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PRODUCTS OF DIGRAPHS AND THEIR COMPETITION GRAPHS

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Abstract

If $D = (V, A)$ is a digraph, its *competition graph* (*with loops*) $CG^l(D)$ has the vertex set V and $\{u, v\} \subseteq V$ is an edge of $CG^l(D)$ if and only if there is a vertex $w \in V$ such that $(u, w), (v, w) \in A$. In $CG^l(D)$, loops $\{v\}$ are allowed only if v is the only predecessor of a certain vertex $w \in V$. For several products $D_1 \circ D_2$ of digraphs D_1 and D_2, we investigate the relations between the competition graphs of the factors D_1, D_2 and the competition graph of their product $D_1 \circ D_2$.

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1. Introduction and Definitions

All graphs $G = (V(G), E(G))$, hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ and digraphs $D = (V(D), A(D))$ considered here may have isolated vertices but no multiple edges and arcs, respectively. Moreover, in digraphs loops are forbidden. In standard terminology concerning digraphs we follow Bang-Jensen and Gutin [1]. With $d^-_D(v), d^+_D(v), N^-_D(v)$ and $N^+_D(v)$ we denote the *in-degree*, *out-degree*, *in-neighbourhood* and *out-neighbourhood* of a vertex v in a digraph D, respectively.

In 1968 Cohen [2] introduced the *competition graph* (*without loops*) $CG(D)$ associated with a digraph $D = (V, A)$ representing a food web of an ecosystem. $CG(D) = (V, E)$ is the graph with the same vertex set as D (corresponding to the species) and $E = \{\{u, v\} \mid u \neq v \wedge \exists\, w \in V : (u, w) \in A \ \wedge (v, w) \in A\}$, i.e. $\{u, v\} \in E$ if and only if u and v compete for a common prey $w \in V$.

Surveys of the large literature around competition graphs can be found in Roberts [6], Kim [4] and Lundgren [5].

In [7] it is shown that in many cases competition hypergraphs yield a better description of the predation relations among the species in $D = (V, A)$ than competition graphs. If $D = (V, A)$ is a digraph its *competition hypergraph*, $\mathcal{CH}(D) = (V, \mathcal{E})$, has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{CH}(D)$ if and only if $|e| \geq 2$ and there is a vertex $w \in V$ such that $e = \{v \in V \mid (v, w) \in A\}$. In this case we say $w \in V = V(D)$ *corresponds to* $e \in \mathcal{E}$ and vice versa.

In our paper [7] we dealt with competition hypergraphs without loops. That way we followed the most usual definition of competition graphs. In the case of digraphs D possessing vertices with only one predecessor, a competition hypergraph *with loops* contains a more detailed information on D (cf. [8]). For that reason, we also include competition hypergraphs (as well as competition graphs) with loops in our investigations and modify the notions given above.

If $D = (V, A)$ is a digraph, its *l-competition hypergraph* (*competition hypergraph with loops*) $\mathcal{CH}^l(D) = (V, \mathcal{E}^l)$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{CH}(D)$ if and only if $e \neq \emptyset$ and there is a vertex $w \in V$ such that $e = \{v \in V \mid (v, w) \in A\}$.

Analogously, the *l-competition graph* (*competition graph with loops*) $CG^l(D) = (V, E^l)$ has the vertex set V and $E^l = E(CG(D)) \cup \{\{v\} \mid v \in V \wedge \exists\, w \in V : N_D^-(w) = \{v\}\}$.

For the sake of brevity, in the following we often use the term *competition graph* (sometimes in connection with the notation $CG^{(l)}(D)$) for the competition graph $CG(D)$ as well as for the l-competition graph $CG^l(D)$ (analogously for competition hypergraphs).

Analogically with [8], for five products $D_1 \circ D_2$ (*Cartesian product* $D_1 \times D_2$, *Cartesian sum* $D_1 + D_2$, *normal product* $D_1 * D_2$, *lexicographic product* $D_1 \cdot D_2$ and *disjunction* $D_1 \vee D_2$) of digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ we investigate the construction of the competition graph $CG^{(l)}(D_1 \circ D_2) = (V, E_\circ^{(l)})$ from $CG^{(l)}(D_1) = (V_1, E_1^{(l)})$, $CG^{(l)}(D_2) = (V_2, E_2^{(l)})$ and vice versa.

The products considered here always have the vertex set $V := V_1 \times V_2$; using the notation $\widetilde{A} := \{((a, b), (a', b')) \mid a, a' \in V_1 \ \wedge\ b, b' \in V_2\}$ their arc sets $A_\circ := A(D_1 \circ D_2)$ are defined as follows:

$$A_\times := \{((a,b),(a',b')) \in \widetilde{A} \mid (a,a') \in A_1 \wedge (b,b') \in A_2\},$$
$$A_+ := \{((a,b),(a',b')) \in \widetilde{A} \mid ((a,a') \in A_1 \wedge b = b') \vee (a = a' \wedge (b,b') \in A_2)\},$$
$$A_* := A(D_1 \times D_2) \cup A(D_1 + D_2),$$
$$A_\cdot := \{((a,b),(a',b')) \in \widetilde{A} \mid (a,a') \in A_1 \vee (a = a' \wedge (b,b') \in A_2)\},$$
$$A_\vee := \{((a,b),(a',b')) \in \widetilde{A} \mid (a,a') \in A_1 \vee (b,b') \in A_2\}.$$

It follows immediately that $A_+ \subseteq A_* \subseteq A_\cdot \subseteq A_\vee$ and $A_\times \subseteq A_*$. Except the lexicographic product all these products are commutative in the sense that $D_1 \circ D_2 \simeq D_2 \circ D_1$, where $\circ \in \{\times, +, *, \vee\}$.

Usually we label the vertices of V_1 and V_2 by $1, 2, \ldots, r_1$ and by $1, 2, \ldots, r_2$, respectively, and arrange the vertices of $V = V_1 \times V_2$ according to the places of an (r_1, r_2)-matrix. Then, for each $\circ \in \{+, *, \cdot, \vee\}$, the subdigraph of $D_1 \circ D_2$ generated by the vertices of a column $S_j := \{(i,j) \mid i \in \{1, \ldots, r_1\}\}$ $(j \in \{1, \ldots, r_2\})$ and a row $Z_i := \{(i,j) \mid j \in \{1, \ldots, r_2\}\}$ $(i \in \{1, \ldots, r_1\})$ of this matrix scheme is isomorphic to D_1 and D_2, respectively.

The factor decomposition of product graphs is an interesting question (cf. Imrich and Klavžar [3]). Related to this problem, the question arises whether or not $CG^{(l)}(D_1 \circ D_2)$ can be obtained from $CG^{(l)}(D_1)$ and $CG^{(l)}(D_2)$ and vice versa. For competition hypergraphs this problem had been investigated in [8].

Since competition hypergraphs include more information than competition graphs, especially in the case of the reconstruction of $\mathcal{CH}^{(l)}(D_1)$ and $\mathcal{CH}^{(l)}(D_2)$ from $\mathcal{CH}^{(l)}(D_1 \circ D_2)$ we achieved better results (cf. [8]) than for competition graphs (see Section 3 in the present paper). In this context, it is interesting that under certain conditions $D_1 \circ D_2$ and even D_1 and D_2 can be reconstructed from $\mathcal{CH}^{(l)}(D_1 \circ D_2)$ (cf. [8], Corollaries 1–3).

