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ON SPECTRA OF VARIANTS OF THE CORONA OF TWO GRAPHS AND SOME NEW EQUIENERGETIC GRAPHS

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Abstract

Let G and H be two graphs. The join $G \vee H$ is the graph obtained by joining every vertex of G with every vertex of H. The corona $G \circ H$ is the graph obtained by taking one copy of G and |V(G)| copies of H and joining the *i*-th vertex of G to every vertex in the *i*-th copy of H. The neighborhood corona $G \star H$ is the graph obtained by taking one copy of G and |V(G)| copies of H and joining the neighbors of the *i*-th vertex of G to every vertex in the *i*-th copy of *H*. The edge corona $G \diamond H$ is the graph obtained by taking one copy of G and |E(G)| copies of H and joining each terminal vertex of *i*-th edge of G to every vertex in the *i*-th copy of H. Let G_1, G_2, G_3 and G_4 be regular graphs with disjoint vertex sets. In this paper we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \star G_3), (G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4),$ $(G_1 \vee G_2) \cup (G_1 \circ G_3), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4), (G_1 \vee G_2) \cup (G_1 \diamond G_3), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_$ $(G_1 \lor G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4), \ (G_1 \lor G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3),$ $(G_1 \lor G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ and $(G_1 \lor G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$. As an application, we show that there exist some new pairs of equienergetic graphs on n vertices for all n > 11.

Keywords: spectrum, corona, neighbourhood corona, edge corona, energy of a graph, equienergetic graphs.

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1. INTRODUCTION

Throughout this paper we consider only undirected simple graphs (i.e., graphs with no loops and multiple edges). Let G be a graph on n vertices. The eigenvalues of the adjacency matrix of G, denoted by $\lambda_i(G)$, i = 1, 2, ..., n, are

the eigenvalues of the graph G and $\sigma(G) = (\lambda_1(G), \lambda_2(G), \dots, \lambda_n(G))$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ is the adjacency spectrum of G [8]. The energy E(G) is the sum of all absolute values of eigenvalues of G. The concept of energy of a graph was introduced by Gutman [12] with an application to chemistry (Huckel molecular orbital approximation for the total π -electron energy [14]). The energy and various forms of energy of a graph G has been extensively studied by many mathematicians and some of their works can be found in [1, 2, 3, 5, 13, 15, 19, 21, 28, 27] and references therein. Two graphs G_1 and G_2 of the same order are said to be equienergetic if $E(G_1) = E(G_2)$. Graphs of the same order are cospectral if they have the same spectrum. Thus, two cospectral graphs are obviously equienergetic. For connected graphs, there are no equienergetic graphs of order $n \leq 5$. In [18] Indulal and Vijayakumar have constructed a pair of equienergetic graphs on n vertices for n = 6, 14, 18 and for all $n \ge 20$. Later Liu et al. [22] and Ramane, Walikar [26] have independently proved that there exists a pair of equienergetic graphs on n vertices for all $n \geq 9$. Studies on equienergetic graphs can be found in [6, 11, 18, 20, 22, 25, 26, 29] and references therein.

The corona of two graphs was first introduced by Frucht and Harary in [10]. Barik *et al.* [4] provided a complete description of the spectrum of corona $G_1 \circ G_2$ using the spectrum of G_1 and G_2 . More about the spectrum of corona can be found in [4, 7, 10, 24]. The neighborhood corona and edge corona was introduced in [17] and in [16], respectively. Complete description of the spectrum of neighborhood corona and edge corona of two graphs are given in [17, 23] and [16], respectively.

Motivated by the above works, in this paper we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \star G_3), (G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4), (G_1 \vee G_2) \cup (G_1 \circ G_3), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3), (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \circ G_4)$, when G_1, G_2, G_3 and G_4 are regular graphs. Here the graphs G_1, G_2, G_3 and G_4 have disjoint vertex sets. As an application of our results we construct some new pairs of equienergetic graphs on n vertices for all $n \geq 11$. Our method of construction and proofs are entirely different from the methods given in [18, 22, 26].

