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ON SPECTRA OF VARIANTS OF THE CORONA OF TWO GRAPHS AND SOME NEW EQUIENERGETIC GRAPHS

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Abstract

Let G and H be two graphs. The join $G \vee H$ is the graph obtained by joining every vertex of G with every vertex of H. The corona $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H. The neighborhood corona $G \star H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the neighbors of the i-th vertex of G to every vertex in the i-th copy of H. The edge corona $G \diamond H$ is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each terminal vertex of i-th edge of G to every vertex in the i-th copy of H. Let G_1, G_2, G_3 and G_4 be regular graphs with disjoint vertex sets. In this paper we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_1 \diamond G_3)$, $(G_1 \vee G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$. As an application, we show that there exist some new pairs of equienergetic graphs on n vertices for all $n \geq 11$.

Keywords: spectrum, corona, neighbourhood corona, edge corona, energy of a graph, equienergetic graphs.

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1. Introduction

Throughout this paper we consider only undirected simple graphs (i.e., graphs with no loops and multiple edges). Let G be a graph on n vertices. The eigenvalues of the adjacency matrix of G, denoted by $\lambda_i(G)$, $i = 1, 2, \ldots, n$, are the eigenvalues of the graph G and $\sigma(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G))$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ is the adjacency spectrum of G [8]. The energy $E(G)$ is the sum of all absolute values of eigenvalues of G. The concept of energy of a graph was introduced by Gutman [12] with an application to chemistry (Huckel molecular orbital approximation for the total π-electron energy [14]). The energy and various forms of energy of a graph G has been extensively studied by many mathematicians and some of their works can be found in [1, 2, 3, 5, 13, 15, 19, 21, 28, 27] and references therein. Two graphs G_1 and G_2 of the same order are said to be equienergetic if $E(G_1) = E(G_2)$. Graphs of the same order are cospectral if they have the same spectrum. Thus, two cospectral graphs are obviously equienergetic. For connected graphs, there are no equienergetic graphs of order $n \leq 5$. In [18] Indulal and Vijayakumar have constructed a pair of equienergetic graphs on n vertices for $n = 6, 14, 18$ and for all $n \geq 20$. Later Liu *et al.* [22] and Ramane, Walikar [26] have independently proved that there exists a pair of equienergetic graphs on n vertices for all $n \geq 9$. Studies on equienergetic graphs can be found in [6, 11, 18, 20, 22, 25, 26, 29] and references therein.

The corona of two graphs was first introduced by Frucht and Harary in [10]. Barik *et al.* [4] provided a complete description of the spectrum of corona $G_1 \circ G_2$ using the spectrum of G_1 and G_2. More about the spectrum of corona can be found in [4, 7, 10, 24]. The neighborhood corona and edge corona was introduced in [17] and in [16], respectively. Complete description of the spectrum of neighborhood corona and edge corona of two graphs are given in [17, 23] and [16], respectively.

Motivated by the above works, in this paper we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_1 \diamond G_3)$, $(G_1 \vee G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$, when G_1, G_2, G_3 and G_4 are regular graphs. Here the graphs G_1, G_2, G_3 and G_4 have disjoint vertex sets. As an application of our results we construct some new pairs of equienergetic graphs on n vertices for all $n \geq 11$. Our method of construction and proofs are entirely different from the methods given in [18, 22, 26].

2. Preliminaries

In this section, we give some definitions and lemmas which are useful to prove our main results.

Definition [10]**.** Let G_1 and G_2 be two graphs on n and m vertices, respectively. The corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2.

Definition [16]**.** Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The edge corona $G_1 \diamond G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining two end vertices of the i-th edge of G_1 to every vertex in the i-th copy of G_2.

Definition [17]**.** Let G_1 and G_2 be two graphs on n and m vertices, respectively. The neighborhood corona $G_1 \star G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining each neighbor of i-th vertex of G_1 to every vertex in the i-th copy of G_2.

