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SUPERMAGIC GENERALIZED DOUBLE GRAPHS [1]

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Abstract

A graph G is called supermagic if it admits a labelling of the edges by pairwise different consecutive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In this paper we will introduce some constructions of supermagic labellings of some graphs generalizing double graphs. Inter alia we show that the double graphs of regular Hamiltonian graphs and some circulant graphs are supermagic.

Keywords: double graphs, supermagic graphs, degree-magic graphs.

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1. Introduction

We consider finite graphs without loops and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G. The subgraph of a graph G induced by $Z \subseteq E(G)$ is denoted by $G[Z]$. For integers p, q we denote by $[p, q]$ the set of all integers z satisfying $p \leq z \leq q$.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every } v \in V(G), \tag{1}$$

where $\eta(v, e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G). \tag{2}$$

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A magic labelling f of G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever there exists a supermagic (magic) labelling of G.

A bijection f from $E(G)$ into $\big[1, |E(G)|\big]$ is called a *degree-magic labelling* (or only *d-magic labelling*) of a graph G if its index-mapping f^* satisfies

$$f^*(v) = \frac{1 + |E(G)|}{2} \deg(v) \quad \text{for all } v \in V(G). \tag{3}$$

We say that a graph G is *degree-magic* (or only *d-magic*) when there exists a d-magic labelling of G.

The concept of magic graphs was introduced by Sedláček [11]. Supermagic graphs were introduced by Stewart [13]. There is by now a considerable number of papers published on magic and supermagic graphs; we single out [8, 9] as being more particularly relevant to the present paper, and refer the reader to [6] for comprehensive references. The concept of degree-magic graphs was introduced in [1]. Degree-magic graphs extend supermagic regular graphs because the following result holds.

Proposition 1 ([1])**.** *Let G be a regular graph. Then G is supermagic if and only if it is degree-magic.*

Suppose that $q \geq 2$ is an integer. A spanning subgraph H of a graph G is called a $\frac{1}{q}$*-factor* of G whenever $\deg_H(v) = \deg_G(v)/q$ for every vertex $v \in V(G)$. A bijection f from $E(G)$ onto $\big[1, |E(G)|\big]$ is called *q-gradual* if the set

$$F_q(f; i) := \{e \in E(G) : (i-1)|E(G)|/q < f(e) \leq i|E(G)|/q\}$$

induces a $\frac{1}{q}$-factor of G for each $i \in [1, q]$. A graph G is called *balanced degree-magic* if there exists a 2-gradual d-magic labelling of G. Some properties of balanced d-magic graphs were described in [1] and [2]. However, the notion of q-gradual labelling seems to be useful also for $q > 2$.

Observation 1. *Let $f : E(G) \longrightarrow \big[1, |E(G)|\big]$ be a q-gradual bijection and let α be a permutation of $[1, q]$. Then $g : E(G) \longrightarrow \big[1, |E(G)|\big]$ defined by*

$$g(e) = f(e) + \big(\alpha(i) - i\big)\frac{|E(G)|}{q} \quad \textit{when } e \in F_q(f; i),$$

is a q-gradual bijection satisfying

(i) *$g^*(v) = f^*(v)$, for every vertex $v \in V(G)$,*
(ii) *$F_q(g; \alpha(i)) = F_q(f; i)$, for each $i \in [1, q]$.*

The graph obtained by replacing each edge uv of a graph G with 2 edges joining u and v is denoted by ${}^2G$. Therefore, $V({}^2G) = V(G)$ and $E({}^2G) = \bigcup_{e\in E(G)}\{(e,1),(e,2)\}$, where an edge (e,i), $i \in \{1,2\}$, is incident with a vertex v in ${}^2G$ whenever e is incident with v in G. In this case, $E_i({}^2G) := \bigcup_{e\in E(G)}\{(e,i)\}$. Evidently, the subgraph of ${}^2G$ induced by $E_i({}^2G)$ is isomorphic to G.

In this paper we will introduce some constructions of supermagic (and degree-magic) labellings of some graphs generalizing double graphs.

2. Generalized Double Graphs

Let G be a graph. Suppose that $U \subseteq V(G)$ and $Z \subseteq E(G)$. Define a graph $D = D(G; Z, U)$ by

$$V(D) = \bigcup_{v\in V(G)} \{v^0, v^1\}$$

and

$$E(D) = \bigcup_{vu\in Z} \{v^0u^0, v^1u^1\} \cup \bigcup_{vu\in E(G)-Z} \{v^0u^1, v^1u^0\} \cup \bigcup_{u\in U} \{u^0u^1\}.$$

Note, that $D(G; E(G), \emptyset)$ consists of two disjoint copies of G, i.e., it is isomorphic to $2G$. The graph $D(G; E(G), V(G))$ is the Cartesian product of G and K_2. The graph $D(G; \emptyset, \emptyset)$ is the categorical product of G and K_2, also called the *bipartite double graph* of a graph G. Similarly, $D({}^2G; E_1({}^2G), \emptyset)$ is the lexicographic product (or composition) of G and $\overline{K_2}$, also called the *double graph* of a graph G (see [10]). Therefore, the graph $D(G; Z, U)$ is a generalization of the double graph of a graph G.