Contrastingly, the results for the construction of $CG^{(l)}(D_1 \circ D_2)$ from $CG^{(l)}(D_1)$ and $CG^{(l)}(D_2)$ (see Section 2) and for the construction of $\mathcal{CH}^{(l)}(D_1 \circ D_2)$ from $\mathcal{CH}^{(l)}(D_1)$ and $\mathcal{CH}^{(l)}(D_2)$ (cf. [8]) are comparable.

2. Determination of $CG^{(l)}(D_1 \circ D_2)$ from $CG^{(l)}(D_1)$ and $CG^{(l)}(D_2)$

In the following, let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be digraphs. By $N_1^-(v)$, $N_2^-(v)$ and $N_\circ^-(v)$ we denote the set of all predecessors of a vertex v in D_1, D_2 and $D_1 \circ D_2$, respectively, where $\circ \in \{\times, +, *, \cdot, \vee\}$.

Theorem 1. *The l-competition graph $CG^l(D_1 \times D_2) = (V, E^l_\times)$ of the Cartesian product can be obtained from the l-competition graphs $CG^l(D_1) = (V_1, E^l_1)$ and $CG^l(D_2) = (V_2, E^l_2)$ of D_1 and D_2:*

$$E^l_\times = \{\{(a,b),(a',b')\} \mid \exists e_1 \in E^l_1\, \exists e_2 \in E^l_2 : \{a,a'\} \subseteq e_1 \wedge \{b,b'\} \subseteq e_2 \\ \wedge (a = a' \wedge b = b' \Rightarrow e_1 = \{a\} \wedge e_2 = \{b\})\}. \tag{1}$$

Proof. The definition of $A_\times$ implies

$$\begin{aligned} E^l_\times &= \{\{(a,b),(a',b')\} \mid \exists\, x \in V_1 \exists\, y \in V_2 : (a,x),(a',x) \in A_1 \wedge (b,y),(b',y) \in A_2 \\ &\qquad \wedge ((a,b) = (a',b') \Rightarrow N_1^-(x) = \{a\} \wedge N_2^-(y) = \{b\})\} \\ &= \{\{(a,b),(a',b')\} \mid \exists\, e_1 \in E^l_1\ \exists\, e_2 \in E^l_2 : \{a,a'\} \subseteq e_1\ \wedge \{b,b'\} \subseteq e_2 \\ &\qquad \wedge (a = a' \wedge b = b' \Rightarrow e_1 = \{a\} \wedge e_2 = \{b\})\}. \end{aligned}$$

■

Clearly, $E_\times$ results from $E^l_\times$ by deleting all loops in $E^l_\times$.

Remark 2. In general, $CG(D_1 \times D_2) = (V, E_\times)$ and therefore $CG^l(D_1 \times D_2) = (V, E^l_\times)$ cannot be obtained from $CG(D_1) = (V_1, E_1)$ and $CG(D_2) = (V_2, E_2)$.

Proof. Consider $D_1 = (V_1 = \{a, x\}, A_1 = \{(a, x)\})$, $D'_1 = (V_1, A'_1 = \emptyset)$ and $D_2 = (V_2 = \{b, b', y\}, A_2 = \{(b, y), (b', y)\})$.

On the one hand, $E(CG(D_1 \times D_2)) = \{\{(a,b),(a,b')\}\} \neq \emptyset = E(CG(D'_1 \times D_2))$, but on the other hand $E(CG(D_1)) = \emptyset = E(CG(D'_1))$. ■

Remark 3. If both D_1 and D_2 contain at least 2 vertices, then $CG^l(D_1 \vee D_2) = CG(D_1 \vee D_2)$, i.e. $CG^l(D_1 \vee D_2)$ contains no loops.

Proof. Assume, $\{(a,b)\} \in E^l_\vee$ is a loop. Then there is a vertex $(x,y) \in V_1 \times V_2$ with $N^-_\vee((x,y)) = \{(a,b)\}$. Consequently, $(a,x) \in A_1$ or $(b,y) \in A_2$.

This implies $\{(a,b') \mid b' \in V_2\} \subseteq N^-_\vee((x,y))$ or $\{(a',b) \mid a' \in V_1\} \subseteq N^-_\vee((x,y))$. Both situations contradict $|N^-_\vee((x,y))| = 1$. ■

Theorem 4. *The l-competition graph $CG^l(D_1 \vee D_2) = (V, E^l_\vee)$ of the disjunction can be obtained from the l-competition graphs $CG^l(D_1) = (V_1, E^l_1)$ and $CG^l(D_2) = (V_2, E^l_2)$ of D_1 and D_2.*

Proof. From the definition of $A_\vee$ it follows $E^l_\vee = \emptyset$ if and only if $E^l_1 = E^l_2 = \emptyset$.

In case of $E^l_1 \neq \emptyset$ and $E^l_2 \neq \emptyset$ we have

$$\begin{aligned} E^l_\vee &= \{\{(a,b),(a',b')\} \mid (a,b) \neq (a',b') \wedge \exists\, x \in V_1 \exists\, y \in V_2 : \\ &\qquad ((a,x) \in A_1 \vee (b,y) \in A_2) \wedge ((a',x) \in A_1 \vee (b',y) \in A_2)\} \\ &= \{\{(a,b),(a',b')\} \mid (a,b) \neq (a',b') \wedge \exists\, e_1 \in E^l_1 \exists\, e_2 \in E^l_2 : \\ &\qquad \{a,a'\} \subseteq e_1 \vee \{b,b'\} \subseteq e_2 \vee (a \in e_1 \wedge b' \in e_2) \vee (a' \in e_1 \wedge b \in e_2)\}. \end{aligned}$$

If exactly one of the sets E^l_1, E^l_2 is empty, then

$$\begin{aligned} E^l_\vee = \{\{(a,b),(a',b')\} \mid (a,b) \neq (a',b') \wedge \exists\, e_1 \in E^l_1 : \{a,a'\} \subseteq e_1 \\ \vee \exists\, e_2 \in E^l_2 : \{b,b'\} \subseteq e_2\}. \end{aligned}$$

■

Note that in the corresponding result for competition hypergraphs (cf. [8], Theorem 2) an additional supposition is needed.

Considering digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ with $|V_1|, |V_2| \geq 2$, $|A_1| = 1$ and $A_2 = \emptyset$ we obtain the following remark.

Remark 5. In general, $CG(D_1 \vee D_2) = (V, E_\vee)$ cannot be obtained from $CG(D_1) = (V_1, E_1)$ and $CG(D_2) = (V_2, E_2)$.