Definition [8]**.** Let $A = (a_{ij})$ be an $n \times m$ matrix, $B = (b_{ij})$ be a $p \times q$ matrix. Then the Kronecker product $A \otimes B$ of A and B is the np by mq matrix obtained by replacing each entry a_{ij} of A by $a_{ij}B$.

Lemma 1 [8]**.** *If M, N, P, Q are matrices with M being a non-singular matrix, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|. \tag{1}$$

Lemma 2 [26]**.** *Let N_1 and N_2 be two graphs as shown in Figure* 1*. Then the line graph $L(N_1)$ of N_1 and the line graph $L(N_2)$ of N_2 are non cospectral and equienergetic.*

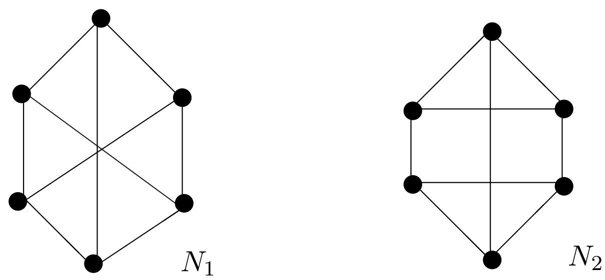
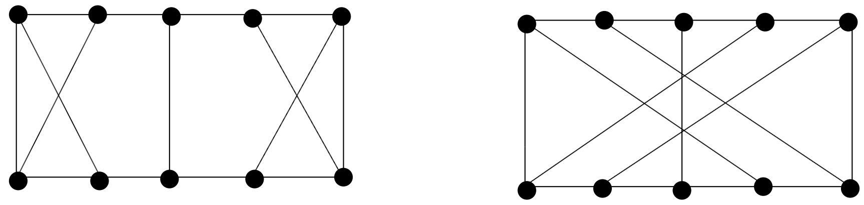


Figure 1

Lemma 3 [8]**.** *The following cubic regular graphs with ten vertices are equienergetic.*



T_1 T_2

Figure 2

3. Spectra of $(G_1 \vee G_2) \cup (G_1 \star G_3)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$

In this section, we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \star G_3)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$, where G_1, G_2, G_3 and G_4 are regular graphs on n, m, l and p vertices, respectively.

Theorem 4. *Let G_i be r_i-regular graphs $(i = 1, 2, 3)$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$ and $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ are the adjacency spectrum of G_1, G_2 and G_3, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_1 \star G_3)$ is*

$$\sigma(G) = \begin{pmatrix} \gamma_i & \mu_j & \left(\lambda_k + r_3 \pm \sqrt{4\,l\lambda_k{}^2 + (\lambda_k - r_3)^2}\right)/2 & x_t \\ n & 1 & 1 & 1 \end{pmatrix},$$

where $i = 2$ to l, $j = 2$ to m, $k = 2$ to n, $t = 1, 2, 3$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial $(x - r_2)\left((x - r_1)(x - r_3) - lr_1^2\right) - nm\,(x - r_3)$.

Proof. With suitable labelling of the vertices of G, the adjacency matrix $A(G)$ can be formulated as follows:

$$A(G) = \begin{pmatrix} I_n \otimes A(G_3) & 0 & A(G_1) \otimes e \\ 0 & A(G_2) & J \\ A(G_1) \otimes e^T & J^T & A(G_1) \end{pmatrix},$$

where $e^T = \overbrace{(1, 1, \ldots, 1)}^{l\ times}$, I_n is the identity matrix of order n and J is the $m \times n$ matrix with all its entries are 1.