Now we prove crucial results of the paper.

Lemma 1. *Let G be a graph such that* $\deg(v) \equiv 0 \pmod 2$ *for every vertex $v \in V(G)$. Suppose that the subgraph of G induced by $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor. Then for any bijection $f : E(G) \longrightarrow \big[1, |E(G)|\big]$ there exists a 2-gradual bijection $g : E(D(G; Z, \emptyset)) \longrightarrow \big[1, 2|E(G)|\big]$ such that for every vertex $v \in V(G)$ it holds*

$$g^*(v^0) = g^*(v^1) = f^*(v) + \frac{1}{2}|E(G)|\deg(v).$$

Proof. The subgraph $G[Z]$ of a graph G induced by $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor. Then there is a set $Z_1 \subseteq Z$ such that the subgraph of $G[Z]$ induced by Z_1 is a $\frac{1}{2}$-factor of $G[Z]$. Clearly, the subgraph of $G[Z]$ induced by $Z_2 = Z - Z_1$ is also a $\frac{1}{2}$-factor of $G[Z]$. Moreover, the degree of each vertex of $G[Z]$ is even. Similarly, the degree of each vertex of $H = G[E(G) - Z]$ is even. Thus, every component of H is Eulerian. Therefore, there is a digraph $\vec{H}$ which we get from H by an orientation of its edges such that the outdegree of every vertex of $\vec{H}$ is equal to its indegree. By $[u,v]$ we denote an arc of $\vec{H}$ and by $A(\vec{H})$ the set of all arcs of $\vec{H}$.

Put $m := |E(G)|$ and $D := D(G; Z, \emptyset)$. Consider the bijection $g : E(D) \longrightarrow [1, 2m]$ given by

$$g(u^i v^j) = \begin{cases} f(uv) & \text{if } \ i = 0,\ j = 1,\ [u,v] \in A(\vec{H}), \\ f(uv) + m & \text{if } \ i = 1,\ j = 0,\ [u,v] \in A(\vec{H}), \\ f(uv) & \text{if } \ i = j = 0,\ uv \in Z_1, \\ f(uv) + m & \text{if } \ i = j = 1,\ uv \in Z_1, \\ f(uv) & \text{if } \ i = j = 1,\ uv \in Z_2, \\ f(uv) + m & \text{if } \ i = j = 0,\ uv \in Z_2. \end{cases}$$

For its index-mapping we have

$$\begin{aligned} g^*(v^0) &= \sum_{[v,w] \in A(\vec{H})} g(v^0 w^1) + \sum_{[w,v] \in A(\vec{H})} g(w^1 v^0) + \sum_{vw \in Z_1} g(v^0 w^0) + \sum_{vw \in Z_2} g(v^0 w^0) \\ &= \sum_{[v,w] \in A(\vec{H})} f(vw) + \sum_{[w,v] \in A(\vec{H})} (f(wv) + m) + \sum_{vw \in Z_1} f(vw) + \sum_{vw \in Z_2} (f(vw) + m) \\ &= \sum_{vw \in E(G)} f(vw) + m \cdot \frac{1}{2} \deg(v) = f^*(v) + \frac{1}{2} m \deg(v) \end{aligned}$$

for every vertex $v^0 \in V(D)$. Similarly, we get $g^*(v^1) = f^*(v) + \frac{1}{2} m \deg(v)$ for every vertex $v^1 \in V(D)$. Since the outdegree of every vertex of $\vec{H}$ is equal to its indegree and the sets Z_1 and Z_2 induce $\frac{1}{2}$-factors of $G[Z]$, the sets $F_2(g;1) = \{u^0 v^1 : [u,v] \in A(\vec{H})\} \cup \{u^0 v^0 : uv \in Z_1\} \cup \{u^1 v^1 : uv \in Z_2\}$ and $F_2(g;2) = \{u^1 v^0 : [u,v] \in A(\vec{H})\} \cup \{u^1 v^1 : uv \in Z_1\} \cup \{u^0 v^0 : uv \in Z_2\}$ induce $\frac{1}{2}$-factors of D. ■

Lemma 2. *Let $q \geq 2$ be a positive integer and let G be a graph such that* $\deg(v) \equiv 0 \pmod{2q}$ *for every vertex $v \in V(G)$. Then for any q-gradual bijection $f : E(G) \longrightarrow \big[1, |E(G)|\big]$ there exists a $2q$-gradual bijection $g : E(D(G; \emptyset, \emptyset)) \longrightarrow \big[1, 2|E(G)|\big]$ such that for every vertex $v \in V(G)$ it holds*

$$g^*(v^0) = g^*(v^1) = f^*(v) + \frac{1}{2} |E(G)| \deg(v).$$

Proof. Since $\deg(v) \equiv 0 \pmod{2q}$ for every vertex $v \in V(G)$, the degree of each vertex of $H_i = G[F_q(f;i)]$, $i \in [1,q]$, is even. Therefore, there is a digraph $\vec{H}_i$ which we get from H_i by an orientation of its edges such that the outdegree of every vertex of $\vec{H}_i$ is equal to its indegree. Let $\vec{H}$ be an orientation of G such that the set $A(\vec{H})$ of all arcs of $\vec{H}$ is equal to $\bigcup_{i=1}^{q} A(\vec{H}_i)$.

