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WEAK TOTAL RESOLVABILITY IN GRAPHS

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Abstract

A vertex $v \in V(G)$ is said to *distinguish* two vertices $x, y \in V(G)$ of a graph G if the distance from v to x is different from the distance from v to y. A set $W \subseteq V(G)$ is a *total resolving set* for a graph G if for every pair of vertices $x, y \in V(G)$, there exists some vertex $w \in W - \{x, y\}$ which distinguishes x and y, while W is a *weak total resolving set* if for every $x \in V(G) - W$ and $y \in W$, there exists some $w \in W - \{y\}$ which distinguishes x and y. A weak total resolving set of minimum cardinality is called a *weak total metric basis* of G and its cardinality the *weak total metric dimension* of G. Our main contributions are the following ones: (a) Graphs with small and large weak total metric bases are characterised. (b) We explore the (tight) relation to independent 2-domination. (c) We introduce a new graph parameter, called *weak total adjacency dimension* and present results that are analogous to those presented for weak total dimension. (d) For trees, we derive a characterisation of the weak total (adjacency) metric dimension. Also, exact figures for our parameters are presented for (generalised) fans and wheels. (e) We show that for Cartesian product graphs, the weak total (adjacency) metric dimension is usually pretty small. (f) The weak total (adjacency) dimension is studied for lexicographic products of graphs.

Keywords: metric dimension, resolving set, weak total metric dimension, weak total resolving set, adjacency dimension, graph operations.

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1. Introduction

A resolving set for a graph G is a set $S \subseteq V(G)$ with the property that every vertex of G is uniquely determined by the distances from the elements of S. The *distance* of two vertices x, y of a graph G is the length of a shortest path between x and y in G, written $d_G(x, y)$. If two vertices belong to different connected components, their distance is infinite. A vertex $v \in V(G)$ is said to *distinguish* two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. More formally, a set $S \subseteq V$ is said to be a *resolving set* for G if any pair of vertices of G is distinguished by some element of S. A minimum resolving set is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension of a graph was introduced by Slater in [9]. This concept was also introduced by Harary and Melter in [4]. Several variations of resolving sets including resolving dominating sets [1], independent resolving sets [2], local metric sets [7], strong resolving sets [8], weak total resolving sets [6], etc. have since been introduced and studied. In this paper we focus on the study of weak total resolving sets.

To begin with, we introduce some notation and terminology. All graphs that we consider in this paper are undirected and without loops or multi-edges. If G is a graph, $V(G)$ is its set of vertices and $E(G)$ is its set of edges. The number $|V(G)|$ is also called the order of G. A graph G with $E(G) = \emptyset$ is known as an empty graph, and the empty graph of order one is also called the trivial graph. We write $G \cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G, $N_G(v)$ will denote the set of neighbours of v in G, i.e., $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is the open neighbourhood of v. Additionally, we denote by $N_G[v] = N_G(v) \cup \{v\}$ the closed neighbourhood of v. Also, given a set $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{v \in S} N_G(v)$. The subgraph induced by a set S of vertices will be denoted by $\langle S \rangle$. The *eccentricity* $\epsilon_G(v)$ of a vertex v in a connected graph G is the maximum distance between v and any other vertex of G. The *diameter* of G is defined as

$$D(G) = \max_{v \in V(G)} \{\epsilon_G(v)\}.$$

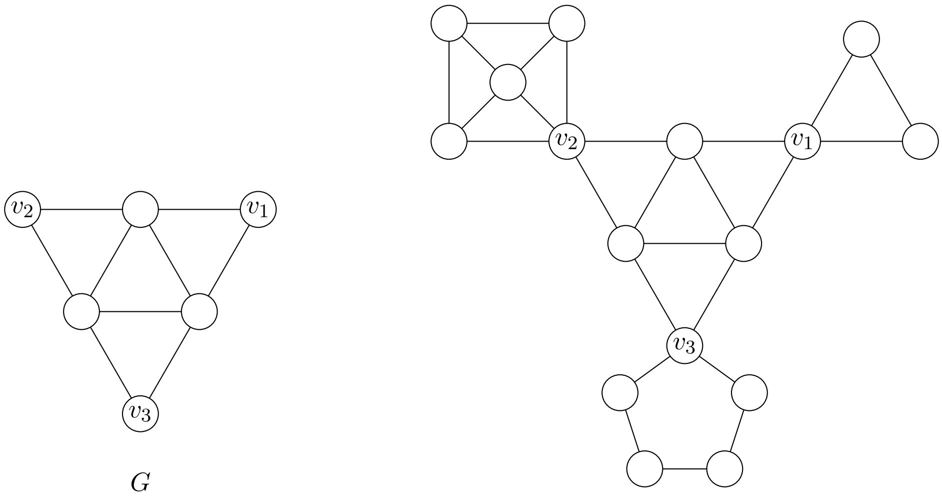
The diameter of a graph is infinite if and only if the graph has more than one connected component. A graph G is 2-*antipodal* if for each vertex $x \in V(G)$ there exists exactly one vertex $y \in V(G)$ such that $d_G(x, y) = D(G)$.

We will use the notation K_n, $K_{r,s}$, C_n, N_n and P_n for complete graphs, complete bipartite graphs, cycle graphs, empty graphs and path graphs of order n, respectively.

The *join* $G + H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H. For instance, the graph $K_1 + C_t$ is a wheel graph, $K_1 + K_r \cong K_{r+1}$ is a complete graph and $K_1 + N_t \cong K_{1,t}$ is a star graph whose central vertex is the vertex of K_1 and whose t leaves are the vertices of the empty graph N_t.

A set $W \subseteq V(G)$ is a *total resolving set* for G if for every pair $x, y \in V(G)$, there exists some vertex $w \in W - \{x, y\}$ such that $d_G(w, x) \neq d_G(w, y)$, while W is a *weak total resolving set* if for every $x \in V(G) - W$ and $y \in W$, there exists some $w \in W - \{y\}$ such that $d_G(w, x) \neq d_G(w, y)$, as defined in [6].[1]

A weak total resolving set of minimum cardinality is called *weak total metric basis* of G, and its cardinality is called the *weak total metric dimension* of G, denoted by $\dim_{wt}(G)$. For instance, $W = \{v_1, v_2, v_3\}$ is a weak total metric basis of the two graphs of Figure 1.



G G'

Figure 1. $W = \{v_1, v_2, v_3\}$ is a weak total metric basis of G and G'.

The remaining definitions are given the first time that the concept is found in the text below, unless they refer to standard notions that can be found in any textbook on graph theory.

2. General Results

In this section, we are (finally) going to derive characterisations of those graphs G that have a weak total metric dimension that equals $|V(G)|$ or $|V(G)| - 1$. We

[1]We are going to discuss more on the history of this notion at the end of this paper.

also identify conditions under which the weak total metric dimension is 2. Moreover, we derive comparisons with other graph parameters, especially related to independent domination.

2.1. Preparations

First, we will argue why studying connected graphs is of major importance to derive our results. We therefore first discuss graphs with at least two components.

Proposition 1. *Let $G = (V, E)$ be a graph with $c \geq 2$ connected components $\langle V_i \rangle$ induced by the vertex sets V_i, $1 \leq i \leq c$. Then, we can conclude:*

1. *Let $W \subseteq V$ be a weak total resolving set for G. If $|V_i| > 1$ and $W_i := W \cap V_i \neq \emptyset$, then W_i is a weak total resolving set for $\langle V_i \rangle$.*
2. *Conversely, if some $U_i \subseteq V_i$ is a weak total resolving set for $\langle V_i \rangle$, then U_i is also a weak total resolving set for G.*

Proof. For the first claim, consider any $x \in W_i$ and $y \in V_i - W_i$ (if $W_i = V_i$, W_i is a trivial weak total resolving set for $\langle V_i \rangle$). Since for all $w \in W - W_i$ $d_G(w, x) = d_G(w, y) = \infty$, there has to be some $u \in W_i - \{x\}$ which distinguishes x and y.

For the second claim, consider some $x \notin U_i$ and some $y \in U_i$. Clearly, if $x \in V_i$, there is some $w \in U_i - \{y\}$ that distinguishes x and y, as U_i is a weak total resolving set for $\langle V_i \rangle$. If $x \notin V_i$, then $d_G(x, w) = \infty$ for any $w \in U_i - \{y\}$, while $d_G(y, w) < \infty$, as y and w are both in the same connected component. ■

This allows us to conclude as follows.

Corollary 2. *Let $G = (V, E)$ be a graph of order $n \geq 2$ with $c \geq 2$ connected components V_i, $1 \leq i \leq c$. Then, one of the following two cases applies:*

- *All components V_i are singleton sets, or, equivalently, G is empty. This means that $G \cong N_n$ and $\dim_{wt}(G) = n$.*
- *G is a non-empty graph. Then,*

$$\dim_{wt}(G) = \min \{\dim_{wt}(\langle V_i \rangle) \colon 1 \leq i \leq c \wedge |V_i| > 1\}.$$

Since every vertex belonging to any weak total resolving set W must be distinguished by at least one other vertex in W, it follows that for any graph of order n the weak total metric dimension satisfies the following inequality:

$$2 \leq \dim_{wt}(G) \leq n. \tag{1}$$

Another simple yet important observation is contained in the following statement.

Lemma 3. *Let $G = (V, E)$ be a graph. If for a vertex $v \in V$ there is no $u \in V - N_G[v]$ such that $N_G(v) \subseteq N_G(u)$, then $N_G[v]$ is a weak total resolving set for G.*

Proof. Let $v \in V$ satisfies the stated condition. Any $x \in N_G(v)$ can be distinguished from any $y \in V - N_G[v]$ by v, since $d_G(v, x) = 1 < d_G(v, y)$. For any $y \in V - N_G[v]$, there is some $u \in N_G(v) - N_G(y)$, so $d_G(u, y) > 1 = d_G(u, v)$, which means that u distinguishes v and y. ■

A special case concerning vertices of degree 2 will be important for trees.