2. Preliminaries

In this section, we give some definitions and lemmas which are useful to prove our main results.

Definition [10]. Let G_1 and G_2 be two graphs on n and m vertices, respectively. The corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n copies of G_2 , and then joining the *i*-th vertex of G_1 to every vertex in the *i*-th copy of G_2 .

Definition [16]. Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The edge corona $G_1 \diamond G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2 , and then joining two end vertices of the *i*-th edge of G_1 to every vertex in the *i*-th copy of G_2 .

Definition [17]. Let G_1 and G_2 be two graphs on n and m vertices, respectively. The neighborhood corona $G_1 \star G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n copies of G_2 , and then joining each neighbor of *i*-th vertex of G_1 to every vertex in the *i*-th copy of G_2 .

Definition [8]. Let $A = (a_{ij})$ be an $n \times m$ matrix, $B = (b_{ij})$ be a $p \times q$ matrix. Then the Kronecker product $A \otimes B$ of A and B is the np by mq matrix obtained by replacing each entry a_{ij} of A by $a_{ij}B$.

Lemma 1 [8]. If M, N, P, Q are matrices with M being a non-singular matrix, then

(1)
$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|.$$

Lemma 2 [26]. Let N_1 and N_2 be two graphs as shown in Figure 1. Then the line graph $L(N_1)$ of N_1 and the line graph $L(N_2)$ of N_2 are non cospectral and equienergetic.



Figure 1

Lemma 3 [8]. The following cubic regular graphs with ten vertices are equienergetic.



Figure 2

3. Spectra of $(G_1 \lor G_2) \cup (G_1 \star G_3)$ and $(G_1 \lor G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$

In this section, we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \star G_3)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$, where G_1, G_2, G_3 and G_4 are regular graphs on n, m, l and p vertices, respectively.

Theorem 4. Let G_i be r_i -regular graphs (i = 1, 2, 3). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$ and $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ are the adjacency spectrum of G_1 , G_2 and G_3 , respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_1 \star G_3)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \mu_j & \left(\lambda_k + r_3 \pm \sqrt{4 l \lambda_k^2 + (\lambda_k - r_3)^2}\right)/2 & x_t \\ n & 1 & 1 & 1 \end{pmatrix},$$

where i = 2 to l, j = 2 to m, k = 2 to n, t = 1, 2, 3. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial $(x - r_2) ((x - r_1) (x - r_3) - lr_1^2) - nm (x - r_3)$.

Proof. With suitable labelling of the vertices of G, the adjacency matrix A(G) can be formulated as follows:

$$A(G) = \left(\begin{array}{ccc} I_n \otimes A(G_3) & 0 & A(G_1) \otimes e \\ \\ 0 & A(G_2) & J \\ \\ A(G_1) \otimes e^T & J^T & A(G_1) \end{array} \right),$$

 $l \ times$

where $e^T = (1, 1, ..., 1)$, I_n is the identity matrix of order n and J is the $m \times n$ matrix with all its entries are 1.

Since $A(G_3)$ is a real symmetric matrix, $A(G_3)$ is orthogonally diagonalizable. Let $A(G_3) = PDP^T$, where $PP^T = I_l$ and $D = diag(\gamma_1, \ldots, \gamma_l)$. Then

$$\begin{split} A(G) &= \begin{pmatrix} I_n \otimes PDP^T & 0 & A(G_1) \otimes e \\ 0 & A(G_2) & J \\ A(G_1) \otimes e^T & J^T & A(G_1) \end{pmatrix} \\ &= \begin{pmatrix} I_n \otimes P & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes P^T e \\ 0 & A(G_2) & J \\ A(G_1) \otimes e^T P & J^T & A(G_1) \end{pmatrix} \begin{pmatrix} I_n \otimes P^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I_n \otimes P & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes \sqrt{le_1} \\ 0 & A(G_2) & J \\ A(G_1) \otimes \sqrt{le_1^T} & J^T & A(G_1) \end{pmatrix} \begin{pmatrix} I_n \otimes P^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{split}$$

where $e_1^T = (1, 0, \dots, 0)$.