Put $m := |E(G)|$, $D := D(G;\emptyset,\emptyset)$ and consider the bijection g from $E(D)$ onto $[1, 2m]$ given by

$$g(u^i v^j) = \begin{cases} f(uv) & \text{if } i = 0,\ j = 1,\ [u,v] \in A(\vec{H}), \\ f(uv) + m & \text{if } i = 1,\ j = 0,\ [u,v] \in A(\vec{H}). \end{cases}$$

Analogously as in the proof of Lemma 1 we can prove that $g^*(v^0) = g^*(v^1) = f^*(v) + \frac{1}{2} m \deg(v)$ for every vertex $v \in V(G)$. Moreover, the outdegree of every vertex of $\vec{H}_i$ is equal to its indegree, and thus the sets $F_{2q}(g;i) = \{u^0v^1 : [u,v] \in A(\vec{H}_i)\}$ and $F_{2q}(g;q+i) = \{u^1v^0 : [u,v] \in A(\vec{H}_i)\}$ induce $\frac{1}{2q}$-factors of D. ■

Lemma 3. *Let $q \geq 3$ be an odd positive integer. Then for any q-gradual bijection $f : E(G) \longrightarrow \big[1, |E(G)|\big]$ there exists a bijection*

$$g : E(D(G;E(G),\emptyset)) \longrightarrow \big[1, |E(G)|\big] \cup \left[1 + \frac{q+1}{q}|E(G)|, \frac{2q+1}{q}|E(G)|\right]$$

such that for every vertex $v \in V(G)$ it holds

$$g^*(v^0) = g^*(v^1) = f^*(v) + \frac{q+1}{2q}|E(G)| \deg(v).$$

Proof. Put $m := |E(G)|$ and $D := D(G;E(G),\emptyset)$. Consider the mapping g from $E(D)$ into the set of integers given by

$$g(u^i v^i) = \begin{cases} f(uv) + 2m & \text{if } i = 0,\ uv \in F_q(f;1), \\ f(uv) & \text{if } i = 1,\ uv \in F_q(f;1), \\ f(uv) & \text{if } i = 0,\ uv \in F_q(f;2), \\ f(uv) + m & \text{if } i = 1,\ uv \in F_q(f;2), \\ f(uv) & \text{if } i = 0,\ uv \in F_q(f;j),\ 3 \leq j \equiv 1 \pmod 2, \\ f(uv) + m & \text{if } i = 1,\ uv \in F_q(f;j),\ 3 \leq j \equiv 1 \pmod 2, \\ f(uv) + m & \text{if } i = 0,\ uv \in F_q(f;j),\ 3 < j \equiv 0 \pmod 2, \\ f(uv) & \text{if } i = 1,\ uv \in F_q(f;j),\ 3 < j \equiv 0 \pmod 2. \end{cases}$$

Evidently, $g : E(D) \longrightarrow [1, m] \cup \left[1 + \frac{q+1}{q} m, \frac{2q+1}{q} m\right]$ is a bijection. Moreover, for its index-mapping we have

$$\begin{aligned} g^*(v^0) &= \sum_{j=1}^{q} \sum_{vw \in F_q(f;j)} g(v^0 w^0) = \sum_{j=1}^{q} \sum_{vw \in F_q(f;j)} f(vw) + \frac{q+1}{2} m \frac{\deg(v)}{q} \\ &= \sum_{vw \in E(G)} f(vw) + \frac{q+1}{2q} m \deg(v) = f^*(v) + \frac{q+1}{2q} m \deg(v) \end{aligned}$$

for every vertex $v^0 \in V(D)$. Similarly, we get $g^*(v^1) = f^*(v) + \frac{q+1}{2q} m \deg(v)$ for every vertex $v^1 \in V(D)$. ■

We say that a q-gradual bijection $f : E(G) \longrightarrow \big[1, |E(G)|\big]$ *respects* a set Z ($Z \subseteq E(G)$) if for each $i \in [1, q]$ either $F_q(f;i) \subseteq Z$ or $F_q(f;i) \subseteq E(G) - Z$. Evidently, a q-gradual bijection f respects a set Z if and only if there exists a subset $I \subset [1, q]$ such that $Z = \bigcup_{i \in I} F_q(f;i)$.