Since $A(G_3)$ is a real symmetric matrix, $A(G_3)$ is orthogonally diagonalizable. Let $A(G_3) = PDP^T$, where $PP^T = I_l$ and $D = diag(\gamma_1, \ldots, \gamma_l)$. Then

$$A(G) = \begin{pmatrix} I_n \otimes PDP^T & 0 & A(G_1) \otimes e \\ 0 & A(G_2) & J \\ A(G_1) \otimes e^T & J^T & A(G_1) \end{pmatrix}$$

$$= \begin{pmatrix} I_n \otimes P & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes P^T e \\ 0 & A(G_2) & J \\ A(G_1) \otimes e^T P & J^T & A(G_1) \end{pmatrix} \begin{pmatrix} I_n \otimes P^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} I_n \otimes P & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes \sqrt{l} e_1 \\ 0 & A(G_2) & J \\ A(G_1) \otimes \sqrt{l} e_1^T & J^T & A(G_1) \end{pmatrix} \begin{pmatrix} I_n \otimes P^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $e_1^T = (1, 0, \ldots, 0)$.

$$\text{Let } B = \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes \sqrt{l} e_1 \\ 0 & A(G_2) & J \\ A(G_1) \otimes \sqrt{l} e_1^T & J^T & A(G_1) \end{pmatrix}.$$

Then by the above equation we have

$$|xI - A(G)| = |xI - B|. \tag{2}$$

Expanding $|xI - B|$ by Laplace's method [9] along $(li + 2), (li + 3), \ldots, (li + l)^{th}$ columns, for $i = 0, 1, \ldots, n - 1$, we see that the only non zero $(l - 1)n \times (l - 1)n$ minor is

$$M = |I_n \otimes diag(x - \gamma_2, \ldots, x - \gamma_l)|. \tag{3}$$

The complementary minor of M is

$$M_1 = \begin{vmatrix} (x - r_3)I_n & 0 & -\sqrt{l} A(G_1) \\ 0 & xI_m - A(G_2) & -J \\ -\sqrt{l} A(G_1) & -J^T & xI_n - A(G_1) \end{vmatrix}.$$

Again as $A(G_1)$ and $A(G_2)$ are orthogonally diagonalizable, one can easily see that the M_1 is the same as

$$(4)\qquad M_1' = \begin{vmatrix} (x-r_3)I_n & 0 & -\sqrt{l}\; diag(\lambda_1,\ldots,\lambda_n) \\ 0 & diag(x-\mu_1,\ldots,x-\mu_m) & -\sqrt{mn}J' \\ -\sqrt{l}\; diag(\lambda_1,\ldots,\lambda_n) & -\sqrt{mn}J'^T & diag(x-\lambda_1,\ldots,x-\lambda_n) \end{vmatrix},$$

where J' is the matrix obtained by replacing every entries of J except the first diagonal entry by 0. Now by (1), we have

$$M_1' = (x-r_3)^n$$

$$(5)\quad \times \begin{vmatrix} diag(x-r_2, x-\mu_2,\ldots,x-\mu_m) & -\sqrt{mn}J' \\ -\sqrt{mn}J'^T & diag(x-\lambda_1-l\lambda_1^2/(x-r_3),\ldots,x-\lambda_n-l\lambda_n^2/(x-r_3)) \end{vmatrix}$$

Applying Laplace method along $2,\ldots,m,m+2,\ldots,m+n$ columns in the above determinant we see that the only non zero $m+n-2\times m+n-2$ minor is $diag(x-\mu_2,\ldots,x-\mu_m,x-\lambda_2-l\lambda_2^2/(x-r_3),\ldots,x-\lambda_n-l\lambda_n^2/(x-r_3))$ and the complementary minor is

$$M_1 = \begin{vmatrix} x-\mu_2 & -\sqrt{mn} \\ -\sqrt{mn} & x-\lambda_2-l\lambda_2^2/(x-r_3) \end{vmatrix}.$$

And so by (2), (3), (4), (5) and from the above equation the result follows. ■

Corollary 5. *Let G_i be r_i-regular graphs $(i=1,2)$. Then*

$$E(G_1\vee G_2\cup G_1\star lK_1) = \sqrt{4l+1}E(G_1)+E(G_2)-r_1(\sqrt{4l+1}-1)-2x_0,$$

where x_0 is the negative root of the polynomial $(x-r_2)\left((x-r_1)\,x-lr_1^2\right)-nmx$.