Let
$$B = \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes \sqrt{l}e_1 \\ 0 & A(G_2) & J \\ A(G_1) \otimes \sqrt{l}e_1^T & J^T & A(G_1) \end{pmatrix}$$

Then by the above equation we have

(2)
$$|xI - A(G)| = |xI - B|.$$

Expanding |xI - B| by Laplace's method [9] along $(li + 2), (li + 3), \ldots, (li + l)^{th}$ columns, for $i = 0, 1, \ldots, n-1$, we see that the only non zero $(l-1)n \times (l-1)n$ minor is

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(3)
$$M = |I_n \otimes diag(x - \gamma_2, \dots, x - \gamma_l)|.$$

The complementary minor of M is

$$M_{1} = \begin{vmatrix} (x - r_{3})I_{n} & 0 & -\sqrt{l}A(G_{1}) \\ 0 & xI_{m} - A(G_{2}) & -J \\ -\sqrt{l}A(G_{1}) & -J^{T} & xI_{n} - A(G_{1}) \end{vmatrix}.$$

Again as $A(G_1)$ and $A(G_2)$ are orthogonally diagonalizable, one can easily see that the M_1 is the same as

(4)
$$M'_{1} = \begin{vmatrix} (x-r_{3})I_{n} & 0 & -\sqrt{l} \operatorname{diag}(\lambda_{1}, \dots, \lambda_{n}) \\ 0 & \operatorname{diag}(x-\mu_{1}, \dots, x-\mu_{m}) & -\sqrt{mn}J' \\ -\sqrt{l} \operatorname{diag}(\lambda_{1}, \dots, \lambda_{n}) & -\sqrt{mn}J'^{T} & \operatorname{diag}(x-\lambda_{1}, \dots, x-\lambda_{n}) \end{vmatrix}$$

where J' is the matrix obtained by replacing every entries of J except the first diagonal entry by 0. Now by (1), we have

$$M_{1}' = (x - r_{3})^{n}$$
(5) $\times \begin{vmatrix} diag(x - r_{2}, x - \mu_{2}, \dots, x - \mu_{m}) & -\sqrt{mn}J' \\ -\sqrt{mn}J'^{T} & diag(x - \lambda_{1} - l\lambda_{1}^{2}/(x - r_{3}), \dots, x - \lambda_{n} - l\lambda_{n}^{2}/(x - r_{3})) \end{vmatrix}$

Applying Laplace method along $2, \ldots, m, m+2, \ldots, m+n$ columns in the above determinant we see that the only non zero $m + n - 2 \times m + n - 2$ minor is $diag(x - \mu_2, \ldots, x - \mu_m, x - \lambda_2 - l\lambda_2^2/(x - r_3), \ldots, x - \lambda_n - l\lambda_n^2/(x - r_3))$ and the complementary minor is

$$M_1 = \begin{vmatrix} x - \mu_2 & -\sqrt{mn} \\ -\sqrt{mn} & x - \lambda_2 - l\lambda_2^2/(x - r_3) \end{vmatrix}.$$

And so by (2), (3), (4), (5) and from the above equation the result follows.

Corollary 5. Let G_i be r_i -regular graphs (i = 1, 2). Then

$$E(G_1 \vee G_2 \cup G_1 \star lK_1) = \sqrt{4l+1}E(G_1) + E(G_2) - r_1(\sqrt{4l+1}-1) - 2x_0,$$

where x_0 is the negative root of the polynomial $(x - r_2) ((x - r_1)x - lr_1^2) - nmx$.

Remark 6. Corollary 5 is a generalization of Theorem 1 in [18]. In fact putting $r_1 = r$, n = p, $r_2 = 0$, m = k, $r_3 = 0$, l = 1 in Corollary 5, we obtain Theorem 1 due to Indulal and Vijayakumar [18].

