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HARDNESS RESULTS FOR TOTAL RAINBOW CONNECTION OF GRAPHS

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Abstract

A total-colored path is *total rainbow* if both its edges and internal vertices have distinct colors. The *total rainbow connection number* of a connected graph G, denoted by $trc(G)$, is the smallest number of colors that are needed in a total-coloring of G in order to make G *total rainbow connected*, that is, any two vertices of G are connected by a total rainbow path. In this paper, we study the computational complexity of total rainbow connection of graphs. We show that deciding whether a given total-coloring of a graph G makes it total rainbow connected is NP-Complete. We also prove that given a graph G, deciding whether $trc(G) = 3$ is NP-Complete.

Keywords: total rainbow connection, computational complexity.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of Bondy and Murty [1] for those not described here.

An *edge-coloring* of a graph G is a mapping from the edges of G to some finite set of colors, where adjacent edges may have the same color. An edge-colored graph is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. This concept of rainbow connection in graphs was introduced by Chartrand *et al.* in [4]. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in an edge-coloring of G in order to make G rainbow connected. Observe that $diam(G) \leq rc(G) \leq n-1$, where $diam(G)$ denotes the diameter of G and n is the order of G. It is easy to verify that $rc(G) = 1$ if and only if G is a complete graph, and that $rc(G) = n-1$ if and only if G is a tree. For an overview of the rainbow connection topic, we refer the reader to some new papers [7, 8, 13], and the survey and the monograph [14, 15].

A *vertex-coloring* of a graph G is a mapping from the vertices of G to some finite set of colors. If all pairs of adjacent vertices have distinct colors, we called the vertex-coloring *proper*. In [9], Krivelevich and Yuster proposed the concept of rainbow vertex-connection. A vertex-colored graph, not necessarily proper, is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in a vertex-coloring of G in order to make G rainbow vertex-connected. An easy observation is that if G is of order n, then $rvc(G) \leq n-2$ and $rvc(G) = 0$ if and only if G is a complete graph. Notice that $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2. There are some approaches to study the bounds of $rvc(G)$, we refer to [9, 12, 16].

Uchizawa *et al.* [18] and Liu *et al.* [16] introduced an analogous definition using total-colorings. A *total-coloring* of a graph G is a mapping from the vertices and edges of G to some finite set of colors. A total-colored graph is *total rainbow connected* if any two vertices are connected by a path whose edges and internal vertices have distinct colors. The *total rainbow connection number* of a connected graph G, denoted by $trc(G)$, is the smallest number of colors that are needed in a total-coloring of G in order to make G total rainbow connected. Observe that $trc(G) = 1$ if and only if G is a complete graph, and $trc(G) \geq 3$ if and only if G is not complete.

For the rainbow connection number and the rainbow vertex-connection number, some examples were given to show that there is no upper bound for one of the parameters in terms of the other, see [9]. In [16], Liu *et al.* compared $trc(G)$

with $rc(G)$ and $rvc(G)$. Notice that $trc(G) \geq \max\{rc(G), rvc(G)\}$. Liu *et al.* showed that for every sufficiently large s, there exists an example of a graph G with $trc(G) = rvc(G) = s$.

The computational complexity of rainbow connectivity and rainbow vertex connectivity has been studied. In [2], Caro *et al.* conjectured that computing $rc(G)$ is an NP-Hard problem, as well as that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. In [3], Chakraborty *et al.* confirmed this conjecture and obtained the following theorems.

Theorem 1. *Given a graph G, deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard.*

Given an edge-coloring of the graph, if the coloring is arbitrary, they showed that checking whether the coloring makes the graph rainbow connected is NP-Complete.

Theorem 2. *The following problem is NP-Complete: Given an edge-colored graph G, check whether the given coloring makes G rainbow connected.*

In [10], Li *et al.* considered bipartite graphs, and obtained the computational complexity results for bipartite graphs. Chen *et al.* [5] investigated the computational complexity of rainbow vertex-connection, and obtained the similar results.

Since computing $rc(G)$ and $rvc(G)$ is NP-Hard, a natural conjecture is that computing $trc(G)$ is also NP-Hard. In this paper, we consider the computational complexity of total rainbow connection of graphs and give some similar results. In Section 2, we show that deciding whether a given total-coloring of a graph G makes it total rainbow connected is NP-Complete. In Section 3, we prove that given a graph G, deciding whether $trc(G) = 3$ is NP-Complete.

2. Total Rainbow Connection

Now, we give our first theorem.

Theorem 3. *The following problem is NP-Complete: Given a total-colored graph G, check whether the given coloring makes G total rainbow connected.*

We define the problem in Theorem 3 as TOTAL RAINBOW CONNECTION, and the problem in Theorem 2 as RAINBOW CONNECTION.