Corollary 4. *For any graph G having at least three vertices $v, x, y \in V(G)$ such that $N_G(v) = \{x, y\}$ and $N_G(x) \cap N_G(y) = \{v\}$, $\dim_{wt}(G) \leq 3$.*

Proposition 5. *Let G be a graph. If there exists some $W \subseteq V(G)$ such that for every $v \in W$ there are two vertices $a, b \in N_G(v) \cap W$ such that $N_G(a) \cap N_G(b) \subset W$, then $\dim_{wt}(G) \leq |W|$.*

Proof. Let $W \subseteq V(G)$ such that for every $v \in W$ there are $a, b \in N_G(v) \cap W$ such that $N_G(a) \cap N_G(b) \subset W$. We claim that W is a weak total resolving set for G. Consider any $v \in W$ and its two neighbours $a, b \in W$ with $N_G(a) \cap N_G(b) \subset W$. For any $u \in V(G) - W$, we have $u \notin N_G(u) \cap N_G(v)$, which yields $d_G(a, u) > 1 = d_G(a, v)$ or $d_G(b, u) > 1 = d_G(b, v)$. ■

A special case which will become an important tool in the study of Cartesian product graphs (in Subsection 5.2) is

Corollary 6. *Let G be a graph. If there exists some $W \subseteq V(G)$ such that $\langle W \rangle \cong C_4$ and $N(v) \cap N(w) \subset W$ for all nonadjacent $v, w \in W$, then $\dim_{wt}(G) \leq 4$.*

2.2. Connection to independent 2-dominating sets

A *k-dominating set* of a graph G is a set S of vertices of G such that every vertex not in S is adjacent to at least k vertices in S. The k-domination number of G, denoted by $\gamma_k(G)$, is the minimum size of a k-dominating set [5]. A set is *independent* (or *stable*) if no two vertices in it are adjacent. An *independent dominating* set in a graph G is a set that is both dominating and independent in G. The *independent k-domination number* of G, denoted by $i_k(G)$, is the minimum size of an independent k-dominating set. The *independence number* of G, denoted $\alpha(G)$, is the maximum size of an independent set in G. For graphs G for which an independent k-dominating set exists, we have

$$\gamma_k(G) \leq i_k(G) \leq \alpha(G). \tag{2}$$

Remark 7. Any independent 2-dominating set of G is a weak total resolving set for G.

Proof. Let $S \in V(G)$ be an independent 2-dominating set in G. Let $s \in S$ and $v \in V(G) - S$. As there exists $w \in S - \{s\}$ such that $d_G(w, s) \geq 2$ and $d_G(w, v) = 1$, the result is immediate. ■

Corollary 8. *If a graph G has at least one independent 2-dominating set, then* $\dim_{wt}(G) \leq i_2(G)$.

To show the tightness of this inequality we take the complete bipartite graph $K_{r,s}$, $r, s \geq 2$. Clearly, $\dim_{wt}(K_{r,s}) = i_2(K_{r,s}) = \min\{r, s\}$.

Notice that any independent 2-dominating set of a graph G is also an independent 2-dominating set of any join graph of the form $G + H$. Thus, we point out the following remark.

Remark 9. Let G and H be two graphs. If G has at least one independent 2-dominating set, then

$$\dim_{wt}(G + H) \leq i_2(G).$$

The above bound is tight. For instance,

(a) $\dim_{wt}(N_r + C_t) = i_2(N_r) = \dim_{wt}(N_r + P_t) = r$, for $2 \leq r \leq 4$ and $t \geq 7$.
(b) $\dim_{wt}(N_r + C_3) = i_2(N_r) = \dim_{wt}(N_r + P_2) = r$, for $r \geq 2$.
(c) $\dim_{wt}(N_r + C_6) = i_2(C_6) = \dim_{wt}(N_r + P_5) = i_2(P_5) = 3$, for $r \geq 3$.
(d) $\dim_{wt}(N_r + C_4) = i_2(C_4) = \dim_{wt}(N_r + P_3) = i_2(P_3) = 2$, for $r \geq 1$.
(e) $\dim_{wt}(K_r + C_4) = i_2(C_4) = \dim_{wt}(K_r + P_3) = i_2(P_3) = 2$, for $r \geq 1$.
(f) $\dim_{wt}(K_r + C_6) = i_2(C_6) = \dim_{wt}(K_r + P_5) = i_2(P_5) = 3$, for $r \geq 1$.

Proposition 10. *Let G be a graph of diameter* 2. *Then the following assertions hold.*

(i) $\dim_{wt}(G) = 2$ *if and only if* $i_2(G) = 2$.
(ii) *If* $i_2(G) = 3$, *then* $\dim_{wt}(G) = 3$.

Proof. By Corollary 8 we have that if G has an independent 2-dominating set, then $2 \leq \dim_{wt}(G) \leq i_2(G)$. So, $i_2(G) = 2$ leads to $\dim_{wt}(G) = 2$.

Now, assume that $W = \{a, b\}$ is a weak total metric basis of G. Notice that a and b cannot be adjacent. Indeed, if they are adjacent, then for any neighbour c of a we have that $d_G(a, c) = 1 = d_G(a, b)$, which is a contradiction. Moreover, since $D(G) = 2$, we deduce that any vertex in $u \in V(G) - W$ must be adjacent to both a and b, as otherwise either $d_G(a, u) = 2 = d_G(a, b)$ or $d_G(b, u) = 2 = d_G(b, a)$, which is a contradiction. Therefore, W is an independent 2-dominating set and so (i) follows.

The proof of (ii) is derived from Corollary 8 and (i). ■

2.3. The role of twin vertices

Two vertices x, y are called *false twins* if $N_G(x) = N_G(y)$ and x, y are called *true twins* if $N_G[x] = N_G[y]$. Two vertices x, y are *twins* if they are false twins or true twins. If two vertices $x, y \in V(G)$ are twins, then for any $z \in V(G) - \{x, y\}$ it holds $d_G(z, x) = d_G(z, y)$. Therefore, the next result follows.

Lemma 11. *Let G be a graph and let W be a weak total metric basis of G. If $x, y \in V(G)$ are twins, then either both x and y are in W or neither x nor y belongs to W.*

We define the *twin equivalence relation* $\mathcal{R}$ on $V(G)$ as follows:

$$x\mathcal{R}y \longleftrightarrow N_G[x] = N_G[y] \text{ or } N_G(x) = N_G(y).$$

Let us see three different examples where every vertex has a twin. An example of a graph where every equivalence class is a true twin equivalence class is $K_r + (K_s \cup K_t)$, $r, s, t \geq 2$. In this case, there are three equivalence classes composed of r, s and t true twins, respectively. As an example where no class is composed of true twins, we take the complete bipartite graph $K_{r,s}$, $r, s \geq 2$. Finally, the graph $K_r + N_s$, $r, s \geq 2$, has two equivalence classes and one of them is composed of r true twins. On the other hand, $K_1 + (K_r \cup N_s)$, $r, s \geq 2$, is an example where one class is singleton, one class is composed of true twins and the other one is composed of false twins.

If U is a twin equivalence class in a connected graph G with $|U| \geq 2$ and there exists a weak total resolving set for G, say W, which contains at least one element from U, then Lemma 11 leads to $U \subseteq W$. Thus, we point out the following result.

Proposition 12. *Let G be a connected graph of order n with $\dim_{wt}(G) < n$ and let $\{U_1, U_2, \ldots, U_k\}$ be the set of twin equivalence classes of G. Then,*

$$\min_{1 \leq i \leq k} |U_i| \leq \dim_{wt}(G) \leq n - \min_{1 \leq i \leq k} |U_i|.$$

Moreover, $\dim_{wt}(G) = \min_{1 \leq i \leq k} |U_i|$ *if and only if there exists a twin equivalence class U, which is an independent $|U|$-dominating set whose cardinality is* $|U| = \min_{1 \leq i \leq k} |U_i| \geq 2$.

Proof. The bounds are directly obtained from Lemma 11. Now, let U be a twin equivalence class, which is an independent $|U|$-dominating set, whose cardinality is $|U| = \min_{1 \leq i \leq k} |U_i| \geq 2$. By the lower bound, we have that $\dim_{wt}(G) \geq |U|$ and, since U is an independent $|U|$-dominating set, by Remark 7 we conclude that $\dim_{wt}(G) = |U|$.

Assume that $\dim_{wt}(G) = \min_{1\leq i\leq k} |U_i|$. Since $\dim_{wt}(G) \geq 2$, we have that no twin equivalence class is a singleton set. So, by Lemma 11, we have that any weak total metric basis U of G is a twin equivalence class, as $|U| = \min_{1\leq i\leq k} |U_i|$. If U is a true twin equivalence class, then given $u \in U$, $v \in N_G(u) - U$ and $w \in U - \{u\}$, we have $d_G(u, v) = d_G(w, v) = 1$, which is a contradiction, and so U is a false twin equivalence class. Then for any $u, v \in U$, $d_G(u, v) = 2$, which implies that U is a $|U|$-dominating set, as the existence of a vertex $w \in V(G) - U$ at distance two from a vertex $u \in U$ leads to a contradiction. ■

For instance, for $r, s \geq 2$, the graph $K_r + N_s$, of order $n = r + s$, is composed of twins and its twin equivalence classes are $U_1 = V(K_r)$ and $U_2 = V(N_s)$. The only weak total metric basis is $U_2 = V(N_s)$ and so the upper bound is achieved for $r < s$, where $\dim_{wt}(K_r + N_s) = s = n - r = n - \min_{1\leq i\leq 2} |U_i|$, and the lower bound is achieved for $r > s$, where $\dim_{wt}(K_r + N_s) = s = \min_{1\leq i\leq 2} |U_i|$. If $r = s$, then both lower and upper bounds are achieved, and U_2 is indeed an independent $|U_2|$-dominating set.

2.4. Characterisations for $\dim_{wt}(G) \in \{n, n-1, 2\}$

We define the following parameter for a non-complete graph G via its graph complement $\overline{G}$

$$\Theta(G) = \max_{u,v\in V(G):\, uv\in E(\overline{G})} |N_G(u) \cap N_G(v)|.$$

We also assume that $\Theta(K_n) = 0$.