Corollary 7. Let G_i (i = 1, 2) be equienergetic regular graphs of the same degree and H_i (i = 1, 2) be equienergetic regular graphs of the same degree. Then

$$E(G_1 \vee H_1 \cup G_1 \star lK_1) = E(G_2 \vee H_2 \cup G_2 \star lK_1).$$

Now we construct some new pairs of equienergetic graphs using Corollary 7.

Theorem 8. There exists a pair of equienergetic graphs on n vertices for all $n \ge 18$.

Proof. From Lemma 2 we have the line graphs $L(N_1)$ and $L(N_2)$ are equienergetic. Now by Corollary 7 it is clear that the graphs $(L(N_1) \vee K_m) \cup (L(N_1) \star K_1)$ and $(L(N_2) \vee K_m) \cup (L(N_2) \star K_1)$, both of order 18 + m (m = 0, 1, ...), are equienergetic. This completes the proof of the theorem.

Theorem 9. There exists a pair of equienergetic graphs on n vertices for all $n \ge 20$.

Proof. From Lemma 3 and Corollary 7 it is clear that the graphs $(T_1 \vee K_m) \cup (T_1 \star K_1)$ and $(T_2 \vee K_m) \cup (T_2 \star K_1)$, both of order 20 + m (m = 0, 1, ...), are equienergetic.

Theorem 10. There exists a pair of equienergetic graphs on n vertices for all $n \ge 13$.

Proof. Case 1. n = 9 + 2m (m = 2, 3, ...). For n = 9 + 2m (m = 2, 3, ...), the graphs $(K_m \vee L(N_1)) \cup (K_m \star K_1)$ and $(K_m \vee L(N_2)) \cup (K_m \star K_1)$ both are of order 9 + 2m (m = 2, 3, ...). Now, Corollary 7 implies that these two graphs are equienergetic.

Case 2. n = 10 + 2m (m = 2, 3, ...). For n = 10 + 2m (m = 2, 3, ...), the graphs $(K_m \vee T_1) \cup (K_m \star K_1)$ and $(K_m \vee T_2) \cup (K_m \star K_1)$ both are of order 10 + 2m (m = 2, 3, ...). Now, Corollary 7 implies that these two graphs are equienergetic.

As the proof of the following theorem is similar to that of Theorem 4, we omit the details.

Theorem 11. Let G_i be r_i -regular graphs (i = 1, 2, 3, 4). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4 , respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & \left(\lambda_k + r_4 \pm \sqrt{4 p \lambda_k^2 + (\lambda_k - r_4)^2}\right)/2 \\ m & n & 1 \\ \left(\mu_s + r_3 \pm \sqrt{4 l \mu_s^2 + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 \end{pmatrix},$$

where i = 2 to l, j = 2 to p, k = 2 to n, s = 2 to m, t = 1, 2, 3, 4. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial

$$\left((x-r_1)(x-r_4)-pr_1^2\right)\left((x-r_2)(x-r_3)-lr_2^2\right)-nm(x-r_3)(x-r_4).$$

Corollary 12. Let G_i be r_i -regular graphs (i = 1, 2). Then

$$E(G_1 \lor G_2 \cup G_2 \star lK_1 \cup G_1 \star pK_1) = \sqrt{4p+1}E(G_1) + \sqrt{4l+1}E(G_2) - r_2(\sqrt{4l+1}-1) - r_1(\sqrt{4p+1}-1) - 2x_0 - 2x_1,$$

where x_0 and x_1 are the negative roots of the polynomial $x^4 - (r_1 + r_2) x^3 + (-r_1^2 p - lr_2^2 + r_1 r_2 - mn) x^2 + (r_1^2 r_2 p + r_1 r_2^2 l) x + r_1^2 r_2^2 l p$.