Problem 1. TOTAL RAINBOW CONNECTION.
Given: Total-colored graph G.
Decide: Whether G is total rainbow connected under the coloring?

Problem 2. RAINBOW CONNECTION.
Given: Edge-colored graph G.
Decide: Whether the coloring makes G rainbow connected?

Clearly, the problem of TOTAL RAINBOW CONNECTION is in NP. By Theorem 2, the problem of RAINBOW CONNECTION is NP-Complete. We reduce Problem 2 to Problem 1, which shows that Problem 1 is NP-Complete, concluding the proof of Theorem 3.

Lemma 4. *Problem* 2 $\preceq$ *Problem* 1.

Proof. Let G be a given graph with an edge-coloring c. We want to construct a graph G' with a total-coloring c' such that G' is total rainbow connected under the coloring c' if and only if G is rainbow connected under the coloring c.

Let $V = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ be the vertex set of G, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n new colors that are not used in c. Let G' be a graph isomorphic to G. We define the coloring c' as follows: $c'(e) = c(e)$, for $e \in E(G)$ and $c'(v_i) = \alpha_i$, for $1 \leq i \leq n$.

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n new colors, one can easily check that G is rainbow connected under the coloring c if and only if G' is total rainbow connected under the coloring c'. ■

3. Total Rainbow Connection Number 3

Notice that $trc(G) = 1$ if and only if G is a complete graph, and $trc(G) \geq 3$ if and only if G is not complete. By Theorem 1, the problem of deciding if $rc(G) = 2$ is NP-Complete. Correspondingly, we show that the problem of deciding if $trc(G) = 3$ is NP-Complete.

Theorem 5. *Given a graph G, deciding whether $trc(G) = 3$ is NP-Complete. Thus, computing $trc(G)$ is NP-Hard.*

We define the problem above as TOTAL RAINBOW CONNECTION NUMBER 3.

Problem 3. TOTAL RAINBOW CONNECTION NUMBER 3.
Given: Graph $G = (V, E)$.
Decide: Whether there is a total-coloring of G with three colors such that all pairs $\{u, v\} \in V^{(2)}$ ($V^{(2)}$ means the unorder pairs from V) are connected by a total rainbow path?

Clearly, Problem 3 is in NP. To show that it is NP-Complete, we need to define another two problems.

Problem 4. SUBSET TOTAL RAINBOW CONNECTION NUMBER 3.

Given: Graph $G = (V, E)$ and a set of pairs $P \subseteq V^{(2)}$.
Decide: Whether there is a total-coloring of G with three colors such that all pairs $\{u, v\} \in P$ are connected by a total rainbow path?

Problem 5. 3-COLORABILITY.
Given: Graph $G = (V, E)$.
Decide: Whether there is a vertex-coloring of G with three colors such that all pairs of adjacent vertices are assigned different colors?

In the following, we will reduce Problem 4 to Problem 3 and then reduce Problem 5 to Problem 4. Since 3-COLORABILITY is NP-Complete [6], we have that the problem of TOTAL RAINBOW CONNECTION NUMBER 3 is NP-Complete, which proves Theorem 5.

Lemma 6. *Problem* 4 $\preceq$ *Problem* 3.

Proof. Given a graph $G = (V, E)$ and a set of pairs $P \subseteq V^{(2)}$, we construct a graph $G' = (V', E')$ as follows.

For each vertex $v \in V$, we introduce a new vertex x_v; for every pair $\{u, v\} \in V^{(2)} \setminus P$, we introduce a new vertex $x_{\{u,v\}}$. Set

$$V' = V \cup \{x_v : v \in V\} \cup \{x_{\{u,v\}} : \{u, v\} \in V^{(2)} \setminus P\}$$

and

$$E' = E \cup \{vx_v : v \in V\} \cup \{ux_{\{u,v\}}, vx_{\{u,v\}} : \{u, v\} \in V^{(2)} \setminus P\} \cup \{xy : x, y \in V' \setminus V\}.$$

In the following, we will prove that there exists a total-coloring of G' with three colors which makes G' total rainbow connected if and only if there is a total-coloring of G with three colors such that all pairs $\{u, v\} \in P$ are connected by a total rainbow path.

First suppose there is a total-coloring of G' with three colors which makes G' total rainbow connected. Observe that G is a subgraph of G'. For each pair $\{u, v\} \in P$, the paths of length not more than 2 that connect u and v have to be in G. Thus, under the coloring, all pairs in P are connected by a total rainbow path.