Proposition 13. *For any non-trivial graph G of order n,* $\dim_{wt}(G) \leq n - \Theta(G)$.

Proof. It is straightforward that the result holds for $G = K_n$, i.e., $\dim_{wt}(K_n) = n - \Theta(K_n) = n$. We now suppose that there are two vertices $x, y \in V(G)$ with $xy \in E(\overline{G})$. Let $W = N_G(x) \cap N_G(y)$. We claim that $W' = V(G) - W$ is a weak total resolving set. To see this, we differentiate the following cases where $w \in W$ and $w' \in W'$.

1. $w' \in N_G(x)$. Since $w' \notin N_G[y]$, we have that $d_G(y, w') \geq 2 > 1 = d_G(y, w)$.
2. $w' \in N_G(y)$. This case is analogous to the previous one.
3. $w' \notin N_G(x) \cup N_G(y)$. If $w' \neq x$, then $d_G(x, w') \geq 2 > 1 = d_G(x, w)$ and, if $w' = x$, then $d_G(y, w') = 2 > 1 = d_G(y, w)$.

According to the three cases above, we conclude that W' is a weak total resolving set and, as a consequence, the result follows. ■

Proposition 13 and Corollary 2 immediately yield the following characterisation.

Corollary 14. *Let G be a graph of order $n \geq 2$. Then* $\dim_{wt}(G) = n$ *if and only if* $G \cong K_n$ *or* $G \cong N_n$.

The following straightforward remark will be useful in describing a procedure to determine the graphs where $\dim_{wt}(G) = 2$.

Remark 15. Let G be a graph. Given $x, y \in V(G)$, let

$$indist(x, y) := \{z \in V(G) : \; d_G(x, y) = d_G(x, z) \text{ or } d_G(x, y) = d_G(y, z)\}.$$

Then $\dim_{wt}(G) = 2$ if and only if there exist $x, y \in V(G)$ such that $indist(x, y) = \{x, y\}$.

In particular, if there is some $z \notin \{x, y\}$ such that $z \in indist(x, y)$, then x does not distinguish y and z, or y does not distinguish x and z. Remark 15 can be used to derive the following algorithmic result.

Proposition 16. *Given a graph G of order $n \geq 2$, in time $O(n^3)$ it can be decided if* $\dim_{wt}(G) = 2$.

Proof. Initially we can compute the distance matrix DistM_G by using the well-known Floyd-Warshall algorithm. DistM_G is a symmetric $n \times n$-matrix whose rows and columns are labelled by vertices, with entries between 0 and $n - 1$ (or ∞). Now observe that $indist(x, y) = \{x, y\}$ if and only if DistM_G possesses a non-zero entry, say, j at position (x, y), i.e., $j = \mathrm{DistM}_G(x, y)$, such that both the row (labelled x) and the column (labelled y) contain j only at position (x, y). Given DistM_G, this condition can be checked in linear time for each pair (x, y), i.e., the overall runnning time of the sketched algorithm is dominated by the cubic time of the Floyd-Warshall algorithm. ∎

Examples of infinite families of graphs where $\dim_{wt}(G) = 2$ are given in Remark 39 and Corollary 40. It remains to study graphs with $3 \leq \dim_{wt}(G) \leq n - 1$.

Theorem 17. *Let G be a connected graph of order $n \geq 5$. Then* $\dim_{wt}(G) = n-1$ *if and only if* $G \cong K_{1,n-1}$ *or* $G \cong K_1 + (K_{n-2} \cup K_1)$.

Proof. First of all, note that $2 \leq \dim_{wt}(K_{1,n-1})$. So, at least one leaf of $K_{1,n-1}$ must belong to any weak total resolving set and, by Lemma 11, we have that $\dim_{wt}(K_{1,n-1}) \geq n - 1$. By Corollary 14, $\dim_{wt}(K_{1,n-1}) = n - 1$. Analogously, $2 \leq \dim_{wt}(K_1 + (K_{n-2} \cup K_1)) \leq n - 1$, and since the set composed of the two vertices associated to the copies of K_1 does not form a weak total resolving set, at least one vertex of degree $n - 2$ must belong to any weak total resolving set and, by Lemma 11, we have that the $n-2$ twins composing the clique K_{n-2} must belong to any weak total resolving set. Let A be this set of twin vertices. Given

$w \in A$, no vertex of $A - \{w\}$ distinguishes w from the vertex of degree $n-1$, thus $\dim_{wt}(K_1 + (K_{n-2} \cup K_1)) \geq n-1$ and, as a consequence, $\dim_{wt}(K_1 + (K_{n-2} \cup K_1)) = n-1$.

Now, let $\dim_{wt}(G) = n-1$. First consider some vertex $v \in V(G)$ with $2 \leq |N_G(v)| \leq n-3$. If there exists some $u \in V(G) - N_G[v]$ with $|N_G(u) \cap N_G(v)| \geq 2$, then Proposition 13 shows that $\dim_{wt}(G) \leq n-2$, which is a contradiction. Hence, for all $u \in V(G) - N_G[v]$, $|N_G(u) \cap N_G(v)| \leq 1$. As $|N_G(v)| \geq 2$, this means that $N_G(v) \subseteq N_G(u)$ cannot hold. Hence, by Lemma 3, $N_G[v]$ is a weak total resolving set with $|N_G[v]| \leq n-2$, contradicting $\dim_{wt}(G) = n-1$. Therefore, G may only contain vertices of degree 1 or of degree at least $n-2$. Since G is connected and $n \geq 5$, there is at least one vertex of degree larger than 1.

Suppose there is a vertex v with $|N_G(v)| = n-2$, which leaves exactly one vertex $u \notin N_G(v)$. Again, by Proposition 13, $N_G(u)$ and $N_G(v)$ are not allowed to intersect in more than one node w implying $N_G(u) = \{w\}$. Since $\{v, u\} \subset N_G(w)$, the node w has to have a degree at least $n-2$. With $n \geq 5$, w has at least one other neighbour $x \in V - \{v, u\}$. Since $\{v, w\} \subseteq N_G(x)$, x also has to have at least $n-2$ neighbours. With $N_G(u) = \{w\}$ this only leaves the possibility $N_G[x] = V - \{u\}$ which yields $\deg(y) \geq 2$ for all remaining vertices $y \in V - \{u, v, w, x\}$. Again, with $N_G(u) = \{w\}$, this only leaves the possibility $N_G[y] = V - \{u\}$, hence $\langle V - \{w, u\} \rangle = K_{n-2}$. Since $\{x, v\} \subset N_G(w) \cap N_G(y)$ for any $y \in V - \{u, v, w, x\}$, Proposition 13 gives $wy \in E(G)$, which means $G \cong K_1 + (K_{n-2} \cup K_1)$.

Suppose there is no vertex of degree $n-2$. Then, at least one vertex v has to have degree $n-1$. If there was another vertex u of degree larger than 1, its degree would have to be $n-1$ as well. Any vertex $w \notin \{u, v\}$ of G is a neighbour of u and of v, so w has a degree $n-1$. Hence, we face the complete graph with weak total dimension n by Corrolary 14. Therefore, the only valid possibility is degree 1 for all vertices $V - \{v\}$, which yields $G \cong K_{1,n-1}$. ■

Checking the few possibilities for $n = 2$, $n = 3$ and $n = 4$, and by Corollary 2, we obtain the following result for not necessarily connected graphs of order at least three.

Theorem 18. *Let G be a graph of order n. Then, $\dim_{wt}(G) = n-1$ if and only if $n \geq 3$ and if one of the following cases applies:*

$$G \cong K_{1,n-1} \text{ or } G \cong K_1 + (K_{n-2} \cup K_1) \text{ or } G \cong K_1 \cup K_{n-1}.$$

3. The Weak Total Adjacency Dimension

We introduce now the weak total adjacency dimension as a tool to study the weak total metric dimension. We say that a set $W \subseteq V(G)$ is a *weak total* *adjacency resolving set* if for every $x \in V(G) - W$ and $y \in W$, there exists some $w \in W - \{y\}$ such that $w \in N_G(x) \Delta N_G(y)$. Henceforth, we will say that a vertex w *distinguishes* a pair of vertices x, y if $w \in N_G(x) \Delta N_G(y)$. A weak total adjacency resolving set of minimum cardinality is called a *weak total adjacency basis* of G, and its cardinality is called the *weak total adjacency dimension* of G, denoted by $\operatorname{adim}_{wt}(G)$. As each weak total adjacency resolving set is a weak total resolving set, we have the following inequalities:

$$2 \le \dim_{wt}(G) \le \operatorname{adim}_{wt}(G) \le n. \tag{3}$$

Moreover, for any graph G of diameter (at most) two,

$$\dim_{wt}(G) = \operatorname{adim}_{wt}(G) \tag{4}$$

and, by definition of weak total adjacency dimension, for any graph G,

$$\operatorname{adim}_{wt}(G) = \operatorname{adim}_{wt}(\overline{G}), \tag{5}$$

where $\overline{G}$ denotes the complement of G. The latter identity, together with the bound of equation (3), shows that the adjacency dimension variant is a suitable tool to study the graph complement operation with respect to the weak total dimension. A set-like notation of the definition of weak total adjacency resolving sets gives:

Lemma 19. *A subset $W \subset V$ is a weak total adjacency resolving set for a graph $G = (V, E)$ if and only if for any $v \in W$,*

$$U(v) := \left(\bigcup_{w \in W - N_G[v]} N_G(w) \right) \cup \left(\bigcup_{w \in W \cap N_G(v)} (V - N_G[w]) \right) \supseteq V - W.$$

With equation (4), this statement also holds for weak total resolving sets of graphs of diameter two.

While most of the results for weak total resolving sets remain true for weak total adjacency resolving sets, point two of Proposition 1 and consequently Corollary 2 do not hold. Consider for example the graph $G = C_4 \cup C_4$: Each component has a weak total adjacency basis of cardinality two but these sets cannot be used for the whole graph G which has a weak total adjacency dimension equal to four. A weaker version of Proposition 1 however still holds.