Corollary 13. Let G_1 , G_2 be equienergetic regular graphs of the same degree and H_1 , H_2 be equienergetic regular graphs of the same degree. Then

$$E(G_1 \lor H_1 \cup H_1 \star lK_1 \cup G_1 \star pK_1) = E(G_2 \lor H_2 \cup H_2 \star lK_1 \cup G_2 \star pK_1).$$

4. Spectra of $(G_1 \lor G_2) \cup (G_1 \circ G_3)$ and $(G_1 \lor G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$

In this section, we simply state some theorems (as the proofs are quite analogous to the proof of Theorem 4) which gives the spectrum of $(G_1 \vee G_2) \cup (G_1 \circ G_3)$ and $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, where G_1, G_2, G_3 and G_4 are regular graphs on n, m, l and p vertices, respectively.

Theorem 14. Let G_i be r_i -regular graphs (i = 1, 2, 3). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$ and $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ are the adjacency spectrum of G_1 , G_2 and G_3 , respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_1 \circ G_3)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \mu_j & \left(\lambda_k + r_3 \pm \sqrt{4l + (\lambda_k - r_3)^2}\right)/2 & x_t \\ n & 1 & 1 & 1 \end{pmatrix},$$

where i = 2 to l, j = 2 to m, k = 2 to n, t = 1, 2, 3. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial $(x - r_2)((x - r_1)(x - r_3) - l) - nm(x - r_3)$.

Theorem 15. Let G be an r-regular graph of order m. Then

$$E(K_n \lor G \cup K_n \circ lK_1) = E(G) + (n-1)\sqrt{4l+1} - 2x_0 + n - 1,$$

where x_0 is the negative root of the polynomial (x - r) (x (x - (n - 1)) - l) - nmx.

Corollary 16. Let G and H be equienergetic regular graphs of the same degree. Then

$$E(K_n \lor G \cup K_n \circ lK_1) = E(K_n \lor H \cup K_n \circ lK_1).$$

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Theorem 17. Let G be an r-regular graph of order m. Then

$$E(nK_1 \vee G \cup nK_1 \circ lK_1) = E(G) + (n-1)\sqrt{4l} - 2x_0,$$

where x_0 is the negative root of the polynomial $(x-r)(x^2-l) - nmx$.

Corollary 18. Let G and H be equienergetic regular graphs of the same degree. Then

$$E(nK_1 \vee G \cup nK_1 \circ lK_1) = E(nK_1 \vee H \cup nK_1 \circ lK_1)$$

Now we construct some new pairs of equienergetic graphs using Corollary 16.

Theorem 19. There exists a pair of equienergetic graphs on n vertices for all $n \ge 11$.

Proof. Case 1. n = 9 + 2m (m = 1, 2, ...). For n = 9 + 2m (m = 1, 2, ...), the graphs $(K_m \vee L(N_1)) \cup (K_m \circ K_1)$ and $(K_m \vee L(N_2)) \cup (K_m \circ K_1)$ both are of order 9 + 2m (m = 1, 2, ...). Now, Corollary 16 implies that these two graphs are equienergetic.

Case 2. n = 10 + 2m (m = 1, 2, ...). For n = 10 + 2m (m = 1, 2, ...), the graphs $(K_m \vee T_1) \cup (K_m \circ K_1)$ and $(K_m \vee T_2) \cup (K_m \circ K_1)$ both are of order 10 + 2m (m = 1, 2, ...). Now, Corollary 16 implies that these two graphs are equienergetic.

Theorem 20. Let G_i be r_i -regular graphs (i = 1, 2, 3, 4). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4 , respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & \left(\lambda_k + r_4 \pm \sqrt{4 \, p + (\lambda_k - r_4)^2}\right)/2 \\ m & n & 1 \\ \left(\mu_s + r_3 \pm \sqrt{4 \, l + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 \end{pmatrix},$$

where i = 2 to l, j = 2 to p, k = 2 to n, s = 2 to m, t = 1, 2, 3, 4. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial

$$((x - r_1)(x - r_4) - p)((x - r_2)(x - r_3) - l) - nm(x - r_3)(x - r_4).$$

5. Spectra of $(G_1 \lor G_2) \cup (G_1 \diamond G_3)$ and $(G_1 \lor G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)$

In Theorems 21 and 25 of this section, we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \diamond G_3)$ and $(G_1 \vee G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)$, where G_1, G_2, G_3 and G_4 are regular graphs on n, m, l and p vertices, respectively. Proofs of these theorems are not given as they are similar to the proof of Theorem 4.