Remark 6. Corollary 5 is a generalization of Theorem 1 in [18]. In fact putting $r_1=r$, $n=p$, $r_2=0$, $m=k$, $r_3=0$, $l=1$ in Corollary 5, we obtain Theorem 1 due to Indulal and Vijayakumar [18].

Corollary 7. *Let G_i $(i=1,2)$ be equienergetic regular graphs of the same degree and H_i $(i=1,2)$ be equienergetic regular graphs of the same degree. Then*

$$E(G_1\vee H_1\cup G_1\star lK_1) = E(G_2\vee H_2\cup G_2\star lK_1).$$

Now we construct some new pairs of equienergetic graphs using Corollary 7.

Theorem 8. *There exists a pair of equienergetic graphs on n vertices for all $n\geq 18$.*

Proof. From Lemma 2 we have the line graphs $L(N_1)$ and $L(N_2)$ are equienergetic. Now by Corollary 7 it is clear that the graphs $(L(N_1) \vee K_m) \cup (L(N_1) \star K_1)$ and $(L(N_2) \vee K_m) \cup (L(N_2) \star K_1)$, both of order $18 + m$ $(m = 0, 1, \ldots)$, are equienergetic. This completes the proof of the theorem. ∎

Theorem 9. *There exists a pair of equienergetic graphs on n vertices for all $n \geq 20$.*

Proof. From Lemma 3 and Corollary 7 it is clear that the graphs $(T_1 \vee K_m) \cup (T_1 \star K_1)$ and $(T_2 \vee K_m) \cup (T_2 \star K_1)$, both of order $20 + m$ $(m = 0, 1, \ldots)$, are equienergetic. ∎

Theorem 10. *There exists a pair of equienergetic graphs on n vertices for all $n \geq 13$.*

Proof. *Case* 1. $n = 9 + 2m$ $(m = 2, 3, \ldots)$. For $n = 9 + 2m$ $(m = 2, 3, \ldots)$, the graphs $(K_m \vee L(N_1)) \cup (K_m \star K_1)$ and $(K_m \vee L(N_2)) \cup (K_m \star K_1)$ both are of order $9 + 2m$ $(m = 2, 3, \ldots)$. Now, Corollary 7 implies that these two graphs are equienergetic.

Case 2. $n = 10 + 2m$ $(m = 2, 3, \ldots)$. For $n = 10 + 2m$ $(m = 2, 3, \ldots)$, the graphs $(K_m \vee T_1) \cup (K_m \star K_1)$ and $(K_m \vee T_2) \cup (K_m \star K_1)$ both are of order $10 + 2m$ $(m = 2, 3, \ldots)$. Now, Corollary 7 implies that these two graphs are equienergetic. ∎

As the proof of the following theorem is similar to that of Theorem 4, we omit the details.

Theorem 11. *Let G_i be r_i-regular graphs $(i = 1, 2, 3, 4)$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$ is*

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & \left(\lambda_k + r_4 \pm \sqrt{4\,p{\lambda_k}^2 + (\lambda_k - r_4)^2}\right)/2 & \left(\mu_s + r_3 \pm \sqrt{4\,l{\mu_s}^2 + (\mu_s - r_3)^2}\right)/2 & x_t \\ m & n & 1 & 1 & 1 \end{pmatrix},$$

where $i = 2$ to l, $j = 2$ to p, $k = 2$ to n, $s = 2$ to m, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial

$$\left((x - r_1)(x - r_4) - p{r_1}^2\right)\left((x - r_2)(x - r_3) - l{r_2}^2\right) - nm\,(x - r_3)(x - r_4).$$