Lemma 4. *Let $q \geq 2$ be a positive integer and let G be a graph such that $\deg(v) \equiv 0 \pmod{2q}$ for every vertex $v \in V(G)$. Let f be a q-gradual bijection from $E(G)$ onto $\big[1, |E(G)|\big]$ which respects a set $Z \subsetneq E(G)$. If $|E(G)|/q < |Z| < |E(G)|$, then there exists a bijection g from $E(D(G; Z, \emptyset))$ onto $\big[1, 2|E(G)|\big]$ such that for every vertex $v \in V(G)$ it holds*

$$g^*(v^0) = g^*(v^1) = f^*(v) + \frac{1}{2}|E(G)| \deg(v).$$

Proof. As f respects the set Z, according to Observation 1, we can assume that there is an integer $k \in [1, q]$ such that $Z = \bigcup_{i=1}^{k} F_q(f;i)$. Moreover, since $|E(G)|/q < |Z| < |E(G)|$, $k \in [2, q-1]$.

Since $\deg(v) \equiv 0 \pmod{2q}$ for every vertex $v \in V(G)$, the degree of each vertex of G is even. If k is even, then the spanning subgraph of G induced by $\bigcup_{i=1}^{k/2} F_q(f;i)$ is a $\frac{1}{2}$-factor of $G[Z]$. According to Lemma 1, there exists a desired bijection $g : E(D(G; Z, \emptyset)) \longrightarrow \big[1, 2|E(G)|\big]$.

Now, suppose that k is odd. Put $m := |E(G)|/q$ and $d(v) := \deg(v)/q$ for every $v \in V(G)$. Clearly, the subgraph $G[F_q(f;i)]$, $i \in [1, q]$, has m edges and each its vertex v has degree $d(v)$. Denote by H_1 the subgraph of G induced by $Z \subseteq E(G)$ (i.e., $H_1 = G[Z]$). The size of H_1 is $|Z| = km$. Evidently, the mapping $h_1 : E(H_1) \longrightarrow [1, km]$, given by

$$h_1(e) := f(e) \quad \text{for every } e \in E(H_1),$$

is a k-gradual bijection. By Lemma 3, there exists a bijection

$$g_1 : E(D(H_1; Z, \emptyset)) \longrightarrow [1, km] \cup [1 + (k+1)m, (2k+1)m]$$

such that for every vertex $v \in V(H_1)$ it holds

$$g_1^*(v^0) = g_1^*(v^1) = h_1^*(v) + \frac{k+1}{2k}|Z| \deg_{H_1}(v) = h_1^*(v) + \frac{k+1}{2} mkd(v).$$

Similarly, denote by H_2 the subgraph of G induced by $E(G) - Z$ (i.e., $H_2 = G[E(G) - Z]$). The size of H_2 is $(q-k)m$. The mapping $h_2 : E(H_2) \longrightarrow [1, (q-k)m]$, given by

$$h_2(e) := f(e) - km \quad \text{for every } e \in E(H_2),$$

is a $(q-k)$-gradual bijection. By Lemma 2 (Lemma 1, if $q-k=1$), there exists a $2(q-k)$-gradual bijection $g_2 : E(D(H_2;\emptyset,\emptyset)) \longrightarrow [1, 2(q-k)m]$ such that for every vertex $v \in V(H_2)$ it holds

$$g_2^*(v^0) = g_2^*(v^1) = h_2^*(v) + \frac{1}{2}|E(H_2)| \deg_{H_2}(v) = h_2^*(v) + \frac{1}{2}m(q-k)^2 d(v).$$

Evidently, $E(D(G;Z,\emptyset)) = E(D(H_1;Z,\emptyset)) \cup E(D(H_2;\emptyset,\emptyset))$. Consider the mapping $g : E(D(G;Z,\emptyset)) \longrightarrow [1, 2qm]$ given by

$$g(e) = \begin{cases} g_1(e) & \text{if } \ e \in E(D(H_1;Z,\emptyset)), \\ g_2(e) + km & \text{if } \ e \in F_{2(q-k)}(g_2;1), \\ g_2(e) + 2km & \text{if } \ e \in E(D(H_2;\emptyset,\emptyset)) - F_{2(q-k)}(g_2;1). \end{cases}$$