Theorem 21. Let G_i be r_i -regular graphs (i = 1, 2, 3) and $r_1 \ge 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n), \sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$ and $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ are the adjacency spectrum of G_1 , G_2 and G_3 , respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_1 \diamond G_3)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & r_3 & \mu_j & \left(\lambda_k + r_3 \pm \sqrt{4 \, l (\lambda_k + r_1) + (\lambda_k - r_3)^2}\right) / 2 & x_t \\ r_1 n / 2 & (r_1 - 2) n / 2 & 1 & 1 \end{pmatrix},$$

where i = 2 to l, j = 2 to m, k = 2 to n, t = 1, 2, 3. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial $(x - r_2)((x - r_1)(x - r_3) - 2lr_1) - nm(x - r_3)$.

Theorem 22. Let G be an r-regular graph of order m. Then

$$E(K_n \lor G \cup K_n \diamond lK_1) = E(G) + (n-1)(\sqrt{4l+n-2}+1) - 2x_0,$$

where x_0 is the negative root of the polynomial

$$x^{3} - (n - 1 + r) x^{2} + ((n - 1) r - 2 (n - 1) l - mn) x + 2 (n - 1) rl.$$

Corollary 23. Let G and H be equienergetic regular graphs of the same degree. Then

$$E(K_n \lor G \cup K_n \diamond lK_1) = E(K_n \lor H \cup K_n \diamond lK_1).$$

Now we construct some new pairs of equienergetic graphs using Corollary 23.

Theorem 24. There exists a pair of equienergetic graphs on n vertices for all $n \ge 15$.

Proof. Case 1. Let n = 9 + 2m (m = 3, 4, ...) and C_m be the cycle of length m. Then, by Corollary 23 and Lemma 2 the graphs $(C_m \vee L(N_1)) \cup (C_m \diamond K_1)$ and $(C_m \vee L(N_2)) \cup (C_m \diamond K_1)$, both of order 9 + 2m (m = 3, 4, ...), are equienergetic.

Case 2. n = 10 + 2m (m = 3, 4, ...). For n = 10 + 2m (m = 3, 4, ...), the graphs $(C_m \lor T_1) \cup (C_m \diamond K_1)$ and $(C_m \lor T_2) \cup (C_m \diamond K_1)$ both are of order 9 + 2m (m = 3, 4, ...). Now, Corollary 23 and Lemma 3 implies that these two graphs are equienergetic.

Theorem 25. Let G_i be r_i -regular graphs $(i = 1, 2, 3, 4), r_1 \ge 2$ and $r_2 \ge 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \dots, \lambda_n), \sigma(G_2) = (\mu_1 = r_2, \mu_2, \dots, \mu_m), \sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \dots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \dots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4 , respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_2 \diamond G_3) \cup (G_1 \diamond G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & r_3 & \eta_j & r_4 \\ r_2 n/2 & (r_2 - 2)n/2 & r_1 n/2 & (r_1 - 2)n/2 \end{pmatrix}$$

$$\binom{\lambda_k + r_4 \pm \sqrt{4 p(\lambda_k + r_1) + (\lambda_k - r_4)^2}}{1} / 2 \quad \binom{\mu_s + r_3 \pm \sqrt{4 l(\mu_s + r_2) + (\mu_s - r_3)^2}}{1} / 2 \quad \frac{x_t}{1}$$

where i = 2 to l, j = 2 to p, k = 2 to n, s = 2 to m, t = 1, 2, 3, 4. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial

$$((x - r_1) (x - r_4) - 2 pr_1) ((x - r_2) (x - r_3) - 2 r_2 l) - nm (x - r_3) (x - r_4).$$

6. Spectra of
$$(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3),$$

 $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$

In this section we just give the spectrum of $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ and $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$, where G_1 , G_2 , G_3 and G_4 are regular graphs on n, m, l and p vertices, respectively. Proofs of Theorems 26–28 are similar to the proof of Theorem 4.