Proposition 20. *Let $G = (V, E)$ be a graph with $c \ge 2$ connected components described by the vertex sets V_i, $1 \le i \le c$. Then*

1. *Let $W \subseteq V$ be a weak total adjacency resolving set. If $|V_i| > 1$ and $W_i := W \cap V_i \neq \emptyset$, then W_i is a weak total adjacency resolving set for $\langle V_i \rangle$.*

2. *Conversely, a weak total adjacency resolving set* $U_i \subseteq V_i$ *for* $\langle V_i \rangle$ *is a weak total adjacency resolving set for* G *if and only if the subgraph* $\langle U_i \rangle$ *has no isolated vertices.*

Proof. The first part follows by analogy to the proof of Proposition 1, part one. For the second part, assume that $U_i \subseteq V_i$ is a weak total adjacency resolving set for $\langle V_i \rangle$. If there exists an isolated vertex u in $\langle U_i \rangle$, then u cannot be distinguished from any $v' \in V - V_i$ and hence U_i is no weak total adjacency resolving set for G.

If the subgraph $\langle U_i \rangle$ has no isolated vertices, then any $u \in U_i$ and $v \in V - V_i$ can be distinguished by a vertex $u' \in N_G(u) \cap U_i$. Any $u \in U_i$ can be distinguished from any $v \in V_i - U_i$ since U_i is a weak total adjacency resolving set for $\langle V_i \rangle$. Altogether U_i is a weak total adjacency resolving set for the whole graph. ∎

Lemma 3 (and hence Corollary 4) remains true for weak total adjacency resolving sets. We explicitly state these results, making use of equation (5) in the following:

Lemma 21. *Let* $G = (V, E)$ *be a graph. If for some* $v \in V$ *there is no* $u \in V - N_G[v]$ *such that* $N_G(v) \subseteq N_G(u)$*, then* $N_G[v]$ *is a weak total adjacency resolving set for* G*. Also, if for some* $v \in V$ *there is no* $u \in N_G(v)$ *such that* $N_G[u] \subseteq N_G[v]$*, then* $\overline{N_G(v)}$ *is a weak total adjacency resolving set for* G*.*

Proposition 5 also remains true and has some interesting variations.

Proposition 22. *Let* G *be a graph and let* $W \subseteq V(G)$*. If for every* $v \in W$*, there are two vertices* $a, b \in \overline{W}$ *such that either* $a, b \in N_G(v)$ *and* $N_G(a) \cap N_G(b) \subset W$ *or* $a, b \in \overline{N_G[v]}$ *and* $\overline{N_G(a)} \cap \overline{N_G(b)} \subset W$*, then* $\operatorname{adim}_{wt}(G) \leq |W|$*.*

By equations (3), (5) and Proposition 22 we deduce the next result.

Corollary 23. *Let* G *be a graph and let* $W \subseteq V(G)$*. If for every* $v \in W$*, there are two vertices* $a, b \in \overline{W}$ *such that either* $a, b \in N_G(v)$ *and* $N_G(a) \cap N_G(b) \subset W$ *or* $a, b \in \overline{N_G[v]}$ *and* $\overline{N_G(a)} \cap \overline{N_G(b)} \subset W$*, then* $\dim_{wt}(\overline{G}) \leq |W|$*.*

By Corollary 4 and equation (5) we have:

Corollary 24. *For any graph* G *having at least three vertices* $v, x, y \in V(G)$ *such that either* $N_G(v) = \{x, y\}$ *and* $N_G(x) \cap N_G(y) = \{v\}$*, or* $N_G(v) = V - \{x, y\}$ *and* $N_G(x) \cup N_G(y) = V - \{v\}$*, we find that* $\operatorname{adim}_{wt}(G) \leq 3$*.*

The connections to independent 2-dominating sets yield even stronger results with respect to the weak total adjacency dimension. To derive the next remark we proceed as in the proof of Proposition 7.

Remark 25. Any independent 2-dominating set of G is a weak total adjacency resolving set for G.

Corollary 26. *Let G be a graph. If G has at least one independent 2-dominating set, then* $\operatorname{adim}_{wt}(G) \leq i_2(G)$.

Proposition 27. *Let G be a graph. Then* $\operatorname{adim}_{wt}(G) = 2$ *if and only if* $i_2(G) = 2$ *or* $i_2(\overline{G}) = 2$.

Proof. By Corollary 26 we conclude that $i_2(G) = 2$ leads to $\operatorname{adim}_{wt}(G) = 2$, and also $i_2(\overline{G}) = 2$ leads to $\operatorname{adim}_{wt}(\overline{G}) = 2$.

Now, assume that $\{a, b\}$ is a weak total adjacency resolving set for G. Note that if $ab \in E(G)$, then $N_G(a) = \{b\}$ and $N_G(b) = \{a\}$, and thus, $G = K_2$ or $G = K_2 \cup H$ where H is an arbitrary graph. So, $\{a, b\}$ is an independent 2-dominating set for $\overline{G}$ and, as a result, $i_2(\overline{G}) = 2$. Conversely, if $ab \in E(\overline{G})$, then in G any $c \in V(G) - \{a, b\}$ should be adjacent to both a and b, and so $\{a, b\}$ is an independent 2-dominating set for G. ■

Characterisations of graphs with weak total adjacency dimension n or $n - 1$ can be shown similarly to those for the weak total metric dimension. Using the parameter $\Theta(G)$ defined prior to Proposition 13 we deduce that for any graph G of order n

$$\operatorname{adim}_{wt}(G) \leq n - \max\{\Theta(G), \Theta(\overline{G})\}.$$

Therefore, the following result immediately follows.

Remark 28. Let G be a non-trivial graph of order n. Then $\operatorname{adim}_{wt}(G) = n$ if and only if $G \cong K_n$ or $G \cong N_n$.

Theorem 29. *Let G be a graph of order $n \geq 5$. Then* $\operatorname{adim}_{wt}(G) = n - 1$ *if and only if* $G \cong H$ *with* $H \in \{K_{1,n-1}, K_1 \cup K_{n-1}, K_1 + (K_{n-2} \cup K_1), K_1 \cup K_{1,n-2}\}$.

Proof. The equality $\operatorname{adim}_{wt}(K_{1,n-1}) = \operatorname{adim}_{wt}(K_1 + (K_{n-2} \cup K_1) = n - 1$ immediately follows from equation (3), Theorem 17 and Remark 28. With equation (5) the same dimension follows for the complement: $\operatorname{adim}_{wt}(K_1 \cup K_{n-1}) = \operatorname{adim}_{wt}(K_1 \cup K_{1,n-2}) = n - 1$.

As this part of the proof to Theorem 17 only uses properties which remain true for weak adjacency resolving sets (Lemma 21 and Proposition 13), any connected graph G of order $n \geq 5$ and $\operatorname{adim}_{wt}(G) = n - 1$ is isomorphic to either $K_{1,n-1}$ or $K_1 + (K_{n-2} \cup K_1)$. Since the complement of any non-connected graph is connected, equation (5) yields that any non-connected graph G of order $n \geq 5$ and with $\operatorname{adim}_{wt}(G) = n - 1$ is isomorphic to either $K_1 \cup K_{n-1}$ or $K_1 \cup K_{1,n-2}$. ■

Observe that this result is not true for $n = 4$ since P_4 has a weak total adjacency dimension equal to three but is not in the stated family of graphs.

4. Special Graph Classes: Trees, Fans and Wheels

Given a tree T, we define $\delta^*(T)$ as the minimum degree among all the internal vertices of T. Given two vertices $u, v \in V(T)$ we say that $B_u(v)$ is the *branch at v containing u* and we define it as the connected component of $T - \{v\}$ containing u. The following proposition is remarkable in the sense that it not only gives a possibility to compute the weak total metric dimension on trees, but also gives a neat characterisation of this parameter on a tree T in terms of $\delta^*(T)$.

Proposition 30. *Let T be a tree.*

(i) *If $\delta^*(T) = 2$, then $\dim_{wt}(T) \leq 3$.*
(ii) *If $\delta^*(T) \geq 3$, then $\dim_{wt}(T) = \delta^*(T)$.*

Proof. Since (i) is a direct consequence of Corollary 4, from now on we assume that $\delta^*(T) \geq 3$. Let $u \in V(T)$ be a vertex of degree $\delta^*(T)$. We claim that $N_T(u)$ is a weak total resolving set for T. On one hand, for any $a, b \in N_T(u)$ we have $d_T(a, u) = 1 \neq 2 = d_T(a, b)$. So, any pair u, a, where $a \in N_T(u)$, is distinguished by some element $b \in N_T(u) - \{a\}$. For any $c \in V(T) - N_T[u]$ and $a \in N_T(u)$ there is a vertex $b \in N_T(u) - \{a\}$ such that $c \notin B_b(u)$; recall that $\deg(u) \geq 3$. Any vertex b of this type distinguishes a and c since $d_T(b, c) = d_T(u, c) + 1 \geq 3 > 2 = d_T(b, a)$. Hence $N_T(u)$ is a weak total resolving set for T and, as a consequence, $\dim_{wt}(T) \leq |N_T(u)| = \delta^*(T)$.

It remains to prove that $\dim_{wt}(T) \geq \delta^*(T)$. Let W be a weak total metric basis of T. Let u, v be two adjacent vertices of T such that $B_u(v) \cap W = \{u\}$. If v is a leaf, W has to be the set $\{u, v\}$. This set however does not allow to distinguish v from any other neighbour of u which means that v is an internal vertex. With $\delta^*(T) \geq 3$ this implies that $|N_T(v)| \geq 3$. Since $d_T(x, y) = d_T(x, u) = d_T(x, v) + 1$ for any $y \in N_T(v)$ and $x \notin B_y(v) \cup B_u(v)$, any $y \in N_T(v) - W$ requires a node $x_y \in B_y(v)$ to distinguish y from u which yields $B_y(v) \cap W \neq \emptyset$ for all $y \in N_T(v)$ and hence $|W| \geq |N_G(v)|$. ■

A similar result can be stated for forests, making use of Corollary 2.