Since $|F_{2(q-k)}(g_2;1)| = 2|E(H_2)|/(2(q-k)) = m$, the mapping g is a bijection. Moreover, for $i \in \{0,1\}$ and every vertex $v \in V(G)$ we have

$$\begin{aligned} g^*(v^i) &= g_1^*(v^i) + g_2^*(v^i) + km\frac{d(v)}{2} + 2km\big(2(q-k)-1\big)\frac{d(v)}{2} \\ &= g_1^*(v^i) + g_2^*(v^i) + (4q-4k-1)km\frac{d(v)}{2}, \end{aligned}$$

because the degree of v^i in a subgraph of $D(H_2;\emptyset,\emptyset)$ (and also $D(G;Z,\emptyset)$) induced by $F_{2(q-k)}(g_2;j)$, $j \in [1, 2(q-k)]$, is $d(v)/2$. Thus,

$$\begin{aligned} g^*(v^i) = &\left(h_1^*(v) + \frac{k+1}{2}mkd(v)\right) + \left(h_2^*(v) + \frac{1}{2}m(q-k)^2 d(v)\right) \\ &+ (4q-4k-1)km\frac{d(v)}{2} = h_1^*(v) + h_2^*(v) + (q^2+2qk-2k^2)m\frac{d(v)}{2}. \end{aligned}$$

As $\deg_{H_2}(v) = (q-k)d(v)$, we have $h_1^*(v) + h_2^*(v) = f^*(v) - km(q-k)d(v)$ and so

$$\begin{aligned} g^*(v^i) &= \big(f^*(v) - km(q-k)d(v)\big) + (q^2+2qk-2k^2)m\frac{d(v)}{2} \\ &= f^*(v) + \frac{1}{2}qmqd(v) = f^*(v) + \frac{1}{2}|E(G)| \deg(v). \end{aligned}$$

Therefore, g is a desired bijection. ∎

3. MAGIC GRAPHS

In this section we present some sufficient conditions for generalized double graphs $D(G;Z,\emptyset)$ to be degree-magic.

Theorem 1. *Let G be a degree-magic graph such that $\deg(v) \equiv 0 \pmod 2$ for every vertex $v \in V(G)$. If the subgraph of G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor, then the graph $D(G; Z, \emptyset)$ is balanced degree-magic.*

Proof. As G is a d-magic graph, there is a d-magic labelling f from $E(G)$ onto $\big[1, |E(G)|\big]$. According to Lemma 1, there exists a 2-gradual bijection $g : E(D(G; Z, \emptyset)) \longrightarrow \big[1, 2|E(G)|\big]$ satisfying

$$g^*(v^0) = g^*(v^1) = f^*(v) + \frac{1}{2}|E(G)| \deg(v),$$

for every vertex $v \in V(G)$. Since f is a d-magic labelling, $f^*(v) = (1 + |E(G)|)$ $\deg(v)/2$. Hence

$$\begin{aligned} g^*(v^0) = g^*(v^1) &= \frac{1}{2}(1 + |E(G)|) \deg(v) + \frac{1}{2}|E(G)| \deg(v) \\ &= \frac{1}{2}(1 + 2|E(G)|) \deg(v) = \frac{1}{2}(1 + |E(D(G; Z, \emptyset))|) \deg(v). \end{aligned}$$

Therefore, g is a 2-gradual d-magic labelling of $D(G; Z, \emptyset)$. ■

Combining Proposition 1 and Theorem 1 we immediately have

Corollary 2. *Let G be a supermagic regular graph of even degree. If the subgraph of G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor, then the graph $D(G; Z, \emptyset)$ is supermagic.*

Corollary 2 provides a copious method to construct supermagic graphs. For example, the complete graph K_7 is supermagic ([14]). One can see that K_7 contains 26 non-isomorphic subgraphs having a $\frac{1}{2}$-factor. By Corollary 2, the graph $D(K_7; E(H), \emptyset)$ is supermagic for each such subgraph H.

A totally disconnected graph has a $\frac{1}{2}$-factor and so we get

Corollary 3. *Let G be a supermagic regular graph of even degree. Then the bipartite double graph $D(G; \emptyset, \emptyset)$ of a graph G is supermagic.*

As the graph $2G$ is isomorphic to $D(G; E(G), \emptyset)$, we have the next corollary.

Corollary 4. ([7]) *Let G be a supermagic regular graph of degree $2d$ which has a d-factor. Then the graph $2G$ is supermagic.*

For double graphs we get the next corollary.

Corollary 5. *Let G be a graph having a $\frac{1}{2}$-factor. Then the double graph $D({}^2G; E_1({}^2G), \emptyset)$ of a graph G is balanced degree-magic.*

Proof. Let h be a bijection from $E(G)$ onto $\big[1, |E(G)|\big]$. Consider the mapping $f : E(^2G) \longrightarrow \big[1, 2|E(G)|\big]$ given by

$$f((e,j)) = \begin{cases} h(e) & \text{if } \ j = 1, \\ 1 + 2|E(G)| - h(e) & \text{if } \ j = 2. \end{cases}$$

Evidently, f is a bijection. Moreover, $f((e,1)) + f((e,2)) = 1 + 2|E(G)|$, for any edge $e \in E(G)$. Therefore,

$$f^*(v) = (1 + 2|E(G)|)\deg_G(v) = (1 + |E(^2G)|)\frac{\deg_{^2G}(v)}{2}.$$

Thus, f is a degree-magic labelling of ^2G. As the subgraph of ^2G induced by $E_1(^2G)$ is isomorphic to G, it contains a $\frac{1}{2}$-factor. By Theorem 1, $D(^2G; E_1(^2G), \emptyset)$ is a balanced d-magic graph. ∎

Combining Proposition 1 and Corollary 5 we immediately have

Corollary 6. *Let G be a regular graph having a $\frac{1}{2}$-factor. Then the double graph $D(^2G; E_1(^2G), \emptyset)$ of a graph G is supermagic.*

Theorem 1 can be only used for subsets Z of even cardinality. The following result can be used also for subsets of odd cardinality.