Theorem 26. Let G_i be r_i -regular graphs (i = 1, 2, 3, 4). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1, G_2, G_3 and G_4 , respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \star G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & \left(\lambda_k + r_4 \pm \sqrt{4 p \lambda_k^2 + (\lambda_k - r_4)^2}\right)/2 \\ n & m & 1 \\ \left(\mu_s + r_3 \pm \sqrt{4 l + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 & 1 \end{pmatrix},$$

where i = 2 to l, j = 2 to p, k = 2 to n, s = 2 to m, t = 1, 2, 3, 4. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial

$$((x-r_2)(x-r_3)-l)((x-r_1)(x-r_4)-pr_1^2)-nm(x-r_3)(x-r_4).$$

Theorem 27. Let G_i be r_i -regular graphs (i = 1, 2, 3, 4) and $r_1 \ge 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n), \ \sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m), \ \sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1 , G_2 , G_3 and G_4 , respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_2 \circ G_3) \cup (G_1 \diamond G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & r_4 & \left(\lambda_k + r_4 \pm \sqrt{4 \, p (\lambda_k + r_1) + (\lambda_k - r_4)^2}\right)/2 \\ n & m & (r_1 - 2)n/2 & 1 \\ & \left(\mu_s + r_3 \pm \sqrt{4 \, l + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 & 1 \end{pmatrix},$$

where i = 2 to l, j = 2 to p, k = 2 to n, s = 2 to m, t = 1, 2, 3, 4. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial

$$((x - r_1)(x - r_4) - 2pr_1)((x - r_2)(x - r_3) - l) - nm(x - r_3)(x - r_4).$$

Theorem 28. Let G_i be r_i -regular graphs (i = 1, 2, 3, 4) and $r_1 \ge 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n), \ \sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m), \ \sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of G_1 , G_2 , G_3 and G_4 , respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_2 \star G_3) \cup (G_1 \diamond G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & r_4 & \left(\lambda_k + r_4 \pm \sqrt{4 \, p(\lambda_k + r_1) + (\lambda_k - r_4)^2}\right)/2 \\ n & m & (r_1 - 2)n/2 & 1 \\ \left(\mu_s + r_3 \pm \sqrt{4 \, l\mu_s^2 + (\mu_s - r_3)^2}\right)/2 & x_t \\ 1 & 1 \end{pmatrix},$$

where i = 2 to l, j = 2 to p, k = 2 to n, s = 2 to m, t = 1, 2, 3, 4. Also, the entries in the first row are the eigenvalues with multiplicity written below, and x_t 's are the roots of the polynomial