Remark 31. Notice that our combinatorial results also lead to a polynomial-time algorithm for computing the weak total dimension of a tree:

1. Determine (in linear time) the value of $\delta^*(T)$.
2. If $\delta^*(T) = 2$, then check whether $\dim_{wt}(T) = 2$ by using Proposition 16; if the check fails, then we conclude that $\dim_{wt}(T) = 3$.
3. If $\delta^*(T) \geq 3$, then $\dim_{wt}(T) = \delta^*(T)$.

Our next result gives the value of the weak total adjacency dimension for any tree T, different from a star, in terms of its minimum internal degree $\delta^*(T)$.

Proposition 32. *For any tree T different from a star,*

$$\operatorname{adim}_{wt}(T) = \delta^*(T) + 1.$$

Proof. Let $u \in V(T)$ be a vertex of degree $\delta^*(T)$. By Lemma 21, $N_T[u]$ is a weak total adjacency resolving set of cardinality $\delta^*(T) + 1$.

It remains to prove that $\operatorname{adim}_{wt}(T) \geq \delta^*(T) + 1$. Let W be a weak total adjacency basis of T. Similar to the proof to Proposition 30, let u, v be two adjacent vertices of T with $B_u(v) \cap W = \{u\}$; again, v is an internal vertex. Distinguishing u from any node $x \in (N_T(v) - \{u\}) - W$ requires a node in $N_T(x) - \{v\}$ which yields $(N_T[x] - \{v\}) \cap W \neq \emptyset$ for all $x \in N_T(v) - \{u\}$. If $v \in W$, this immediately yields $|W| \geq |N_T[v]| \geq \delta^*(T) + 1$. Suppose $v \notin W$. Since T is not a star, at least one neighbour y of v is an interior vertex implying the existence of a vertex $z \in N_T(y) - \{v\}$. Distinguishing z from y requires either a neighbour of z other than y or a neighbour of y other than z, which yields $|W \cap B_y(v)| \geq 2$ and hence

$$|W| = |W \cap B_y(v)| + \sum_{x \in N_T(v) - \{y\}} |W \cap N_T[x]| \geq 2 + |N_T(v)| - 1 \geq \delta^*(T) + 1.$$

■

Now we derive some results on the weak total metric dimension for complements of trees.

Proposition 33. *The following assertions hold.*

(i) *For any tree T,* $\dim_{wt}(\overline{T}) \leq \delta^*(T) + 1$.

(ii) *For any tree T of diameter three,* $\dim_{wt}(\overline{T}) = 2$.

(iii) *For any tree T of diameter $D(T) \geq 4$,* $\dim_{wt}(\overline{T}) \geq 3$.

Proof. (i) By Theorem 18, together with equation (4), and by Proposition 32, we have $\operatorname{adim}_{wt}(T) \leq \delta^*(T) + 1$ for all trees T. Thus, by equations (3) and (5), we get $\dim_{wt}(\overline{T}) \leq \operatorname{adim}_{wt}(\overline{T}) = \operatorname{adim}_{wt}(T) \leq \delta^*(T) + 1$.

(ii) The two central vertices x, y of any tree T of diameter three are antipodal vertices in $\overline{T}$, and for any $v \in V(T) - \{x, y\}$ we have that $d_{\overline{T}}(v, x) < 3$ and $d_{\overline{T}}(v, y) < 3$. Hence, $\{a, b\}$ is a weak total metric basis of $\overline{T}$.

(iii) First of all, notice that $\overline{T}$ has diameter two. Suppose that $\{x, y\}$ is a weak total metric basis of $\overline{T}$. Since x and y cannot be adjacent in $\overline{T}$, they are adjacent in T and so, for any vertex z adjacent to x in T we have that $d_{\overline{T}}(x, y) = 2 = d_{\overline{T}}(x, z)$ and so x does not distinguish the pair y, z in $\overline{T}$, which is a contradiction. ■

By equations (3), (5) and Corollary 24 we obtain the following corollaries.

Corollary 34. *Let n be an integer.*

- *If $n \geq 1$, then* $\dim_{wt}(\overline{P_n}) \leq \operatorname{adim}_{wt}(\overline{P_n}) = \operatorname{adim}_{wt}(P_n) \leq 3$.

- *If* $n \geq 5$, *then* $\dim_{wt}(\overline{C_n}) \leq \operatorname{adim}_{wt}(\overline{C_n}) = \operatorname{adim}_{wt}(C_n) \leq 3$.

Corollary 35. *For any tree* T *having a vertex of degree two,* $\dim_{wt}(\overline{T}) \leq 3$.

We now consider generalisations of the wheel graph and fan graph: the *complete-core generalised wheel* $W_{r,t} = K_r + C_t$ $(r \geq 1, t \geq 4)$, and the *complete-core generalised fan* $F_{r,t} = K_r + P_t$, $(r \geq 1, t \geq 3)$. The complete-core generalised fan $F_{3,4}$ is shown in Figure 2.

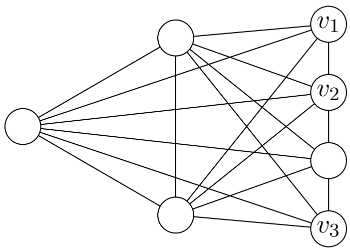


Figure 2. $W = \{v_1, v_2, v_3\}$ is a weak total metric basis of the complete-core generalised fan $F_{3,4}$.

By performing some simple calculations, we have that

$$\dim_{wt}(F_{r,4}) = \dim_{wt}(W_{r,5}) = 3,$$

while the values of $\dim_{wt}(F_{r,t})$ for $t \in \{3, 5\}$ and $\dim_{wt}(W_{r,t})$ for $t \in \{4, 6\}$ have been shown in Remark 9. For the remaining values of t, Lemma 19 gives the following result.

Theorem 36. *The following assertions hold.*

(i) *For any integer* $t \geq 6$, $\dim_{wt}(F_{r,t}) = \operatorname{adim}_{wt}(F_{r,t}) = 4$.

(ii) *For any integer* $t \geq 7$, $\dim_{wt}(W_{r,t}) = \operatorname{adim}_{wt}(W_{r,t}) = 4$.

Proof. First observe that $\dim_{wt}(F_{r,t}) = \operatorname{adim}_{wt}(F_{r,t})$ and that $\dim_{wt}(W_{r,t}) = \operatorname{adim}_{wt}(W_{r,t})$ by equation (4). Suppose W is a weak total resolving set for $W_{r,t} = (V, E)$ (or $F_{r,t} = (V, E)$). By Lemma 19 for any vertex $v \in W$, the set $U(v)$ has to contain at least all vertices of $V - W$. Since $N[w] = V$ for all $w \in V(K_r)$, these kinds of vertices do not contribute to the set $U(v)$, what allows us to assume $W - V(K_r) = W$. Suppose $W = \{x, y\}$. Since $r + 2 \leq |N[y]| \leq r + 3$, neither the set $N(y)$ nor the set $V - N[y]$ can contain $r + t - 2$ vertices, which means $|U(x)| < |V - W|$. Suppose $W = \{v, x, y\} \subset V(C_t)$ $(V(P_t))$. Since $V(K_r) \subset N(w)$ for every vertex $w \in V - V(K_r)$, each vertex $z \in W$ needs a vertex $z' \in W - N_{W_{r,t}}(z)$ $(z' \in W - N_{F_{r,t}}(z))$ for $U(z)$ to contain $V(K_r) \subset V - W$. Assume $v \notin N_G(x) \cap N_G(y)$, which gives $U(v) = N[x] \cup N[y]$. The requirement $V - W = U(v)$ yields that $\{x, y\}$ is a dominating set for $\langle V(C_t) - v \rangle$ $(\langle V(P_t) - v \rangle)$, which is impossible for $t \geq 7$ $(t \geq 6)$.

Any induced P_4 in G on the other hand yields a weak total resolving set W of cardinality 4, since every vertex in W has one neighbour $u \in W$ with $V(C_t) - W \subset V - N(u)$ ($P_t - W \subset V - N(u)$) and one non-neighbour w with $V(K_r) \subset N(w)$. ■

We now consider the *empty-core generalised wheel* $W_{\overline{r},t} = N_r + C_t$ ($r \geq 1$, $t \geq 3$), and the *empty-core generalised fan* $F_{\overline{r},t} = N_r + P_t$, ($r \geq 1, t \geq 2$). Notice that the first number, giving the size of the core, is overlined to differentiate this notion from the (complete-core) generalised wheel, resp. fan.

Some cases of $\dim_{wt}(F_{\overline{r},t})$ and $\dim_{wt}(W_{\overline{r},t})$ have been shown after Remark 9. Also, by performing some simple calculations, we have that

$$\dim_{wt}(F_{\overline{2},4}) = \dim_{wt}(F_{\overline{2},5}) = 2 \text{ and } \dim_{wt}(F_{\overline{r},4}) = 3, \text{ for } r \geq 3.$$

Moreover,

$$\dim_{wt}(W_{\overline{2},5}) = \dim_{wt}(W_{\overline{2},6}) = 2 \text{ and } \dim_{wt}(W_{\overline{r},5}) = 3, \text{ for } r \geq 3.$$

By Remark 7, for $r \geq 2$, the set of vertices of N_r is a weak total resolving set for $W_{\overline{r},t}$ and $F_{\overline{r},t}$. However, if we take $r \geq 4$ and we proceed analogously to the proof of Theorem 36 we deduce the following result.

Remark 37. The following assertions hold.

(i) For any integers $r \geq 4$ and $t \geq 6$, $\dim_{wt}(F_{\overline{r},t}) = \text{adim}_{wt}(F_{\overline{r},t}) = 4$.

(ii) For any integers $r \geq 4$ and $t \geq 7$, $\dim_{wt}(W_{\overline{r},t}) = \text{adim}_{wt}(W_{\overline{r},t}) = 4$.

5. Operations on Graphs

In this section, we study operations on graphs in connection with the weak total metric/adjacency dimension. Henceforth, in the case of ordered pairs (x, y) we will write $N_G(x, y)$, $N_G[x, y]$ and $U(x, y)$ rather than $N_G((x, y))$, $N_G[(x, y)]$ and $U((x, y))$, respectively.