Theorem 2. *Let $q \geq 2$ be a positive integer and let G be a graph such that $\deg(v) \equiv 0 \pmod{2q}$ for every vertex $v \in V(G)$. Let Z be a subset of $E(G)$ such that $|E(G)|/q < |Z| < |E(G)|$. If G admits a q-gradual degree-magic labelling which respects Z, then the graph $D(G; Z, \emptyset)$ is degree-magic.*

Proof. Suppose that f is a q-gradual d-magic labelling of G which respects a set Z. According to Lemma 4, there exists a bijection g from $E(D(G; Z, \emptyset))$ onto $\big[1, 2|E(G)|\big]$ satisfying

$$g^*(v^0) = g^*(v^1) = f^*(v) + \frac{1}{2}|E(G)|\deg(v),$$

for every vertex $v \in V(G)$. Since f is a d-magic labelling, $f^*(v) = (1 + |E(G)|)\deg(v)/2$. Hence

$$\begin{aligned} g^*(v^0) = g^*(v^1) &= \frac{1}{2}(1 + |E(G)|)\deg(v) + \frac{1}{2}|E(G)|\deg(v) \\ &= \frac{1}{2}(1 + 2|E(G)|)\deg(v) = \frac{1}{2}(1 + |E(D(G; Z, \emptyset))|)\deg(v). \end{aligned}$$

Therefore, g is a d-magic labelling of $D(G; Z, \emptyset)$. ∎

For double graphs we have the following result.

Corollary 7. *Let G be a graph such that* $\deg(v) \equiv 0 \pmod 2$ *for every vertex $v \in V(G)$ and let $q \geq 2$ be a positive integer. If G can be decomposed into q pairwise edge-disjoint $\frac{1}{q}$-factors, then the double graph $D({}^2G; E_1({}^2G), \emptyset)$ of a graph G is degree-magic.*

Proof. If q is even, then the union of $q/2$ edge-disjoint $\frac{1}{q}$-factors induces a $\frac{1}{2}$-factor of G and the result follows from Corollary 5. Therefore, next we suppose that q is odd. Evidently, $\deg(v) \equiv 0 \pmod{2q}$ for every vertex v of G in this case. Let H_1, H_2, ..., H_q be pairwise edge-disjoint $\frac{1}{q}$-factors of a graph G. Put $m := |E(G)|/q$. Clearly, the subgraph H_i, $i \in [1, q]$, has m edges. Suppose that h_i is a bijection from $E(H_i)$ onto $[1, m]$, for $i \in [1, q]$. Consider the mapping $f : E({}^2G) \longrightarrow [1, 2qm]$ given by

$$f((e,j)) = \begin{cases} h_i(e) + (i-1)m & \text{if } j = 1 \text{ and } e \in E(H_i), \\ 1 + (1 + 2q - i)m - h_i(e) & \text{if } j = 2 \text{ and } e \in E(H_i). \end{cases}$$

Evidently, $f((e,1)) + f((e,2)) = 1 + 2qm$, for any edge $e \in E(G)$. Therefore,

$$f^*(v) = (1 + 2qm)\deg(v) = (1 + |E({}^2G)|)\frac{\deg_{{}^2G}(v)}{2}.$$

Moreover, for $i \in [1, q]$, we have

$$F_{2q}(f; i) = \{(e,1) \in E({}^2G) : e \in E(H_i)\} \quad \text{and}$$
$$F_{2q}(f; i+q) = \{(e,2) \in E({}^2G) : e \in E(H_{1+q-i})\}.$$

Thus, the mapping f is a $2q$-gradual degree-magic labelling of ${}^2G$ which respects the set $E_1({}^2G)$. According to Theorem 2, $D({}^2G; E_1({}^2G), \emptyset)$ is a d-magic graph. ■

As any regular graph of even degree d is decomposable into $d/2$ pairwise edge-disjoint 2-factors (i.e., $\frac{1}{d/2}$-factors), we immediately get

Corollary 8. *Let G be a regular graph of degree d, where $4 \leq d \equiv 0 \pmod 2$. Then the double graph $D({}^2G; E_1({}^2G), \emptyset)$ of a graph G is supermagic.*

4. Antimagic Graphs

Bodendiek and Walther [3] introduced the special case of antimagic graphs. For positive integers a, d, a graph G is said to be (a, d)-*antimagic* if it admits a bijection f from $E(G)$ onto $[1, |E(G)|]$ such that

$$\{f^*(v) : v \in V(G)\} = \{a, a+d, \ldots, a + (|V(G)| - 1)d\}.$$

The mapping f is then called an (a,d)-*antimagic labelling* of G. Obviously, $a = \frac{|E(G)|(|E(G)|+1)}{|V(G)|} - \frac{(|V(G)|-1)d}{2}$ in this case.

