holds for a graph G, but it does not necessarily hold for a subgraph formed by a nice cycle and its interior.

In addition, since G' has a perfect matching M such that the faces with labels 2 and the infinite face of G' are M-resonant, by Theorem 1.2 we have $F(G') \geq 7$. Since there exist no 8 pairwise disjoint cycles in G', we have $F(G') \leq 7$. So $F(G') = 7$ and $F(G') \neq res(G')$. But $F(G) = res(G) = 12$. This shows that the converse of Theorem 4.4 does not hold.

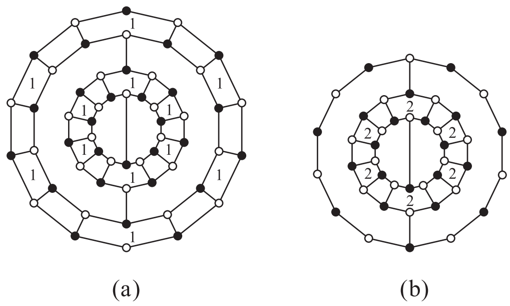

Figure 7. (a) An elementary graph G. (b) A subgraph G' of G.

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