Proposition 6. *In general, $CG(D_1 \circ D_2) = (V, \mathcal{E}_\circ)$, and therefore $CG^l(D_1 \circ D_2)$, cannot be obtained from $CG^l(D_1)$ and $CG^l(D_2)$, for $\circ \in \{+, *, \cdot\}$.*

Proof. For illustration, we use figures of $CG(D_1 \circ D_2)$. In these figures several large cliques (each of them induced by the in-neighbourhood $N_\circ^-(v)$ of a vertex v of $D_1 \circ D_2$) will occur. Some of these cliques contain many edges what could be confusing in the drawings. Therefore we represent such cliques (i.e. cliques of cardinality greater than 2) as closed curves around the vertices of $N_\circ^-(v)$, i.e. as a kind of hyperedges in the competition hypergraph $\mathcal{CH}(D_1 \circ D_2)$. Of course, if $N_\circ^-(v') \subseteq N_\circ^-(v)$, it would be sufficient to draw the clique induced by the larger in-neighbourhood $N_\circ^-(v)$. But for a better traceability of the structure of $CG(D_1 \circ D_2)$ we decided to draw all "hyperedges" representing such cliques.

We make use of an example from our paper [8].

Example 7. Consider the digraphs $D_1 = (V_1, A_1), D_1' = (V_1, A_1')$ and $D_2 = (V_2, A_2)$ with $V_1 = \{1, 2, 3, 4\}, V_2 = \{1, 2, 3\}, A_1 = \{(1, 2), (3, 2), (4, 3)\}$, $A_1' = \{(1, 4), (3, 4), (4, 2)\}$ and $A_2 = \{(1, 3), (2, 3)\}$, respectively (cf. Figure 1).

Then $E(CG^l(D_1)) = \{\{1, 3\}, \{4\}\} = E(CG^l(D_1'))$.

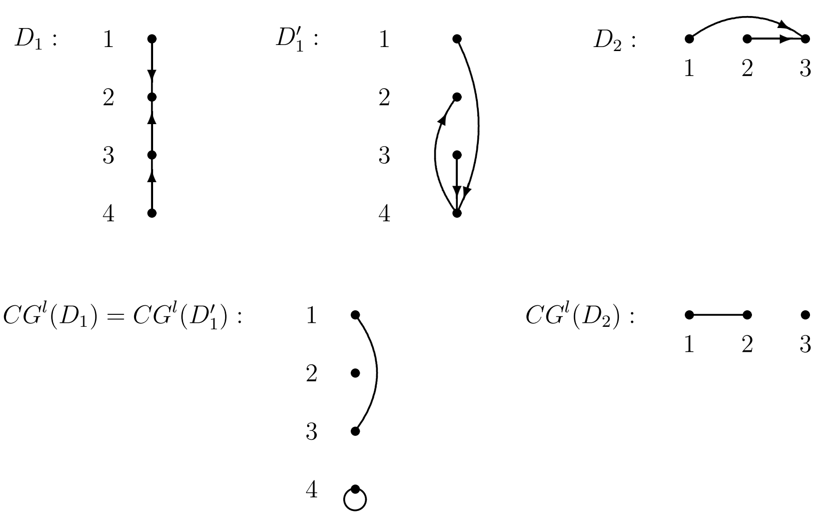

Figure 1

On the other hand, $CG(D_1 + D_2) \neq CG(D_1' + D_2)$, since the vertices $(4,1)$ and $(1,3)$ are adjacent in $CG(D_1' + D_2)$ but non-adjacent in $CG(D_1 + D_2)$ (cf. Figure 2).

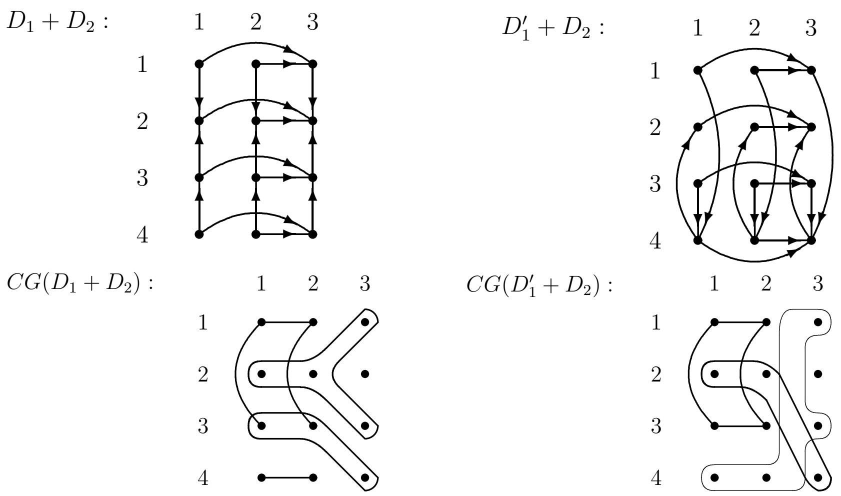

Figure 2

Moreover, $CG(D_1 * D_2) \neq CG(D_1' * D_2)$, since the vertices $(2,1)$ and $(4,1)$ are adjacent in $CG(D_1' * D_2)$ but non-adjacent in $CG(D_1 * D_2)$ (cf. Figure 3).

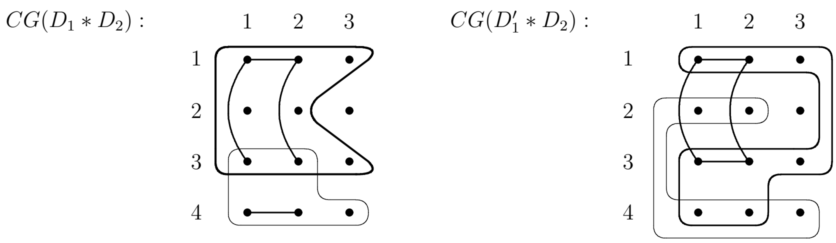

Figure 3

Finally, $CG(D_1 \cdot D_2) \neq CG(D_1' \cdot D_2)$, since the vertices $(2,1)$ and $(4,1)$ are adjacent in $CG(D_1' \cdot D_2)$ but non-adjacent in $CG(D_1 \cdot D_2)$ (cf. Figure 4). ∎

Looking at Figures 3 and 4, replacing the "hyperedges" by cliques of ordinary edges (of cardinality 2) and identifying multiple edges we make a nice observation.

Remark 8. For the digraphs D_1, D_1', D_2, we obtain $CG(D_1 * D_2) = CG(D_1 \cdot D_2)$ and $CG(D_1' * D_2) = CG(D_1' \cdot D_2)$.

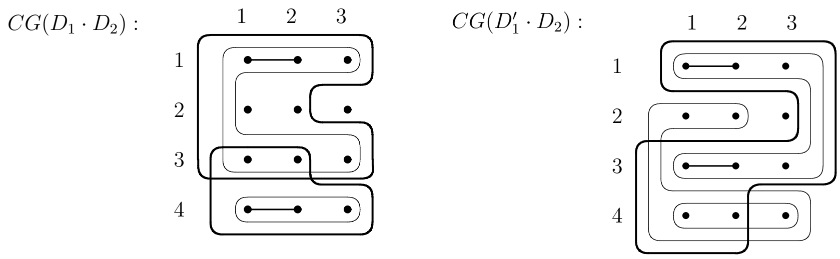

Figure 4

3. RECONSTRUCTION OF $CG^{(l)}(D_1)$ AND $CG^{(l)}(D_2)$ FROM $CG^{(l)}(D_1 \circ D_2)$

Whereas in Section 2 the results for constructing $CG^{(l)}(D_1 \circ D_2)$ from $CG^{(l)}(D_1)$ and $CG^{(l)}(D_2)$ are very closely related to the corresponding results for competition hypergraphs (cf. [8]), in the present Section 3 we will find more significant differences between the graph and the hypergraph case. So it is worth mentioning that under certain conditions it is even possible to reconstruct the digraphs D_1 and D_2 themselves from $\mathcal{CH}^l(D_1 + D_2)$ or $\mathcal{CH}(D_1 * D_2)$. In general, being premised on competition graphs, this is impossible.