In this section we will deal with $(a,1)$-antimagic graphs and their connection with supermagic generalized double graphs.

There is known an effective method to construct an $(a',1)$-antimagic labelling of a supergraph of an $(a,1)$-antimagic graph (see [9]). The following assertion is a purpose overwriting of this method.

Lemma 5. *Let $H_1, H_2, \dots, H_q$ be pairwise edge-disjoint 2-factors which form a decomposition of a graph G. If H_1 is an $(a,1)$-antimagic graph, then there exists a q-gradual $(a',1)$-antimagic labelling f of G such that $F_q(f;i) = E(H_i)$ for each $i \in [1,q]$.*

Proof. For $k \in [1,q]$ we define a graph G_k by $V(G_k) = V(G)$ and $E(G_k) = \bigcup_{i=1}^{k} E(H_i)$. Evidently, G_k is a $2k$-regular graph, $G_1 = H_1$ and $G_q = G$. Put $n := |V(G)|$. Then $|E(H_k)| = n$ and $|E(G_k)| = kn$. Using induction on k we prove that there is a k-gradual $(a_k,1)$-antimagic labelling f_k of G_k such that $F_k(f_k;i) = E(H_i)$ for each $i \in [1,k]$.

If $k = 1$, then $G_1 = H_1$ is an $(a_1,1)$-antimagic graph and so there is a (1-gradual) $(a_1,1)$-antimagic labelling of G_1 such that $F_1(f_1;1) = E(H_1)$.

Now assume that there is a $(k-1)$-gradual $(a_{k-1},1)$-antimagic labelling f_{k-1} of G_{k-1} such that $F_{k-1}(f_{k-1};i) = E(H_i)$ for each $i \in [1,k-1]$. Let $\vec{H}_k$ be a digraph which we get from H_k by an orientation of its edges such that the outdegree of every vertex of $\vec{H}_k$ is equal to 1. By $[u,v]$ we denote an arc of $\vec{H}_k$ and by $A(\vec{H}_k)$ the set of all arcs of $\vec{H}_k$. Consider a mapping $f_k : E(G_k) \longrightarrow [1,kn]$ defined by

$$f_k(e) = \begin{cases} f_{k-1}(e) & \text{if } e \in E(G_{k-1}), \\ a_{k-1} + kn - f^*_{k-1}(u) & \text{if } e = uv \in E(H_k) \text{ and } [u,v] \in A(\vec{H}_k). \end{cases}$$

It is easy to see that f_k is a bijection and $f^*_k(v) = a_{k-1} + nk + f_k(uv)$, where $[u,v]$ is an arc of $\vec{H}_k$. As $\{f_k(e) : e \in E(H_k)\} = [(k-1)n+1, nk]$, the labelling f_k is $(a_k,1)$-antimagic, where $a_k = a_{k-1} + kn + (k-1)n + 1$. Moreover, $F_k(f_k;i) = F_{k-1}(f_{k-1};i) = E(H_i)$ for each $i \in [1,k-1]$ and $F_k(f_k;k) = E(G_k) - E(G_{k-1}) = E(H_k)$. ■

As any regular graph of even degree $2r$ is decomposable into r pairwise edge-disjoint 2-factors and the cycle C_n of odd order n is $(a,1)$-antimagic (see [4]), we immediately get

Corollary 9. *Every $2r$-regular Hamiltonian graph of odd order admits an r-gradual $(a,1)$-antimagic labelling.*

In [9] there is proved that the graph $G^{\bowtie} = D(G;\emptyset, V(G))$ is supermagic for every $(a,1)$-antimagic $2r$-regular graph G, and that the Cartesian product $G\Box K_2 = D(G;E(G),V(G))$ is supermagic for every $(a,1)$-antimagic $2r$-regular graph G with an r-factor. The following theorem generalizes these results.