On the other hand, assume that $c : V \cup E \to \{1, 2, 3\}$ is a total-coloring of G such that all pairs $\{u, v\} \in P$ are connected by a total rainbow path. We extend the coloring c to a total-coloring $c' : V' \cup E' \to \{1, 2, 3\}$ in the following way: $c'(vx_v) = 1$ for all $v \in V$, $c'(ux_{\{u,v\}}) = 1$ and $c'(vx_{\{u,v\}}) = 2$ for all $\{u, v\} \in V^{(2)} \setminus P$, $c'(x_v) = c'(x_{\{u,v\}}) = 3$, $c'(xy) = 2$ for all $x, y \in V' \setminus V$. Now we show that G' is indeed total rainbow connected under this coloring. For any two vertices u and v, if $\{u, v\} \in P$, the total rainbow path connecting u and v in G is also the total rainbow path in G'. If $\{u, v\} \in V^{(2)} \setminus P$, then $ux_{\{u,v\}}v$ is a total rainbow path connecting u and v. If $u \in V$, $v \in V' \setminus V$, then ux_uv is a total rainbow path connecting u and v. If $u, v \in V' \setminus V$, then uv is the total rainbow path. Hence, G' is total rainbow connected under the coloring c'.

This completes the proof of the lemma. ∎

Lemma 7. *Problem* 5 $\preceq$ *Problem* 4.

Proof. Let $G = (V, E)$ be an instance of the 3-COLORABILITY problem. We construct an instance $G' = (V', E')$ of the SUBSET TOTAL RAINBOW CONNECTION NUMBER 3 problem.

Let $V = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ be the vertex set of G. For each $v_i \in V$, we introduce three new vertices v_i^1, v_i^2, v_i^3. For each edge $e = v_iv_j \in E$, we introduce a new vertex v_{ij}. We set

$$V' = \{v_i^1, v_i^2, v_i^3 : 1 \leq i \leq n\} \cup \{v_{ij} : v_iv_j \in E\}$$

and

$$E' = \{v_i^1v_i^2, v_i^2v_i^3 : 1 \leq i \leq n\} \cup \{v_i^2v_{ij}, v_{ij}v_j^2 : v_iv_j \in E\}.$$

Now we define the set $P \subseteq V'^{(2)}$ as follows.

$$P = \{\{v_i^1, v_i^3\}, \{v_i^3, v_{ij}\}, \{v_j^1, v_j^3\}, \{v_j^3, v_{ij}\}, \{v_i^2, v_j^2\} : v_iv_j \in E\}.$$

If there is a vertex-coloring $c : V(G) \to \{1, 2, 3\}$ such that the adjacent vertices are assigned different colors, then we define a total-coloring c' of G' as follows. For $1 \leq i \leq n$, let $c'(v_i^1v_i^2) = c(v_i)$, $c'(v_i^2) \in \{1, 2, 3\} \setminus \{c(v_i)\}$, $c'(v_i^2v_i^3) \in \{1, 2, 3\} \setminus \{c(v_i), c'(v_i^2)\}$. For each $v_iv_j \in E$, let $c'(v_i^2v_{ij}) = c(v_i)$, $c'(v_j^2v_{ij}) = c(v_j)$, $c'(v_{ij}) \in \{1, 2, 3\} \setminus \{c(v_i), c(v_j)\}$. For all other vertices, we assign them the colors arbitrarily. It is easy to check that all $\{u, v\} \in P$ are connected by a total rainbow path.

On the other hand, suppose there is a total-coloring c' of G' with three colors such that all pairs $\{u, v\} \in P$ are connected by a total rainbow path, we define a vertex-coloring c of G as follows. For $1 \leq i \leq n$, $c(v_i) = c'(v_i^1v_i^2)$. We show that this coloring is proper, that is, if $v_iv_j \in E$, then $c(v_i) \neq c(v_j)$. Indeed, for $v_iv_j \in E$, since $\{v_i^2, v_j^2\} \in P$, there is a total rainbow path connecting v_i^2 and v_j^2, this path must be $v_i^2v_{ij}v_j^2$, thus $c'(v_i^2v_{ij}) \neq c'(v_j^2v_{ij})$. Since $\{v_i^1, v_i^3\} \in P$, $\{v_i^3, v_{ij}\} \in P$, and the total rainbow paths connecting v_i^1 and v_i^3, v_i^3 and v_{ij} are $v_i^1v_i^2v_i^3$, $v_i^3v_i^2v_{ij}$, we have $c'(v_i^1v_i^2) = c'(v_i^2v_{ij})$ since only three colors are used. Similarly, $c'(v_j^1v_j^2) = c'(v_j^2v_{ij})$. Then $c'(v_i^1v_i^2) \neq c'(v_j^1v_j^2)$. Therefore, $c(v_i) \neq c(v_j)$, thus c is a proper coloring of G, concluding the proof. ∎

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