5.1. Point attaching graphs

Let $G_W[\mathcal{H}]$ be a graph constructed from a graph G, a set $W = \{v_1, \ldots, v_k\} \subseteq V(G)$ and a family of pairwise disjoint (non-trivial) connected graphs $\mathcal{H} = \{G_1, \ldots, G_k\}$ as follows. Select one vertex u_i of G_i and identify u_i with $v_i \in W$, for every $i \in \{1, \ldots, k\}$. In Figure 1, the graph $G' = G_W[\mathcal{H}]$ is obtained by so-called point attaching from G, the family $\mathcal{H} = \{K_3, K_1 + C_4, C_5\}$ and the set $W = \{v_1, v_2, v_3\}$. Note that for any $G_i \in \mathcal{H}$ we have $\dim_{wt}(G_i) \geq \dim_{wt}(G)$ and $\dim_{wt}(G_W[\mathcal{H}]) = \dim_{wt}(G) = 3$.

We would point out the following remark which follows from Lemma 21.

Remark 38. Let G be a graph and let $W \subseteq V(G)$. Then for any family $\mathcal{H}$ composed of $|W|$ pairwise disjoint non-trivial connected graphs and any $v \in W$, the set $N_{G_W[\mathcal{H}]}[v]$ is a weak total adjacency resolving set for $G_W[\mathcal{H}]$.

The following remark is straightforward.

Remark 39. Let $W = \{a, b\}$ be a weak total metric basis of a graph G and let $\mathcal{H} = \{G_1, G_2\}$ be a family of disjoint connected graphs. If for $u_i \in V(G_i)$ it holds that $\epsilon_{G_i}(u_i) < d_G(a, b)$, $i \in \{1, 2\}$, then

$$\dim_{wt}(G_W[\mathcal{H}]) = 2.$$

Corollary 40. *Let G be a 2-antipodal graph and let $\mathcal{H} = \{G_1, G_2\}$ be a family of disjoint connected graphs. If for $u_i \in V(G_i)$ it holds that $\epsilon_{G_i}(u_i) < D(G)$, $i \in \{1, 2\}$, then for any set $W = \{v_1, v_2\} \subset V(G)$ of antipodal vertices,*

$$\dim_{wt}(G_W[\mathcal{H}]) = 2.$$

Proposition 41. *Let G be a connected graph with $\dim_{wt}(G) \geq 3$ such that there exists a weak total metric basis W' such that $d_G(v, w) = d_G(x, y)$ for all $v, w, x, y \in W'$ with $v \neq w$ and $x \neq y$. For any non-empty set $W \subseteq W'$ and any family $\mathcal{H}$ composed of $|W|$ pairwise disjoint connected graphs,*

$$\dim_{wt}(G_W[\mathcal{H}]) \leq \dim_{wt}(G).$$

Moreover, if for all $G_i \in \mathcal{H}$ it holds that $\dim_{wt}(G_i) \geq \dim_{wt}(G)$, then

$$\dim_{wt}(G_W[\mathcal{H}]) = \dim_{wt}(G).$$

Proof. Consider G, W' and W as described in the statement of the proposition. We will show that W' is a weak total resolving set for $G_W[\mathcal{H}]$. Since W' is a weak total metric basis of G, we only need to show that for any $w \in W'$ and any $v \in V(G_j)$, there exists some $w' \in W' - \{w\}$ such that $d_{G_W[\mathcal{H}]}(w', w) \neq d_{G_W[\mathcal{H}]}(w', v)$. With $v_j \in V(G) \cap V(G_j)$, any $w' \in W' - \{v_j, w\}$ satisfies $d_{G_W[\mathcal{H}]}(w', w) = d_G(w', w) = d_G(w', v_j) < d_G(w', v_j) + d_{G_j}(v_j, v) = d_{G_W[\mathcal{H}]}(w', v)$. Thus, the upper bound follows.

Now, assume that $\dim_{wt}(G_i) \geq \dim_{wt}(G)$, for all $G_i \in \mathcal{H}$, and let $\{v_i\} = W \cap V(G_i)$. Let X be a weak total metric basis of $G_W[\mathcal{H}]$. We claim that $|X| \geq \dim_{wt}(G)$. To see this, we define $X_i = X \cap V(G_i)$ for all $G_i \in \mathcal{H}$. If $X_i \subseteq \{v_i\}$, for all $G_i \in \mathcal{H}$, then X is a weak total resolving set for G so especially $|X| \geq \dim_{wt}(G)$. Also, if there exists $G_i \in \mathcal{H}$ such that $|X| \geq \dim_{wt}(G_i)$, also $|X| \geq \dim_{wt}(G)$. So, assume that there exists $G_i \in \mathcal{H}$ such that $0 < |X_i| < \dim_{wt}(G_i)$. Then there exists $x \in X_i$ and $y \in V(G_i) - X_i$ such that $d_{G_i}(u, x) = d_{G_i}(u, y)$ for any $u \in X_i - \{x\}$. Since X is a weak total metric basis of $G_W[\mathcal{H}]$, there must exist some $u' \in X - V(G_i)$ such that $d_{G_W[\mathcal{H}]}(u',x) \neq d_{G_W[\mathcal{H}]}(u',y)$. Hence, $d_{G_W[\mathcal{H}]}(u',v_i) + d_{G_i}(v_i,x) = d_{G_W[\mathcal{H}]}(u',x) \neq d_{G_W[\mathcal{H}]}(u',y) = d_{G_W[\mathcal{H}]}(u',v_i) + d_{G_i}(v_i,y)$ and, as a result, $d_{G_i}(v_i,x) \neq d_{G_i}(v_i,y)$. Thus, $X_i \cup \{v_i\}$ must be a weak total resolving set for G_i and so $\dim_{wt}(G_W[\mathcal{H}]) = |X| \geq |X_i \cup \{v_i\}| \geq \dim_{wt}(G_i) \geq \dim_{wt}(G)$. Therefore, the result follows. ∎

An example of application of Proposition 41 is shown in Figure 1.

5.2. Cartesian product graphs

In this section we study the weak total metric dimension of Cartesian product graphs. We recall that the *Cartesian product* of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G\Box H$ such that $V(G\Box H) = V_1 \times V_2$, and two vertices $(a,b),(c,d)$ are adjacent in $G\Box H$ if and only if, either ($a = c$ and $bd \in E_2$) or ($b = d$ and $ac \in E_1$).

Proposition 42. *Let G and H be two connected non-trivial graphs. Then $\dim_{wt}(G\Box H) = 2$ if and only if there exists a weak total metric basis $\{a,c\}$ of G such that $d_G(a,c) = \epsilon_G(a) = \epsilon_G(c)$ and there exists a weak total metric basis $\{b,d\}$ of H such that $d_H(b,d) = \epsilon_H(b) = \epsilon_H(d)$.*

Proof. Assume that there exists a weak total metric basis $\{a,c\}$ of G such that $d_G(a,c) = \epsilon_G(a) = \epsilon_G(c)$ and there exists a weak total metric basis $\{b,d\}$ of H such that $d_H(b,d) = \epsilon_H(b) = \epsilon_H(d)$.

We claim that $W = \{(a,b),(c,d)\}$ is a weak total metric basis of $G\Box H$. Since $d_G(a,x) < d_G(a,c)$ for any vertex $x \in V(G)$ and $d_H(b,y) < d_H(b,d)$ for any vertex $y \in V(H)$, we can conclude that for any vertex $(x,y) \in V(G\Box H)$

$$d_{G\Box H}((a,b),(x,y)) = d_G(a,x) + d_H(b,y) < d_G(a,c) + d_H(b,d) = d_{G\Box H}((a,b),(c,d)).$$

Analogously, we deduce that $d_{G\Box H}((c,d),(x,y)) < d_{G\Box H}((a,b),(c,d))$. Therefore, W is a weak total metric basis of $G\Box H$ and so we conclude that $\dim_{wt}(G\Box H) = 2$.

On the other hand, assume that $\dim_{wt}(G\Box H) = 2$ and let $\{(x,y),(u,v)\}$ be a weak total metric basis of $G\Box H$. Suppose that $y = v$. Let u' be a vertex adjacent to u, lying on a shortest path between x and u, and let $w \in N_H(y)$. Then $d_{G\Box H}((x,y),(u,v)) = d_{G\Box H}((x,y),(u',w))$, which is a contradiction, and as a consequence $y \neq v$. By analogy we deduce that $x \neq u$.

Suppose that $\epsilon_G(x) > d_G(x,u)$. Thus, there exists $u' \in V(G)$ such that $d_G(x,u') = d_G(x,u) + 1$. Let y' be a vertex adjacent to v lying on a shortest path between y and v. Then $d_{G\Box H}((x,y),(u,v)) = d_{G\Box H}((x,y),(u',y'))$, which is a contradiction. Thus, $\epsilon_G(x) = d_G(x,u)$. Analogously, it follows that $\epsilon_G(u) = d_G(x,u)$. Now, if there exists $z \in V(G)$ such that $d_G(x,z) = d_G(x,u)$, then

$d_{G\square H}((x,y),(u,v)) = d_{G\square H}((x,y),(z,v))$, which is a contradiction again. By analogy we deduce that for every $z \in V(G)$ it holds $d_G(u,z) \neq d_G(x,u)$. So, $\{x,u\}$ is a weak total metric basis of G.

By symmetry it holds that $d_H(y,v) = \epsilon_G(y) = \epsilon_G(v)$ and $\{y,v\}$ is a weak total metric basis of H. Therefore, the result follows. ■

Corollary 43. *For any connected non-trivial graph G and any integer $n \geq 3$,*

$$\dim_{wt}(G\square K_n) \geq 3.$$

By Proposition 5 and Corollary 43 we deduce our next result.

Proposition 44. *For any connected graph G,*

$$\dim_{wt}(G\square K_3) = 3.$$

Proposition 45. *For any integers $k \geq 1$ and $n \geq 2$,*

$$\dim_{wt}(C_{2k+1}\square K_n) = 3.$$

Proof. The case $k = 1$ was previously studied in Proposition 44, so we can assume that $k \geq 2$. Since $\dim_{wt}(C_{2k+1}) = 3$, by Proposition 42, we deduce $\dim_{wt}(C_{2k+1}\square K_n) \geq 3$.