$$((x-r_1)(x-r_4)-2pr_1)((x-r_2)(x-r_3)-lr_1^2)-nm(x-r_3)(x-r_4).$$

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References

- C. Adiga, R. Balakrishnan and W. So, *The skew energy of a digraph*, Linear Algebra Appl. **432** (2010) 1825–1835. doi:10.1016/j.laa.2009.11.034
- [2] R. Balakrishnan, The energy of a graph, Linear Algebra Appl. 387 (2004) 287–295. doi:10.1016/j.laa.2004.02.038
- [3] R.B. Bapat, Energy of a graph is never an odd integer, Bull. Kerala Math. Assoc. 1 (2004) 129–132.
- S. Barik, S. Pati and B.K. Sarma, The spectrum of the corona of two graphs, SIAM J. Discrete Math. 21 (2007) 47–56. doi:10.1137/050624029
- [5] S.B. Bozkurt, A.D. Gungor and I. Gutman, Note on distance energy of graphs, MATCH. Commun. Math. Comput. Chem. 64 (2010) 129–134.
- [6] V. Brankov, D. Stevanović and I. Gutman, Equienergetic chemical trees, J. Serb. Chem. Soc. 69 (2004) 549–553. doi:10.2298/JSC0407549B
- S.-Y. Cui and G.-X. Tian, The spectrum and the signless Laplacian spectrum of coronae, Linear Algebra Appl. 437 (2012) 1692–1703. doi:10.1016/j.laa.2012.05.019
- [8] D. Cvetković, M. Doob and H. Sachs, Spectra of Graphs: Theory and Application (Academic Press, New York, 1980).
- [9] W.L. Ferrar, A Text-Book of Determinants, Matrices and Algebraic Forms (Oxford University Press, 1953).
- [10] R. Frucht and F. Harary, On the corona of two graphs, Aequationes Math. 4 (1970) 322–325. doi:10.1007/BF01844162
- [11] S. Gong, X. Li, G. Xu, I. Gutman and B. Furtula, *Borderenergetic graphs*, MATCH Commun. Math. Comput. Chem. **74** (2015) 321–332.
- [12] I. Gutman, The energy of a graph, Ber. Math.-Statist. Sekt. Forschungsz. Graz 103 (1978) 1–22.
- [13] I. Gutman, The energy of a graph: old and new results, in: Algebraic Combinatorics and Applications, A. Betten, A. Kohnert, R. Laue and A. Wassermann (Ed(s)), (Berlin, Springer, 2000) 196–211.
- [14] I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology, J. Serb. Chem. Soc. 70 (2005) 441–456. doi:10.2298/JSC0503441G
- [15] I. Gutman, D. Kiani, M. Mirzakhah and B. Zhou, On incidence energy of a graph, Linear Algebra Appl. 431 (2009) 1223–1233. doi:10.1016/j.laa.2009.04.019

- [16] Y. Hou and W.-C. Shiu, *The spectrum of the edge corona of two graphs*, Electron. J. Linear Algebra **20** (2010) 586–594. doi:10.13001/1081-3810.1395
- [17] G. Indulal, The spectrum of neighborhood corona of graphs, Kragujevac J. Math. 35 (2011) 493–500.
- [18] G. Indulal and A. Vijayakumar, On a pair of equienergetic graphs, MATCH. Commun. Math. Comput. Chem. 55 (2006) 83–90.
- [19] X. Li, Y. Shi and I. Gutman, Graph Energy (Springer, New York, 2012). doi:10.1007/978-1-4614-4220-2
- [20] X. Li, M. Wei and S. Gong, A computer search for the borderenergetic graphs of order 10, MATCH Commun. Math. Comput. Chem. 74 (2015) 333–342.
- [21] B. Liu, Y. Huang and Z. You, A survey on the Laplacian-energy-like invariant, MATCH. Commun. Math. Comput. Chem. 66 (2011) 713-730.
- [22] J. Liu and B. Liu, On a pair of equienergetic graphs, MATCH. Commun. Math. Comput. Chem. 59 (2008) 275–278.
- [23] X. Liu and S. Zhou, Spectra of the neighbourhood corona of two graphs, Linear Multilinear Algebra 62 (2014) 1205–1219. doi:10.1080/03081087.2013.816304
- [24] C. McLeman and E. McNicholas, Spectra of coronae, Linear Algebra Appl. 435 (2011) 998–1007. doi:10.1016/j.laa.2011.02.007
- [25] H.S. Ramane, I. Gutman, H.B. Walikar and S.B. Halkarni, Equienergetic complement graphs, Kragujevac J. Sci. 27 (2005) 67–74.
- [26] H.S. Ramane and H.B. Walikar, Construction of equienergetic graphs, MATCH Commun. Math. Comput. Chem. 57 (2007) 203–210.
- [27] D. Stevanović, Energy and NEPS of graphs, Linear Multilinear Algebra 53 (2005) 67–74. doi:10.1080/03081080410001714705
- [28] D. Stevanović and I. Stanković, Remarks on hyperenergetic circulant graphs, Linear Algebra Appl. 400 (2005) 345–348. doi:10.1016/j.laa.2005.01.001
- [29] L. Xu and Y. Hou, Equienergetic bipartite graphs, MATCH Commun. Math. Comput. Chem. 57 (2007) 363–370.

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