Corollary 12. *Let G_i be r_i-regular graphs $(i = 1, 2)$. Then*

$$E(G_1 \vee G_2 \cup G_2 \star lK_1 \cup G_1 \star pK_1) = \sqrt{4p+1}E(G_1) + \sqrt{4l+1}E(G_2) - r_2(\sqrt{4l+1} - 1) - r_1(\sqrt{4p+1} - 1) - 2x_0 - 2x_1,$$

where x_0 and x_1 are the negative roots of the polynomial
$x^4 - (r_1 + r_2)\, x^3 + \left(-r1^2 p - l r_2{}^2 + r_1 r_2 - mn\right) x^2 + \left(r_1{}^2 r_2 p + r_1 r_2{}^2 l\right) x + r_1{}^2 r_2{}^2 lp.$

Corollary 13. *Let G_1, G_2 be equienergetic regular graphs of the same degree and H_1, H_2 be equienergetic regular graphs of the same degree. Then*

$$E(G_1 \vee H_1 \cup H_1 \star lK_1 \cup G_1 \star pK_1) = E(G_2 \vee H_2 \cup H_2 \star lK_1 \cup G_2 \star pK_1).$$

4. Spectra of $(G_1 \vee G_2) \cup (G_1 \circ G_3)$ and $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$

In this section, we simply state some theorems (as the proofs are quite analogous to the proof of Theorem 4) which gives the spectrum of $(G_1 \vee G_2) \cup (G_1 \circ G_3)$ and $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, where G_1, G_2, G_3 and G_4 are regular graphs on n, m, l and p vertices, respectively.

Theorem 14. *Let G_i be r_i-regular graphs $(i = 1, 2, 3)$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$ and $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ are the adjacency spectrum of G_1, G_2 and G_3, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_1 \circ G_3)$ is*

$$\sigma(G) = \begin{pmatrix} \gamma_i & \mu_j & \left(\lambda_k + r_3 \pm \sqrt{4\,l + (\lambda_k - r_3)^2}\right)/2 & x_t \\ n & 1 & 1 & 1 \end{pmatrix},$$

where $i = 2$ to l, $j = 2$ to m, $k = 2$ to n, $t = 1, 2, 3$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial $(x - r_2)\left((x - r_1)(x - r_3) - l\right) - nm\,(x - r_3)$.

Theorem 15. *Let G be an r-regular graph of order m. Then*

$$E(K_n \vee G \cup K_n \circ lK_1) = E(G) + (n-1)\sqrt{4l+1} - 2x_0 + n - 1,$$

where x_0 is the negative root of the polynomial $(x - r)\left(x\,(x - (n-1)) - l\right) - nmx$.

Corollary 16. *Let G and H be equienergetic regular graphs of the same degree. Then*

$$E(K_n \vee G \cup K_n \circ lK_1) = E(K_n \vee H \cup K_n \circ lK_1).$$

Theorem 17. *Let G be an r-regular graph of order m. Then*

$$E(nK_1 \vee G \cup nK_1 \circ lK_1) = E(G) + (n-1)\sqrt{4l} - 2x_0,$$

where x_0 is the negative root of the polynomial $(x-r)\left(x^2-l\right) - nmx$.

Corollary 18. *Let G and H be equienergetic regular graphs of the same degree. Then*

$$E(nK_1 \vee G \cup nK_1 \circ lK_1) = E(nK_1 \vee H \cup nK_1 \circ lK_1).$$

Now we construct some new pairs of equienergetic graphs using Corollary 16.

Theorem 19. *There exists a pair of equienergetic graphs on n vertices for all $n \geq 11$.*

Proof. *Case* 1. $n = 9 + 2m$ $(m = 1, 2, \ldots)$. For $n = 9 + 2m$ $(m = 1, 2, \ldots)$, the graphs $(K_m \vee L(N_1)) \cup (K_m \circ K_1)$ and $(K_m \vee L(N_2)) \cup (K_m \circ K_1)$ both are of order $9 + 2m$ $(m = 1, 2, \ldots)$. Now, Corollary 16 implies that these two graphs are equienergetic.