Let $u_1, u_2, u_3 \in V(C_{2k+1})$ such that $u_1u_2 \in E(C_{2k+1})$ and $d_{C_{2k+1}}(u_3,u_1) = d_{C_{2k+1}}(u_3,u_2) = k$. Given $v \in V(K_n)$ and $W = \{u_1,u_2,u_3\}$, we claim that $W' = W \times \{v\}$ is a weak total resolving set for $C_{2k+1}\square K_n$. To this end, we differentiate three cases for any $(a,b) \in V(C_{2k+1}\square K_n) - W'$ and $(u,v) \in W'$.

• In case $a = u$, immediately $b \neq v$ and for any $u' \in W - \{u\}$, we have $d_{C_{2k+1}\square K_n}((u',v),(a,b)) = d_{C_{2k+1}}(u',a) + 1 > d_{C_{2k+1}\square K_n}((u',v),(u,v))$.

• In case $a \in W - \{u\}$, the definition of W' yields $b \neq v$. Let $u' \in W - \{a,u\}$. If $u' = u_3$, we have $d_{C_{2k+1}\square K_n}((u',v),(a,b)) = d_{C_{2k+1}\square K_n}((u',v),(a,v)) + 1 > d_{C_{2k+1}\square K_n}((u',v),(u,v))$. Since $k \geq 2$, for $u' \neq u_3$ we immediately obtain that $d_{C_{2k+1}\square K_n}((a,v),(u,v)) = k > d_{C_{2k+1}\square K_n}((a,v),(a,b)) = 1$.

• In case $a \notin W$, if $b = v$, $W \times \{v\}$ is a weak total resolving set for $\langle V(C_{2k+1}) \times \{v\}\rangle \cong C_{2k+1}$, so consider $b \neq v$. Let $\{u',u''\} = W - \{u\}$. Suppose that vertex (u',v) does not distinguish the pair $(u,v),(a,b)$, i.e., $d_{C_{2k+1}\square K_n}((u',v),(u,v)) = d_{C_{2k+1}\square K_n}((u',v),(a,b))$. Thus, $d_{C_{2k+1}}(u',u) = d_{C_{2k+1}}(u',a) + 1$ and, as a consequence, either $u = u_3$ or $u' = u_3$. Note that in both cases $d_{C_{2k+1}}(u',a) = k-1$. If $u = u_3$, we observe that $d_{C_{2k+1}\square K_n}((u'',v),(u,v)) = k \neq d_{C_{2k+1}\square K_n}((u'',v),(a,b)) \in \{k-1,k+1\}$. Otherwise, $u' = u_3$ which means that $d_{C_{2k+1}\square K_n}((u'',v),(u,v)) = 1 < d_{C_{2k+1}\square K_n}((u'',v),(a,b)) \in \{2,3\}$.

According to the three cases above, we conclude that W' is a weak total resolving set for $C_{2k+1}\square K_n$ and, as a consequence, $\dim_{wt}(C_{2k+1}\square K_n) \leq |W'| = 3$. Therefore, the result follows. ■

Since the Cartesian product of two 2-antipodal graphs is a 2-antipodal graph, we have that for any 2-antipodal graphs G and H it holds that $\dim_{wt}(G\Box H) = 2$. For instance, the following graphs all have a weak total metric dimension of two: $P_r\Box P_t$, $C_{2k}\Box P_r$, $C_{2k}\Box C_{2t}$, $C_{2k}\Box Q_t$ $P_r\Box Q_t$, where Q_t denotes the hypercube of degree t.

Proposition 46. *For any 2-antipodal graph G and for any integer $k \geq 1$,*

$$\dim_{wt}(G\Box C_{2k+1}) = 3.$$

Proof. Let $a, b \in V(G)$ such that $d_G(a,b) = D(G)$, and let $x, y, z \in V(C_{2k+1})$ such that $yz \in E(C_{2k+1})$ and $d_{C_{2k+1}}(x,y) = d_{C_{2k+1}}(x,z) = k$. Since for any $(u,v) \in V(G\Box C_{2k+1})$ it holds that (u,v) lies on a shortest path between (a,x) and (b,y) or (u,v) lays on a shortest path between (a,x) and (b,z), we conclude that $\{(a,x),(b,y),(b,z)\}$ is a weak total resolving set for $G\Box C_{2k+1}$ and, as a consequence, $\dim_{wt}(G\Box C_{2k+1}) \leq 3$. On the other hand, by Proposition 42 we deduce that $\dim_{wt}(G\Box C_{2k+1}) \geq 3$. ■

Notice that Proposition 46 can be extended to any pair of graphs G and H satisfying the following restrictions:

- There exist $a, b \in V(G)$ such that $d_G(a,b) = D(G)$ and for any $c \in V(G) - \{a,b\}$, $d_G(a,c) < D(G)$ and $d_G(b,c) < D(G)$.
- There exist $x, y, z \in V(H)$ such that $yz \in E(H)$, $d_H(x,y) = d_H(x,z) = D(H)$ and for any $v \in V(H) - \{x,y,z\}$, $d_H(x,v) < D(H)$, $d_H(y,v) < D(H)$ and $d_H(z,v) < D(H)$.

The following result is a direct consequence of Corollary 6.

Corollary 47. *For any non-trivial graphs G and H,*

$$\dim_{wt}(G\Box H) \leq 4.$$

Looking at the weak total adjacency dimension, Corollary 23 and equation (3) give bounds for complements of Cartesian products[2].

Remark 48. For any non-trivial graphs G and H, $\dim_{wt}(\overline{G\Box H}) \leq 4$. Moreover, $\dim_{wt}(\overline{G\Box K_3}) \leq 3$.

Proposition 49. *For any graph G of order $n \geq 4$ and for any integer $r \geq 4$, $\mathrm{adim}_{wt}(K_r\Box G) = 3$ if and only if there exists some $W = \{x,y,z\} \subset V(G)$ such that $\langle W\rangle \cong K_3$ and $N_G(v) \cap N_G(w) \subset W$ for all $v, w \in W$.*

[2]Notice that for $G \cong H \cong K_2$, Corollary 23 does not apply, although $\dim_{wt}(\overline{K_2\Box K_2}) = \dim_{wt}(K_2 \cup K_2) = 2 \leq 4$.

Proof. For any $(v,w) \in V(K_r) \times V(G)$, by the structure of $H := K_r \Box G$, the set of neighbours is given by $N_H[v,w] = (\{v\} \times N_G(w)) \cup (V(K_r) \times \{w\})$. Since $|N_H(v,w)| \leq n+r-2$ and $|V(K_r) \times V(G) - N_H[v,w]| \leq rn - r$, a weak total adjacency basis of cardinality two for H would be a contradiction to Lemma 19 (recall that $n, r \geq 4$).

Assume that $S = \{(v_1,w_1),(v_2,w_2),(v_3,w_3)\}$ is a weak total adjacency basis for H. Counting cardinalities, $N_H(v_i,w_i) \cup N_H(v_j,w_j)$ cannot contain $V(K_r) \times V(G) - S$ for any $i,j \in \{1,2,3\}$. Lemma 19 hence yields $|N_H(v_i,w_i) \cap S| \geq 1$ for all $i = 1,2,3$.

Suppose that (v_1,w_1) and (v_3,w_3) are not adjacent, which implies that $w_1 \neq w_3$ and that (v_1,w_1) and (v_2,w_2) are adjacent. $U(v_1,w_1) \supseteq V(K_r) \times V(G) - S$ requires $N_H(v_2,w_2) - S \subset N_H[v_3,w_3]$. Since $r,n \geq 4$, there is a $v \notin \{v_1,v_2,v_3\}$. The vertex $(v,w_2) \in N_H(v_2,w_2) - S$ requires $w_3 = w_2$. For the set $U(v_3,w_3)$, the neighbourhood $N_H(v_1,w_1)$ similarly has to contain (v,w_2), what would require $w_1 = w_2$ and hence $w_1 = w_3$, so that (v_1,w_1) and (v_3,w_3) are indeed adjacent, a contradiction.

Then we have that (v_i,w_i) and (v_j,w_j) are adjacent for all $i,j = 1,2,3$, so that either $v_1 = v_2 = v_3$ or $w_1 = w_2 = w_3$. Notice that the set $U(v_1,w_1)$ equals $V(G) \times V(K_r) - (N_H[v_2,w_2] \cap N_H[v_3,w_3])$ which, by Lemma 19, means that $N_H[v_2,w_2] \cap N_H[v_3,w_3] \subseteq S$. Thus, if $w_1 = w_2 = w_3$, then $N_H[v_2,w_2] \cap N_H[v_3,w_3] = V(K_r) \times \{w_2\} \nsubseteq S$, which is a contradiction. Hence, $v_1 = v_2 = v_3$ and the subgraph induced by $\{w_1,w_2,w_3\}$ is isomorphic to K_3. Now we find that $\{v_2\} \times (N_G(w_2) \cap N_G(w_3)) = (\{v_2\} \times N_G(w_2)) \cap (\{v_3\} \times N_G(w_3)) = N_H[v_2,w_2] \cap N_H[v_3,w_3] \subseteq S$ and, as a result, $N_G(w_2) \cap N_G(w_3) = \{w_1\}$. Looking at $U(v_i,w_i)$ for $i = 2,3$ in the same way, this argument gives $N_G(w_i) \cap N_G(w_j) \in \{w_1,w_2,w_3\}$ for all $i \neq j$ which shows that $\{w_1,w_2,w_3\}$ is the required set.

Conversely, if we take $W = \{x,y,z\} \subset V(G)$ such that $\langle W \rangle \cong K_3$ and $N_G(v) \cap N_G(w) \subset W$ for all $v,w \in W$, then the choice $S' := \{v\} \times W$ for any $v \in V(K_r)$ yields a weak total adjacency set for H by Proposition 22. ■

Corollaries 43 and 47 lead to $3 \leq \dim_{wt}(K_r \Box K_s) \leq 4$ for $r,s \geq 3$, and hence Proposition 49 and equation (4) give:

Corollary 50. *For any integers $r,s \geq 4$,*

$$\text{adim}_{wt}(K_r \Box K_s) = \dim_{wt}(K_r \Box K_s) = 4.$$

The previous propositions show that the weak total metric and adjacency dimensions behave differently from other graph parameters, as usually it can be expected that the parameter of a Cartesian product graph depends on the parameter of its constituents.