In the following, for a set $e = \{(i_1, j_1), \ldots, (i_k, j_k)\} \subseteq V_1 \times V_2$ we define $\pi_1(e) := \{i_1, \ldots, i_k\}$ and $\pi_2(e) := \{j_1, \ldots, j_k\}$, respectively, i.e. π_i denotes the projection of the vertices of $CG^{(l)}(D_1 \circ D_2)$ onto their ith component, for $i \in \{1, 2\}$.

3.1. The Cartesian product $D_1 \times D_2$

First of all, if $E^l_\times = E(CG^l(D_1 \times D_2)) = \emptyset$ then $A(D_1 \times D_2) = \emptyset$ and, therefore, $A_1 = \emptyset$ or $A_2 = \emptyset$. But considering only $CG^l(D_1 \times D_2)$ (or even $D_1 \times D_2$) it is impossible to detect which of the arc sets A_1 or A_2 is empty. The same holds for $E^l_1 = E(CG^l(D_1)) = \emptyset$ and $E^l_2 = E(CG^l(D_2)) = \emptyset$, respectively, since $A_i = \emptyset$ if and only if $E(CG^l(D_i)) = \emptyset$ $(i \in \{1, 2\})$.

The following example shows digraphs D_1, D_1' and D_2 with $CG^l(D_1 \times D_2) = CG^l(D_1' \times D_2)$, but $CG^l(D_1) \neq CG^l(D_1')$.

Example 9. Let $D_1 = (V_1 = \{1, 2, 3, 4\}, A_1 = \{(1, 2), (3, 2), (3, 4)\})$, $D_1' = (V_1, A_1' = A_1 \cup \{(1, 4)\})$ and $D_2 = (V_2 = \{1, 2, 3\}, A_2 = \{(1, 2), (3, 2)\})$. Then

$$\begin{aligned} E(CG^l(D_1 \times D_2)) &= \{\{(1,1),(1,3)\},\{(1,1),(3,1)\},\{(1,1),(3,3)\}, \\ &\qquad \{(1,3),(3,1)\},\{(1,3),(3,3)\},\{(3,1),(3,3)\}\} \\ &= E(CG^l(D_1' \times D_2)), \end{aligned}$$

but $E(CG^l(D_1)) = \{\{1, 3\}, \{3\}\} \neq \{\{1, 3\}\} = E(CG^l(D_1'))$.

Theorem 10. *For the Cartesian product $D_1 \times D_2$ it holds:*

(a) *If $E_\times \neq \emptyset$, then the competition graphs $CG(D_1)$ and $CG(D_2)$ can be reconstructed from $CG(D_1 \times D_2)$.*

(b) *In general, the l-competition graphs $CG^l(D_1)$ and $CG^l(D_2)$ cannot be reconstructed from $CG^l(D_1 \times D_2)$.*

(c) *If $CG^l(D_1 \times D_2)$ contains a loop, then the l-competition graphs $CG^l(D_1)$ and $CG^l(D_2)$ can be reconstructed from $CG^l(D_1 \times D_2)$.*

Proof. (a) Let $e \in E_\times$ and $(a,b) \in e$. Then there exists a vertex $(x,y) \in V_1 \times V_2$ with $(a,x) \in A_1$ and $(b,y) \in A_2$.

Suppose $\{a', a''\} \in E_1$ and $x' \in V_1$ such that $(a', x'), (a'', x') \in A_1$. Clearly, $(a', b), (a'', b) \in N_\times^-((x', y))$, $\{(a', b), (a'', b)\} \in E_\times$ and $\{a', a''\} = \pi_1(\{(a', b), (a'', b)\})$. So it follows $E_1 = \{\pi_1(e) \mid e \in E_\times \wedge |\pi_1(e)| = 2\}$ and, analogously $E_2 = \{\pi_2(e) \mid e \in E_\times \wedge \ |\pi_2(e)| = 2\}$.

(b) See Example 9.

(c) It suffices to show that all loops in $CG^l(D_1)$ and $CG^l(D_2)$ can be reconstructed. Let $\{(a,b)\} \in E_\times^l$ be a loop. Consequently, there is a vertex $(x,y) \in V_1 \times V_2$ such that $N_\times^-((x,y)) = \{(a,b)\}$ and we obtain the loops $N_1^-(x) = \{a\}$ and $N_2^-(y) = \{b\}$ in E_1^l and E_2^l, respectively.

Now let $\{a'\} \in E_1^l$ be a loop in $CG^l(D_1)$ and $x' \in V_1$ with $N_1^-(x') = \{a'\}$. Clearly, $N_\times^-((x', y)) = \{(a', b)\} \in E_\times^l$ is a loop and $\{a'\} = \pi_1(\{(a', b)\})$. Analogously, every loop $\{b'\} \in E_2^l$ can be obtained as the projection $\pi_2(e)$ of a certain loop $e \in E_\times^l$. ■

Note that there is a loop in $CG^l(D_1 \times D_2)$ if and only if both $CG^l(D_1)$ and $CG^l(D_2)$ contain a loop, which is equivalent to the fact that in D_1 as well as in D_2 there is at least one vertex with in-degree 1.

3.2. The Cartesian sum $D_1 + D_2$

Based on the definition of $D_1 + D_2 = (V_1 \times V_2, A_+)$ we get the edge set of $CG(D_1 + D_2)$ as follows.

$$\begin{aligned} E_+ &= \{\{(a,b),(a',b')\} \mid \exists (x,y) \in V_1 \times V_2 : (a,b) \neq (a',b') \\ &\qquad \wedge \{(a,b),(a',b')\} \subseteq N_+^-((x,y))\} \\ &= \{\{(a,b),(a',b')\} \mid \exists (x,y) \in V_1 \times V_2 : (a,b) \neq (a',b') \\ &\qquad \wedge ((a = x \wedge (b,y) \in A_2) \vee ((a,x) \in A_1 \wedge b = y)) \\ &\qquad \wedge ((a' = x \wedge (b',y) \in A_2) \vee ((a',x) \in A_1 \wedge b' = y))\} \end{aligned}$$

$$= \{\{(a,b),(a',b')\} \mid \exists (x,y) \in V_1 \times V_2 : (a = a' = x \wedge b \neq b' \wedge \{(b,y),(b',y)\} \subseteq A_2)$$
$$\vee\ (a = x \wedge (a',x) \in A_1 \wedge (b,y) \in A_2 \wedge b' = y)$$
$$\vee\ (a' = x \wedge (a,x) \in A_1 \wedge (b',y) \in A_2 \wedge b = y)$$
$$\vee\ (a \neq a' \wedge \{(a,x),(a',x)\} \subseteq A_1 \wedge b = b' = y)\}$$
$$= \{\{(a,b),(a',b')\} \mid (a = a' \wedge \{b,b'\} \in E_2) \vee ((a',a) \in A_1 \wedge (b,b') \in A_2)$$
$$\vee\ ((a,a') \in A_1 \wedge (b',b) \in A_2) \vee (\{a,a'\} \in E_1 \wedge b = b')\}.$$