Case 2. $n = 10 + 2m$ $(m = 1, 2, \ldots)$. For $n = 10 + 2m$ $(m = 1, 2, \ldots)$, the graphs $(K_m \vee T_1) \cup (K_m \circ K_1)$ and $(K_m \vee T_2) \cup (K_m \circ K_1)$ both are of order $10 + 2m$ $(m = 1, 2, \ldots)$. Now, Corollary 16 implies that these two graphs are equienergetic. ■

Theorem 20. *Let G_i be r_i-regular graphs $(i = 1, 2, 3, 4)$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ is*

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & \left(\lambda_k + r_4 \pm \sqrt{4\,p + (\lambda_k - r_4)^2}\right)/2 & \left(\mu_s + r_3 \pm \sqrt{4\,l + (\mu_s - r_3)^2}\right)/2 & x_t \\ m & n & 1 & 1 & 1 \end{pmatrix},$$

where $i = 2$ to l, $j = 2$ to p, $k = 2$ to n, $s = 2$ to m, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial

$$((x - r_1)(x - r_4) - p)((x - r_2)(x - r_3) - l) - nm(x - r_3)(x - r_4).$$

5. SPECTRA OF $(G_1 \vee G_2) \cup (G_1 \diamond G_3)$ AND $(G_1 \vee G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)$

In Theorems 21 and 25 of this section, we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \diamond G_3)$ and $(G_1 \vee G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)$, where G_1, G_2, G_3 and G_4 are regular graphs on n, m, l and p vertices, respectively. Proofs of these theorems are not given as they are similar to the proof of Theorem 4.

Theorem 21. *Let G_i be r_i-regular graphs $(i = 1, 2, 3)$ and $r_1 \geq 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$ and $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ are the adjacency spectrum of G_1, G_2 and G_3, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_1 \diamond G_3)$ is*

$$\sigma(G) = \begin{pmatrix} \gamma_i & r_3 & \mu_j & \left(\lambda_k + r_3 \pm \sqrt{4\,l(\lambda_k + r_1) + (\lambda_k - r_3)^2}\right)/2 & x_t \\ r_1 n/2 & (r_1 - 2)n/2 & 1 & 1 & \end{pmatrix},$$

where $i = 2$ to l, $j = 2$ to m, $k = 2$ to n, $t = 1, 2, 3$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial $(x - r_2)\left((x - r_1)(x - r_3) - 2\,lr_1\right) - nm\,(x - r_3)$.

Theorem 22. *Let G be an r-regular graph of order m. Then*

$$E(K_n \vee G \cup K_n \diamond lK_1) = E(G) + (n-1)(\sqrt{4l + n - 2} + 1) - 2x_0,$$

where x_0 is the negative root of the polynomial

$$x^3 - (n - 1 + r)\,x^2 + ((n-1)\,r - 2\,(n-1)\,l - mn)\,x + 2\,(n-1)\,rl.$$

Corollary 23. *Let G and H be equienergetic regular graphs of the same degree. Then*

$$E(K_n \vee G \cup K_n \diamond lK_1) = E(K_n \vee H \cup K_n \diamond lK_1).$$

Now we construct some new pairs of equienergetic graphs using Corollary 23.

Theorem 24. *There exists a pair of equienergetic graphs on n vertices for all $n \geq 15$.*

Proof. *Case* 1. Let $n = 9 + 2m$ $(m = 3, 4, \ldots)$ and C_m be the cycle of length m. Then, by Corollary 23 and Lemma 2 the graphs $(C_m \vee L(N_1)) \cup (C_m \diamond K_1)$ and $(C_m \vee L(N_2)) \cup (C_m \diamond K_1)$, both of order $9 + 2m$ $(m = 3, 4, \ldots)$, are equienergetic.