5.3. Lexicographic product graphs

The *lexicographic product* of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G \circ H$ with vertex set $V = V_1 \times V_2$ and two vertices $(a,b), (c,d) \in V$ are adjacent in $G \circ H$ if and only if either $ac \in E_1$, or ($a = c$ and $bd \in E_2$).

Note that the lexicographic product of two graphs is not a commutative operation. Moreover, $G \circ H$ is a connected graph if and only if G is connected. We would point out the following known results.

Fact 51 [3]**.** Let G be a connected graph and H be a non-trivial graph. Then the following assertions hold for any $a, c \in V(G)$ and $b, d \in V(H)$ such that $a \neq c$.

(i) $d_{G\circ H}((a,b),(c,d)) = d_G(a,c)$

(ii) $d_{G\circ H}((a,b),(a,d)) = \min\{d_H(b,d), 2\}$.

Proposition 52. *Let G be a connected graph and H be a non-trivial graph. Let S be a weak total resolving set for $G \circ H$, and let $S_i = \{v_j : (u_i, v_j) \in S\}$. If $S_i \neq \emptyset$, then S_i is a weak adjacency resolving set for H and, as a consequence,*

$$\dim_{wt}(G \circ H) \geq \operatorname{adim}_{wt}(H).$$

Proof. We take $S_i \neq \emptyset$. Thus, since S is a weak total metric basis of $G \circ H$, by Fact 51 (i), for any $v_j \in S_i$ and $v_k \notin S_i$, there exists $(u_i, v_s) \in S - \{(u_i, v_j)\}$ such that $d_{G\circ H}((u_i,v_s),(u_i,v_j)) \neq d_{G\circ H}((u_i,v_s),(u_i,v_k))$. Hence, $v_s \in S_i - \{v_j\}$ and, by Fact 51 (ii), either ($v_s v_j \in E(H)$ and $v_s v_k \notin E(H)$) or ($v_s v_j \notin E(H)$ and $v_s v_k \in E(H)$). Therefore, S_i is a weak total adjacent set for H. Moreover, taking S as a weak total metric basis of $G \circ H$ we have $\dim_{wt}(G \circ H) = |S| \geq |S_i| \geq \operatorname{adim}_{wt}(H)$. ∎

Now we shall show that the above bound is tight.

Proposition 53. *Let G be a connected graph and H be a non-trivial graph. If there exists a weak adjacency basis S of H such that for every $a \in S$ it holds that $N_H(a) \cap S \neq \emptyset$ and $N_H[a] \not\supset S$, then*

$$\dim_{wt}(G \circ H) = \operatorname{adim}_{wt}(H).$$

Proof. By Proposition 52, we have that $\dim_{wt}(G \circ H) \geq \operatorname{adim}_{wt}(H)$. It only remains to prove that $\dim_{wt}(G \circ H) \leq \operatorname{adim}_{wt}(H)$, and to this end, we will show that $S_i = \{u_i\} \times S$ is a weak total resolving set for $G \circ H$, where $u_i \in V(G)$ is arbitrary and S is a weak total adjacency basis of H that satisfies the conditions stated above. Let $(u_i, v_j) \in S_i$ and $(u_r, v_s) \in V(G \circ H) - S_i$. We differentiate three cases.

• $i = r$. Since S is a weak total adjacency basis, there exists $v_k \in S$ such that either ($v_j v_k \in E(H)$ and $v_s v_k \notin E(H)$) or ($v_j v_k \notin E(H)$ and $v_s v_k \in E(H)$). Thus, $d_{G\circ H}((u_i,v_j),(u_i,v_k)) \neq d_{G\circ H}((u_i,v_k),(u_i,v_s))$.

• $d_{G\circ H}((u_i,v_j),(u_r,v_s)) = 1$ and $i \neq r$. Since $N_H[a] \not\supset S$ for all $a \in S$, there exists $v_t \in S$ with $v_jv_t \notin E(H)$, so (u_i,v_j) and (u_i,v_t) are not adjacent in $G \circ H$. Hence, $d_{G\circ H}((u_i,v_j),(u_i,v_t)) = 2 \neq 1 = d_{G\circ H}((u_i,v_j),(u_r,v_s))$.

• $d_{G\circ H}((u_i,v_j),(u_r,v_s)) \geq 2$ and $i \neq r$. Since $N_H(a) \cap S \neq \emptyset$ for all $a \in S$, there exists $v_t \in S$ such that $v_jv_t \in E(H)$, which means that (u_i,v_j) and (u_i,v_t) are adjacent in $G\circ H$. Hence, $d_{G\circ H}((u_i,v_j),(u_i,v_t)) = 1 \neq 2 \leq d_{G\circ H}((u_i,v_j),(u_r,v_s)) = d_{G\circ H}((u_i,v_t),(u_r,v_s))$.

Therefore, the result follows. ■

If, given two connected graphs H_1 and H_2 we have $\mathrm{adim}_{wt}(H_1 \Box H_2) = 4$, then we can construct a weak total adjacency basis W in such a way that $\langle W\rangle \cong C_4$. Hence, our next result is a direct consequence of Proposition 53.

Corollary 54. *Let H_1, H_2, G be three connected graphs. If $\mathrm{adim}_{wt}(H_1 \Box H_2) = 4$, then*

$$\dim_{wt}(G \circ (H_1 \Box H_2)) = \dim_{wt}(G \circ \overline{(H_1 \Box H_2)}) = 4.$$

By equation (4) and Theorem 36 we have that for any complete-core generalised fan graph $F_{r,t}$ such that $t \geq 6$, it holds that $\dim_{wt}(F_{r,t}) = \mathrm{adim}_{wt}(F_{r,t}) = 4$. Analogously, for any complete-core generalised wheel graph $W_{r,t}$ such that $t \geq 7$, it holds that $\dim_{wt}(W_{r,t}) = \mathrm{adim}_{wt}(W_{r,t}) = 4$. In both cases, the weak total adjacency metric bases are composed by four consecutively adjacent vertices of the corresponding path or cycle. Hence, these weak total metric basis satisfy the premises of Proposition 53 for $H = F_{r,t}$ and $H = W_{r,t}$, as well as for their complements. Therefore, we can state the following result.

Proposition 55. *Let r, t be two positive integers and let G be a connected graph.*

- *If $t \geq 6$, then $\dim_{wt}(G \circ F_{r,t}) = \dim_{wt}(G \circ \overline{F}_{r,t}) = 4$.*
- *If $t \geq 7$, then $\dim_{wt}(G \circ W_{r,t}) = \dim_{wt}(G \circ \overline{W}_{r,t}) = 4$.*

Proposition 56. *Let G be a non-trivial graph and let $r \geq 2$ be an integer. Given a weak total adjacency basis $W \subseteq V(G)$ of G and $u \in V(K_r)$, the set $\{u\} \times W$ is a weak total adjacency basis of $K_r \circ G$ if and only if $W - N_G(v) \neq \emptyset$ for all $v \in W$.*

Proof. Assume that $\{u\} \times W$ is a weak total adjacency basis of $K_r \circ G$. Since for any $u' \in V(K_r) - \{u\}$ and any $v \in W$, the vertices (u,v) and (u',v) are adjacent, there must exists $v' \in W - N_G(v)$ such that (u,v') distinguishes (u,v) from (u',v). Therefore, $W - N_G(v) \neq \emptyset$ for all $v \in W$.

Conversely, assume that $W - N_G(v) \neq \emptyset$ for all $v \in W$. Consider any $(x,y) \in V(K_r \circ G) - (\{u\} \times W)$ and $v \in W$. If $x = u$, then there exists (u,v') which distinguishes (u,v) from (u,y), as W is a weak total adjacency basis of G.

Also, since there exists $v'' \in W - N_G(v)$, then for $x \neq u$, $(u, v'') \in N_{K_r \circ G}(x, y) - N_{K_r \circ G}(u, v)$. Hence, $\{u\} \times W$ is a weak total adjacency resolving set for $K_r \circ G$. Now, suppose that S is a weak total adjacency resolving set for $K_r \circ G$ such that $|S| < |W|$. Then taking $S_a = \{b \in V(G) : (a, b) \in S\} \neq \emptyset$, by Proposition 52 we obtain that S_a is a weak total adjacency resolving set for G and $|S_a| \leq |S| < |W| = \mathrm{adim}_{wt}(G)$, which is a contradiction. Therefore, $\{u\} \times W$ is a weak total adjacency basis of $K_r \circ G$. ■

6. Discussion

This paper was devoted to the study of combinatorial properties of graph parameters called weak total metric, resp. adjacency, dimension. These parameters have some very peculiar properties, for instance, the latter one is indifferent against graph complementation, something we do not know for any other graph parameter. Also, it seems to be difficult to build gadgets with these parameters as this is commonly done to prove computational hardness results, as piecing graphs together from smaller graphs can dramatically decrease these parameters. So, we leave it as an open question whether (or not) a weak total metric, resp. adjacency, basis for a given graph can be computed in polynomial time.

There is one more peculiarity about the first parameter: We followed with our definition the one contained in the abstract of [6]. In other places, the definition of a weak total resolving set furthermore requires that the set is indeed a resolving set. Of course, this changes the properties completely. In a sense, we studied a pure version of this notion, not combining it with the previously and intensively studied notion of metric dimension. In order to differentiate both notions and avoid further confusions, we therefore propose to call the version that we studied in this paper *pure weak total resolving set* and keep the notion of weak total resolving set for sets that are also resolving sets. To put it positively, given the fact that we observed quite interesting properties, it might be an idea to study other "pure versions" of graph parameters to better understand the cause of certain combinatorial and computational results. This would allow to study certain effects in isolation.

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