Moreover, if the edge set of $CG^l(D_1+D_2)$ is $E^l_+ = \{\{(a,b)\} \mid (a,b) \in V_1 \times V_2\}$, then either D_1 is a directed cycle and $A_2 = \emptyset$ or D_2 is a directed cycle and $A_1 = \emptyset$. In this case it cannot be decided whether $E^l_1 = \{\{a\} \mid a \in V_1\}$ and $E^l_2 = \emptyset$ or $E^l_2 = \{\{b\} \mid b \in V_2\}$ and $E^l_1 = \emptyset$. Therefore, the existence of a loop in $CG^l(D_1+D_2)$ (as in Theorem 10(c) for the Cartesian product) is not sufficient for the reconstructibility of $CG^l(D_1)$ and $CG^l(D_2)$.

In analogy with Example 9, in our next example we give digraphs D_1, D_1' and D_2 with $CG^l(D_1+D_2) = CG^l(D_1'+D_2)$, but $CG^l(D_1) \neq CG^l(D_1')$.

Example 11. Let $D_1 = (V_1 = \{1,2,3\},\ A_1 = \{(1,2),(1,3),(2,3)\})$, $D_1' = (V_1, A_1' = \{(2,1),(1,3),(2,3)\})$ and $D_2 = (V_2 = \{1,2,3\},\ A_2 = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\})$.

Clearly, $E(CG^l(D_1)) = \{\{1,2\},\{1\}\} \neq \{\{1,2\},\{2\}\} = E(CG^l(D_1'))$.

Lets consider $E(CG^l(D_1+D_2)) = \bigcup\{EN^-_+((i,j)) \mid (i,j) \in V_1 \times V_2\}$, where $EN^-_+((i,j))$ includes all edges in $CG^l(D_1+D_2)$ generated by the predecessors of the vertex (i,j) in D_1+D_2. Denoting the corresponding edge sets in $E(CG^l(D_1'+D_2))$ by $EN'^-_+((i,j))$, we observe the following:

- In $E(CG^l(D_1+D_2))$ we have the sets
  $EN^-_+((1,1)) = \{\{(1,2),(1,3)\}\}$,
  $EN^-_+((1,2)) = \{\{(1,1),(1,3)\}\}$,
  $EN^-_+((1,3)) = \{\{(1,1),(1,2)\}\}$,
  $EN^-_+((2,1)) = \{\{(1,1),(2,2)\},\{(1,1),(2,3)\},\{(2,2),(2,3)\}\}$,
  $EN^-_+((2,2)) = \{\{(1,2),(2,1)\},\{(1,2),(2,3)\},\{(2,1),(2,3)\}\}$,
  $EN^-_+((2,3)) = \{\{(1,3),(2,1)\},\{(1,3),(2,2)\},\{(2,1),(2,2)\}\}$,
  $EN^-_+((3,1)) = EN'^-_+((3,1))$,
  $EN^-_+((3,2)) = EN'^-_+((3,2))$, and
  $EN^-_+((3,3)) = EN'^-_+((3,3))$.

The transition from D_1+D_2 to $D_1'+D_2$ does not change the last three edge sets.

- The deletion of the arc $(1,2) \in A_1$ induces that in D_1+D_2 the arcs $((1,1),(2,1))$, $((1,2),(2,2))$ and $((1,3),(2,3))$ vanish and, consequently, the edge set

$\widetilde{E} = \{\{(1,1),(2,2)\},\{(1,1),(2,3)\},\{(1,2),(2,1)\},\{(1,2),(2,3)\},$
$\{(1,3),(2,1)\},\{(1,3),(2,2)\}\}$
disappears in the l-competition graph.

• On the other hand, if we add the arc $(2,1)$ to $D_1 \setminus \{(1,2)\}$, then we obtain D_1' and the same set $\widetilde{E}$ of edges emerges in the l-competition graph $CG^l(D_1' + D_2)$. In detail, for $E(CG^l(D_1' + D_2))$ we get
$EN_+^{'-}((1,1)) = \{\{(1,2),(1,3)\},\{(1,2),(2,1)\},\{(1,3),(2,1)\}\}$,
$EN_+^{'-}((1,2)) = \{\{(1,1),(1,3)\},\{(1,1),(2,2)\},\{(1,3),(2,2)\}\}$,
$EN_+^{'-}((1,3)) = \{\{(1,1),(1,2)\},\{(1,1),(2,3)\},\{(1,2),(2,3)\}\}$,
$EN_+^{'-}((2,1)) = \{\{(2,2),(2,3)\}\}$,
$EN_+^{'-}((2,2)) = \{\{(2,1),(2,3)\}\}$, and
$EN_+^{'-}((2,3)) = \{\{(2,1),(2,2)\}\}$.
Therefore, $E(CG^l(D_1 + D_2)) = E(CG^l(D_1' + D_2))$ in spite of $E(CG^l(D_1)) \neq E(CG^l(D_1'))$.

In our next theorem we need an additional notation. Let $\{\alpha,\beta\} = \{1,2\}$, $D_1 = (V_1, A_1), D_2 = (V_2, A_2)$, $V_1 = \{1,\ldots,r_1\}, V_2 = \{1,\ldots,r_2\}$, $k \in \{1,\ldots,r_\beta\}$ and

$$R_k^\beta = \begin{cases} Z_k & \text{if } \beta = 1, \\ S_k & \text{if } \beta = 2. \end{cases}$$

Theorem 12. *For the Cartesian sum $D_1 + D_2$ it holds:*

(a) *The competition graphs $CG(D_1)$ and $CG(D_2)$ can be reconstructed from $CG(D_1 + D_2)$.*

(b) *In general, the l-competition graphs $CG^l(D_1)$ and $CG^l(D_2)$ cannot be reconstructed from $CG^l(D_1 + D_2)$.*

(c) *Let $\{\alpha,\beta\} = \{1,2\}$. If*

(c1) *$E_+^l \neq \{\{(a,b)\} \mid (a,b) \in V_1 \times V_2\}$ and for all edges $e \in E_+^l$ it holds $|\pi_\beta(e)| = 1$ or*

(c2) *there exists an edge $\widetilde{e} \in E_+^l$ with $|\pi_1(\widetilde{e})| = |\pi_2(\widetilde{e})| = 2$ and*

$$\exists j \in \{1,\ldots,r_\beta\}\, \forall e \in E_+^l : R_j^\beta \cap e \neq \emptyset \Rightarrow e \subseteq R_j^\beta, \tag{2}$$

then $CG^l(D_\alpha)$, in case (c1) *also $CG^l(D_\beta)$, can be reconstructed from $CG^l(D_1 + D_2)$.*

Proof. (a) From the above expression for E_+ we obtain
$E_1 = \{\pi_1(e) \mid e \in E_+ \wedge \mid \pi_1(e)| = 2 \wedge \mid \pi_2(e)| = 1\}$ and, analogously,
$E_2 = \{\pi_2(e) \mid e \in E_+ \wedge \mid \pi_2(e)| = 2 \wedge \mid \pi_1(e)| = 1\}$.