Case 2. $n = 10 + 2m$ $(m = 3, 4, \ldots)$. For $n = 10 + 2m$ $(m = 3, 4, \ldots)$, the graphs $(C_m \vee T_1) \cup (C_m \diamond K_1)$ and $(C_m \vee T_2) \cup (C_m \diamond K_1)$ both are of order $9 + 2m$ $(m = 3, 4, \ldots)$. Now, Corollary 23 and Lemma 3 implies that these two graphs are equienergetic. ∎

Theorem 25. *Let G_i be r_i-regular graphs $(i = 1, 2, 3, 4)$, $r_1 \geq 2$ and $r_2 \geq 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)$ is*

$$\sigma(G) = \left(\begin{array}{cccc} \gamma_i & r_3 & \eta_j & r_4 \\ r_2 n/2 & (r_2-2)n/2 & r_1 n/2 & (r_1-2)n/2 \end{array} \right.$$

$$\left. \begin{array}{ccc} \left(\lambda_k + r_4 \pm \sqrt{4\,p(\lambda_k + r_1) + (\lambda_k - r_4)^2}\right)/2 & \left(\mu_s + r_3 \pm \sqrt{4\,l(\mu_s + r_2) + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 & 1 \end{array} \right),$$

where $i = 2$ to l, $j = 2$ to p, $k = 2$ to n, $s = 2$ to m, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial

$$((x - r_1)(x - r_4) - 2\,pr_1)((x - r_2)(x - r_3) - 2\,r_2 l) - nm\,(x - r_3)(x - r_4)\,.$$

6. SPECTRA OF $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ AND $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$

In this section we just give the spectrum of $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$, where G_1, G_2, G_3 and G_4 are regular graphs on n, m, l and p vertices, respectively. Proofs of Theorems 26–28 are similar to the proof of Theorem 4.

Theorem 26. *Let G_i be r_i-regular graphs $(i = 1, 2, 3, 4)$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_4)$ is*

$$\sigma(G) = \left(\begin{array}{ccc} \gamma_i & \eta_j & \left(\lambda_k + r_4 \pm \sqrt{4\,p{\lambda_k}^2 + (\lambda_k - r_4)^2}\right)/2 \\ n & m & 1 \end{array} \right.$$

$$\left. \begin{array}{cc} \left(\mu_s + r_3 \pm \sqrt{4\,l + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 \end{array} \right),$$

where $i = 2$ to l, $j = 2$ to p, $k = 2$ to n, $s = 2$ to m, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial

$$((x - r_2)(x - r_3) - l)\left((x - r_1)(x - r_4) - p{r_1}^2\right) - nm(x - r_3)(x - r_4).$$

Theorem 27. *Let G_i be r_i-regular graphs $(i = 1, 2, 3, 4)$ and $r_1 \geq 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ is*

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & r_4 & \left(\lambda_k + r_4 \pm \sqrt{4\,p(\lambda_k + r_1) + (\lambda_k - r_4)^2}\right)/2 \\ n & m & (r_1 - 2)n/2 & 1 \end{pmatrix.$$
$$\begin{matrix} \left(\mu_s + r_3 \pm \sqrt{4\,l + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 \end{matrix}\Bigg),$$

where $i = 2$ to l, $j = 2$ to p, $k = 2$ to n, $s = 2$ to m, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial

$$((x - r_1)\,(x - r_4) - 2\,pr_1)\,((x - r_2)\,(x - r_3) - l) - nm(x - r_3)(x - r_4).$$

Theorem 28. *Let G_i be r_i-regular graphs $(i = 1, 2, 3, 4)$ and $r_1 \geq 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$ is*

$$\sigma(G) = \Bigg(\begin{matrix} \gamma_i & \eta_j & r_4 & \left(\lambda_k + r_4 \pm \sqrt{4\,p(\lambda_k + r_1) + (\lambda_k - r_4)^2}\right)/2 \\ n & m & (r_1 - 2)n/2 & 1 \end{matrix}$$
$$\begin{matrix} \left(\mu_s + r_3 \pm \sqrt{4\,l{\mu_s}^2 + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 \end{matrix}\Bigg),$$

where $i = 2$ to l, $j = 2$ to p, $k = 2$ to n, $s = 2$ to m, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t's are the roots of the polynomial

$$((x - r_1)\,(x - r_4) - 2\,pr_1)\left((x - r_2)\,(x - r_3) - lr_1^2\right) - nm(x - r_3)(x - r_4).$$

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