(b) See Example 11.

(c) First of all, it is clear that, in the case of $|V_1| = 1$ or $|V_2| = 1$, the reconstruction is trivial; the same holds for $|E^l_+| = 0$. So, let $|V_1|, |V_2| \geq 2$ and $|E^l_+| \geq 1$. Since every arc $(a, a') \in A_1$ induces $|V_2|$ arcs in $D_1 + D_2$ (analogously for arcs $(b, b') \in A_2$ and $|V_1|$), we know $|E^l_+| \geq 2$.

(c1) First, let E^l_+ contain only loops. Then either $A_1 = \emptyset = E^l_1$ or $A_2 = \emptyset = E^l_2$.

Let $\{(a, b)\} \notin E^l_+$. This implies $\{a\} \notin E^l_1$, i.e. $N^+_1(a) = \emptyset$ as well as $\{b\} \notin E^l_2$, i.e. $N^+_2(b) = \emptyset$.

If $\{\{(a, b')\} \mid b' \in V_2\} \cap E^l_+ = \emptyset$, then $E^l_2 = \emptyset$ and $E^l_1 = \{\pi_1(e) \mid e \in E^l_+\}$.

Otherwise, $\{\{(a', b)\} \mid a' \in V_1\} \cap E^l_+ = \emptyset$ and $E^l_1 = \emptyset$ as well as $E^l_2 = \{\pi_2(e) \mid e \in E^l_+\}$.

Secondly, let $\widetilde{e} \in E^l_+$ with $|\widetilde{e}| = 2$. It suffices to consider $\alpha = 1$ and $\beta = 2$. Then the case $\alpha = 2$ and $\beta = 1$ follows from $D_1 + D_2 \simeq D_2 + D_1$.

Because of $|\pi_2(e)| = 1$ for all $e \in E^l_+$, we have $\widetilde{e} = \{(a, b), (a', b)\}$ with distinct $a, a' \in V_1$ and $b \in V_2$. For the same reason, $A_2 = \emptyset = E^l_2$ and $E^l_1 = \{\pi_1(e) \mid e \in E^l_+\}$.

(c2) The existence of an edge $\widetilde{e} \in E^l_+$ with $|\pi_1(\widetilde{e})| = |\pi_2(\widetilde{e})| = 2$ implies $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. For $\widetilde{e} = \{(a, b), (a', b')\}$ this follows from $a \neq a'$, $b \neq b'$ and the definition of $D_1 + D_2$, since $a \in N^-_1(a')$ and $b' \in N^-_2(b)$ or $a' \in N^-_1(a)$ and $b \in N^-_2(b')$ must be valid.

Again, we consider only the case $\alpha = 1$ and $\beta = 2$.

Let $j \in \{1, \ldots, r_2\}$ fulfil the condition (2) and, moreover, $(i, i') \in A_1$.

Assume there is a vertex $j' \in V_2 \setminus \{j\}$ with $(j, j') \in A_2$. Then $e' := \{(i, j'), (i', j)\} \subseteq N^-_+((i', j'))$ is an edge in E^l_+ with $e' \cap S_j \neq \emptyset$ and $e' \not\subseteq S_j$, in contradiction to (2).

Consequently, $N^+_2(j) = \emptyset$.

Therefore, all edges $e \in E^l_+$ with $e \cap S_j \neq \emptyset$ (or, equivalently, $e \subseteq S_j$) are induced by the sets of predecessors $N^-_+((i', j))$ of vertices $(i', j) \in S_j$. Hence the subgraph $\langle S_j \rangle_{CG^l(D_1+D_2)} = (S_j, \{e \mid e \in E^l_+ \wedge e \subseteq S_j\})$ of $CG^l(D_1 + D_2)$ is isomorphic to $CG^l(D_1)$ and

$$E^l_1 = \{\pi_1(e) \mid e \in E^l_+ \wedge e \cap S_j \neq \emptyset\} = \{\pi_1(e) \mid e \in E^l_+ \wedge e \subseteq S_j\}.$$

■

Remark 13. The condition (2) is equivalent to $\exists j \in V_\beta : N^+_\beta(j) = \emptyset$.

In Theorem 12 we prefer (2), since (2) uses only the edge set E^l_+ of the l-competition graph of $D_1 + D_2$, which is given by the assumptions of the theorem, and not properties of the (unknown) digraph D_β.

3.3. The normal product $D_1 * D_2$

In case of the normal product we obtain an analogous result as Theorem 12 for the Cartesian sum.

Theorem 14. *For the normal product $D_1 * D_2$ it holds:*

(a) *The competition graphs $CG(D_1)$ and $CG(D_2)$ can be reconstructed from $CG(D_1 * D_2)$.*

(b) *In general, the l-competition graphs $CG^l(D_1)$ and $CG^l(D_2)$ cannot be reconstructed from $CG^l(D_1 * D_2)$.*

(c) *Let $\{\alpha, \beta\} = \{1, 2\}$. If*

(c1) $E_*^l \neq \{\{(a,b)\} \mid (a,b) \in V_1 \times V_2\}$ *and for all edges $e \in E_*^l$ it holds $|\pi_\beta(e)| = 1$ or*

(c2) *there exists an edge $\widetilde{e} \in E_*^l$ with $|\pi_1(\widetilde{e})| = |\pi_2(\widetilde{e})| = 2$ and*

$$\exists j \in \{1, \ldots, r_\beta\} \, \forall e \in E_*^l : R_j^\beta \cap e \neq \emptyset \Rightarrow e \subseteq R_j^\beta, \tag{3}$$

then $CG^l(D_\alpha)$, in case (c1) *also $CG^l(D_\beta)$, can be reconstructed from $CG^l(D_1 * D_2)$.*

Proof. (a) Because of $A(D_1 + D_2) \subseteq A(D_1 * D_2)$, we obtain $E_+ \subseteq E_*$. Moreover, we have

$$\{e \mid e \in E_+ \wedge |\pi_1(e)| = 2 \wedge |\pi_2(e)| = 1\} = \{e \mid e \in E_* \wedge |\pi_1(e)| = 2 \wedge |\pi_2(e)| = 1\}$$

and

$$\{e \mid e \in E_+ \wedge |\pi_2(e)| = 2 \wedge |\pi_1(e)| = 1\} = \{e \mid e \in E_* \wedge |\pi_2(e)| = 2 \wedge |\pi_1(e)| = 1\}.$$

Therefore, analogously to the proof of Theorem 12 (a) we get

$$E_1 = \{\pi_1(e) \mid e \in E_* \ \wedge \ |\pi_1(e)| = 2 \ \wedge \ |\pi_2(e)| = 1\}, \text{ and}$$
$$E_2 = \{\pi_2(e) \mid e \in E_* \ \wedge \ |\pi_2(e)| = 2 \ \wedge \ |\pi_1(e)| = 1\}.$$

(b) We use the digraphs D_1, D_1' and D_2 given in Example 11. Since $CG^l(D_1) \neq CG^l(D_1')$, to disprove the reconstructibility of $CG^l(D_i)$ it suffices to verify $CG^l(D_1 * D_2) = CG^l(D_1' * D_2)$. We will show that both $CG^l(D_1 * D_2)$ and $CG^l(D_1' * D_2)$ are complete graphs.

So let $(i,j), (i',j') \in V_1 \times V_2$ with $(i,j) \neq (i',j')$ and $j'' \in V_2 \setminus \{j, j'\}$. Because the digraph D_2 is complete, we have $j, j' \in N_2^-(j'')$. This implies that in $D_1 * D_2$ (as well as in $D_1' * D_2$) the vertex $(3, j'')$ is a common successor of the vertices (i,j) and (i',j'), since in D_1 (as well as in D_1') the vertex 3 is a successor of the vertices 1 and 2 (note that also for $i = 3$ and $i' = 3$ trivially $(3, j'')$ is a successor of (i,j) and (i',j'), respectively). Therefore, in both $CG^l(D_1 * D_2)$ and $CG^l(D_1' * D_2)$ any two vertices $(i,j), (i',j') \in V_1 \times V_2$ are adjacent, i.e. both l-competition graphs are complete graphs.

(c) Since nearly the whole argumentation from the proof of case (c) of Theorem 12 remains valid also for the normal product if we replace $D_1 + D_2$ by $D_1 * D_2$, we discuss only the differences between the proofs of Theorem 12(c) and Theorem 14(c).

The considerations at the beginning of the proof of case (c) of Theorem 12 are the same for $D_1 * D_2$.

(c1) Here as well as in Theorem 12(c1) either $A_1 = \emptyset$ or $A_2 = \emptyset$, therefore $D_1 + D_2 = D_1 * D_2$ and no additional thinking is necessary.

(c2) Analogously to the proof of Theorem 12(c2), we obtain $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$ from $\widetilde{e} = \{(a,b),(a',b')\}$, where $a \neq a'$, $b \neq b'$, but now using the definition of $D_1 * D_2$, since

$$\exists\, x \in V_1 : \{a,a'\} \subseteq N_1^-(x) \vee a \in N_1^-(a') \vee a' \in N_1^-(a) \ \text{ and}$$
$$\exists\, y \in V_2 : \{b,b'\} \subseteq N_2^-(y) \vee b \in N_2^-(b') \vee b' \in N_2^-(b)$$

must be valid.

Replacing (2) by (3) and "+" by "$*$", the rest of the proof can be taken over word by word and we obtain

$$E_1^l = \{\pi_1(e) \mid e \in E_*^l \wedge e \cap S_j \neq \emptyset\} = \{\pi_1(e) \mid e \in E_+^l \wedge e \subseteq S_j\}. \qquad \blacksquare$$

Now we consider an example, where (3) does not hold and the reconstruction of E_1^l described at the end of the proof of Theorem 14 fails.

Example 15. Let $D_1 = (V_1 = \{1,2\},\ A_1 = \{(1,2),(2,1)\})$ and $D_2 = (V_2 = \{1,2\}$, $A_2 = \{(1,2)\})$. Then

$$\begin{aligned} D_1 * D_2 = (V_1 \times V_2, A_* = \{&((1,1),(1,2)),((1,1),(2,1)),((1,1),(2,2)),\\ &((1,2),(2,2)),((2,1),(1,1)),((2,1),(1,2)),((2,1),(2,2)),((2,2),(1,2))\}),\end{aligned}$$

$$\begin{aligned} E_*^l = \{&\{(1,1)\},\{(2,1)\},\{(1,1),(1,2)\},\{(1,1),(2,1)\},\{(1,1),(2,2)\},\\ &\{(1,2),(2,1)\},\{(2,1),(2,2)\}\},\end{aligned}$$

but

$$\begin{aligned} \{\pi_1(e) \mid e \in E_*^l \wedge e \cap S_1 \neq \emptyset\} &= \{\{1\},\{2\},\{1,2\}\}\\ &= \{\pi_1(e) \mid e \in E_*^l \wedge e \subseteq S_1\} \neq \{\{1\},\{2\}\} = E_1^l.\end{aligned}$$

Moreover,

$$\begin{aligned} \{\pi_1(e) \mid e \in E_*^l \wedge e \cap S_2 \neq \emptyset\} &= \{\{1\},\{2\},\{1,2\}\}\\ &\neq \emptyset = \{\pi_1(e) \mid e \in E_*^l \wedge e \subseteq S_2\} \neq E_1^l.\end{aligned}$$

3.4. The lexicographic product $D_1 \cdot D_2$

Note that if $(i,i') \in A_1$, then it holds $((i,j),(i',j')) \in A.$ (for all $j,j' \in V_2$) and, consequently, $\mathcal{P}^2(Z_i) := \{\{(i,j),(i,j')\} \mid j,j' \in V_2 \wedge j \neq j'\} \subseteq E_.$.

(Conversely, this means that $\mathcal{P}^2(Z_i) \setminus E_\cdot \neq \emptyset$ includes $N_1^+(i) = \emptyset$.)

Hence, the existence of a vertex $i \in V_1$ having the outdegree 0 in D_1 is necessary to get some information on E_2^l from $CG^l(D_1 \cdot D_2)$; this will be used in (a) of our next theorem.

Concerning the reconstruction of $CG(D_1)$ we discuss (3) from Theorem 14. Replacing E_*^l by $E_\cdot^l$ in (3), for $\beta = 2$ (i.e. R_j^β is the column S_j) the condition (3) makes no sense for the lexicographic product, because in the case $(i, i'') \in A_1$ the vertex $(i'', j) \in V_1 \times V_2$ has predecessors (i, j') in all columns $S_{j'}$ $(j' \in \{1, \ldots, r_2\})$. Hence, for $j' \in \{1, \ldots, r_2\} \setminus \{j\}$, we get $\{(i, j), (i, j')\} \in E_\cdot$, i.e. there are a lot of edges $e \in E_\cdot$ fulfilling $e \cap S_j \neq \emptyset$ and $e \not\subseteq S_j$.

Therefore, for the lexicographic product we have to make use of another condition ((5), see below), which implies that there are no "horizontal arcs" $((i, j), (i, j')) \in A_\cdot$ in $D_1 \cdot D_2$ (for all $j' \in \{1, \ldots, r_2\} \setminus \{j\}$).

Whereas (5) will be proven to allow the reconstruction of $CG(D_1)$ from $CG(D_1 \cdot D_2)$, our next example shows that, in general, in the case of the l-competition graph the corresponding reconstruction is impossible.

Example 16. Let $D_1 = (V_1 = \{1, 2, 3\}, A_1 = \{(1, 2), (1, 3), (3, 2)\})$, $D_1' = (V_1, A_1' = \{(1, 2), (3, 2)\})$ and $D_2 = (V_2 = \{1, 2\}, A_2 = \emptyset)$. Obviously, $A(D_1' \cdot D_2) \subseteq A(D_1 \cdot D_2)$ and $E(CG^l(D_1' \cdot D_2)) \subseteq E(CG^l(D_1 \cdot D_2))$. On the other hand, the only edge $e \in E(CG^l(D_1 \cdot D_2))$ which is induced by the arc $(1, 3) \in A_1 \setminus A_1'$ is $e = N^-_{CG^l(D_1 \cdot D_2)}((3, 1)) = N^-_{CG^l(D_1 \cdot D_2)}((3, 2)) = \{(1, 1), (1, 2)\}$. Because of $e \subseteq N^-_{CG^l(D_1' \cdot D_2)}((2, 1))$, it follows $e \in E(CG^l(D_1' \cdot D_2))$ and, therefore, $E(CG^l(D_1' \cdot D_2)) = E(CG^l(D_1 \cdot D_2))$.

Consequently, $E(CG^l(D_1)) = \{\{1\}, \{1, 3\}\} \neq \{\{1, 3\}\} = E(CG^l(D_1'))$ cannot be reconstructed from $CG^l(D_1 \cdot D_2) = CG^l(D_1' \cdot D_2)$.

Theorem 17. *For the lexicographic product $D_1 \cdot D_2$ it holds:*

(a) *If*
(4) $\quad \exists i \in \{1, \ldots, r_1\} : \mathcal{P}^2(Z_i) \setminus E_\cdot \neq \emptyset$,
then $CG(D_2)$ can be reconstructed from $CG(D_1 \cdot D_2)$.

(b) *If D_1 contains an isolated vertex $i \in V_1$, then $CG^l(D_2)$ can be reconstructed from $CG^l(D_1 \cdot D_2)$.*

(c) *If*
(5) $\quad \exists j \in \{1, \ldots, r_2\} : N_2^+(j) = \emptyset$,
then $CG(D_1)$ can be reconstructed from $CG(D_1 \cdot D_2)$.

(d) *In general, the l-competition graph $CG^l(D_1)$ cannot be reconstructed from $CG^l(D_1 \cdot D_2)$.*

Proof. (a) Let $i \in \{1, \ldots, r_1\}$ with $\mathcal{P}^2(Z_i) \setminus E_\cdot \neq \emptyset$, i.e. $N_1^+(i) = \emptyset$. Further let $e \in \mathcal{P}^2(Z_i) \cap E_\cdot$ be an edge in $CG(D_1 \cdot D_2)$ containing only vertices of the row Z_i.

Since $i \in V_1$ has no successor in D_1, there is a vertex $(i, j'') \in Z_i$ with $e \subseteq N_{\cdot}^{-}((i, j''))$. Therefore, for $e = \{(i, j), (i, j')\}$ it follows $\{j, j'\} \subseteq N_2^{-}(j'')$, i.e. $\pi_2(e) \in E_2$. Consequently, we obtain $E_2 = \{\pi_2(e) \mid e \in E_{\cdot} \wedge e \subseteq Z_i\}$.

(Note that in the case of the l-competition graphs $CG^l(D_1 \cdot D_2)$ and $CG^l(D_2)$ the following problem occurs: if $i' \in V_1$ is a predecessor of the vertex i and $N_2^{-}(j'') = \{j\}$ is a loop in $CG^l(D_2)$, then $\{(i, j), (i', j'')\} \subseteq N_{\cdot}^{-}((i, j''))$ and therefore there is no edge $e \in E_{\cdot}^l$ with $e \subseteq Z_i$ and $\pi_2(e) = \{j\}$. In other words, there is no loop $e = \{(i, j)\}$ in $E_{\cdot}^l$.)

(b) Now let $i \in V_1$ be an isolated vertex in D_1 (i.e. $N_1^{-}(i) = N_1^{+}(i) = \emptyset$).

Further let $j, j'' \in V_2$ such that $N_2^{-}(j'') = \{j\}$ is a loop in $CG^l(D_2)$. Since $N_1^{-}(i) = \emptyset$, $e = \{(i, j)\} = N_{\cdot}^{-}((i, j'')) \in E_{\cdot}^l$ is a loop in $CG^l(D_1 \cdot D_2)$ with $e \subseteq Z_i$ and $\pi_2(e) = \{j\}$. This way, all loops in $CG^l(D_2)$ can be reconstructed. Because of $N_1^{+}(i) = \emptyset$, the reconstruction of the edges $e \in E_2$ can be done analogously to part (a) of the proof and we have $E_2^l = \{\pi_2(e) \mid e \in E_{\cdot}^l \wedge e \subseteq Z_i\}$.

(c) Let j fulfil the condition in (5) and consider an edge $\{(i, j), (i', j)\} \in E_{\cdot}$. Then, in $D_1 \cdot D_2$ for all successors (i'', j'') of the vertices $(i, j), (i', j)$ the vertex i'' must be a common successor of the vertices i and i', since $j'' \notin N_2^{+}(j) = \emptyset$.

Consequently, $\{i, i'\} \subseteq N_1^{-}(i'')$ and $\{i, i'\} \in E_1$. On the other hand, for all $\{i, i'\} \in E_1$, even in the subgraph $\langle S_j \rangle_{CG(D_1 \cdot D_2)} = (S_j, \{e \mid e \in E_{\cdot} \wedge e \subseteq S_j\})$ of $CG(D_1 \cdot D_2)$, we find the edge $e = \{(i, j), (i', j)\}$ with $\pi_1(e) = \{i, i'\}$. Therefore, we get $E_1 = \{\pi_1(e) \mid e \in E_{\cdot} \wedge e \subseteq S_j\}$.

(d) In Example 16, the digraph D_2 fulfills condition (5), but for the l-competition hypergraphs we have $CG^l(D_1 \cdot D_2) = CG^l(D_1' \cdot D_2)$ whereas $CG^l(D_1) \neq CG^l(D_1')$ holds. ∎

3.5. The disjunction $D_1 \vee D_2$

In analogy with Theorem 17, (c) and (d), we obtain a corresponding result for the disjunction.

Theorem 18. *Let $\{\alpha, \beta\} = \{1, 2\}$. For the disjunction $D_1 \vee D_2$ it holds:*

(a) *If*
(6) $\exists j \in \{1, \ldots, r_\beta\} : N_\beta^{+}(j) = \emptyset,$
then $CG(D_\alpha)$ can be reconstructed from $CG(D_1 \vee D_2)$.

(b) *In general, the l-competition graph $CG^l(D_\alpha)$ cannot be reconstructed from $CG^l(D_1 \vee D_2)$.*

Proof. Owing to $D_1 \vee D_2 \simeq D_2 \vee D_1$ it is sufficient to consider the case $\alpha = 1$ and $\beta = 2$.

Then, replacing $D_1 \cdot D_2$ by $D_1 \vee D_2$, the argumentation from the proof of Theorem 17 (parts (c) and (d)) as well as the counterexample (Example 16) can be used without any changes. ∎

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