Theorem 3. *Let G be an $(a,1)$-antimagic $2r$-regular graph. If the subgraph of G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor, then the graph $D(G;Z,V(G))$ is supermagic.*

Proof. Put $n := |V(G)|$. Since G is a $2r$-regular graph, $|E(G)| = rn$. As G is an $(a,1)$-antimagic graph, there is an $(a,1)$-antimagic labelling f from $E(G)$ onto $[1,rn]$. According to Lemma 1, there exists a bijection $g : E(D(G;Z,\emptyset)) \longrightarrow [1,2rn]$ satisfying

$$g^*(v^0) = g^*(v^1) = f^*(v) + r^2n,$$

for every vertex $v \in V(G)$. Since f is an $(a,1)$-antimagic labelling, the set $\{f^*(v) : v \in V(G)\}$ consists of consecutive integers. It means that the bijection $h : E(D(G;Z,V(G))) \longrightarrow [1,(2r+1)n]$, given by

$$h(e) = \begin{cases} g(e) & \text{if } e \in E(D(G;Z,\emptyset)), \\ (2r+1)n + a - f^*(v) & \text{if } e = v^0v^1 \text{ for } v \in V(G), \end{cases}$$

is a supermagic labelling of $D(G;Z,V(G))$. ■

One can see that Theorem 3 (similarly as, Corollary 2) provides a copious method to construct supermagic graphs.

In the same manner as above (using Lemma 2 instead of Lemma 1) we can prove the following result.

Theorem 4. *Let G be a $2r$-regular graph. If G admits an r-gradual $(a,1)$-antimagic labelling, then the graph $D(G;\emptyset,V(G))$ admits a $(2r+1)$-gradual supermagic labelling.*

Combining Corollary 9 and Theorem 4 we obtain

Corollary 10. *Let G be a $2r$-regular Hamiltonian graph of odd order. Then the graph $D(G;\emptyset,V(G))$ admits a $(2r+1)$-gradual supermagic labelling.*

The following assertion uses a gradual $(a,1)$-antimagic labelling.

Theorem 5. *Let G be a $2r$-regular graph, where $r \geq 2$. Let Z be a subset of $E(G)$ such that $|Z| > |E(G)|/r$. If G admits an r-gradual $(a,1)$-antimagic labelling which respects Z, then the graph $D(G;Z,V(G))$ is supermagic.*

Proof. Put $n := |V(G)|$. Clearly, $|E(G)| = rn$. Suppose that f is an r-gradual $(a, 1)$-antimagic labelling of G which respects a set Z.

According to Lemma 4 (if $|Z| < |E(G)|$) or Lemma 1 (if $|Z| = |E(G)|$ and r is even), there exists a bijection g_1 from $E(D(G; Z, \emptyset))$ onto $[1, 2rn]$ satisfying

$$g_1^*(v^0) = g_1^*(v^1) = f^*(v) + r^2 n,$$

for every vertex $v \in V(G)$. Since f is an $(a, 1)$-antimagic labelling, the set $\{f^*(v) : v \in V(G)\}$ consists of consecutive integers. It means that the bijection $h_1 : E(D(G; Z, V(G))) \longrightarrow [1, (2r + 1)n]$, given by

$$h_1(e) = \begin{cases} g_1(e) & \text{if } e \in E(D(G; Z, \emptyset)), \\ (2r + 1)n + a - f^*(v) & \text{if } e = v^0 v^1 \text{ for } v \in V(G), \end{cases}$$

is a supermagic labelling of $D(G; Z, V(G))$.

Finally, if $|Z| = |E(G)|$ and r is odd, then by Lemma 3 there is a bijection g_2 from $E(D(G; Z, \emptyset))$ onto $[1, rn] \cup [1 + (r + 1)n, (2r + 1)n]$ satisfying

$$g_2^*(v^0) = g_2^*(v^1) = f^*(v) + (r + 1)rn,$$

for every vertex $v \in V(G)$. Since f is an $(a, 1)$-antimagic labelling, the set $\{f^*(v) : v \in V(G)\}$ consists of consecutive integers. It means that the bijection $h_2 : E(D(G; Z, V(G))) \longrightarrow [1, (2r + 1)n]$, given by

$$h_2(e) = \begin{cases} g_2(e) & \text{if } e \in E(D(G; Z, \emptyset)), \\ (r + 1)n + a - f^*(v) & \text{if } e = v^0 v^1 \text{ for } v \in V(G), \end{cases}$$

is a supermagic labelling of $D(G; Z, V(G))$. ■

In [9] it is proved that the Cartesian product $G \square K_2 = D(G; E(G), V(G))$ is supermagic for every $4r$-regular Hamiltonian graph G of odd order. Combining Theorem 5 and Corollary 9, we have

Corollary 11. *Let G be a $2r$-regular Hamiltonian graph of odd order, where $r \geq 2$. Then the Cartesian product $G \square K_2$ is a supermagic graph.*

Let n, m and $1 \leq s_1 < \cdots < s_m \leq \left\lfloor \frac{n}{2} \right\rfloor$ be positive integers. A graph $C_n(s_1, \ldots, s_m)$ with the vertex set $\{v_0, \ldots, v_{n-1}\}$ and the edge set $\{v_i v_{i+s_j} : 0 \leq i \leq n - 1, 1 \leq j \leq m\}$, the indices are being taken modulo n, is called a *circulant graph.* It is easy to see that the circulant graph $C_n(s_1, \ldots, s_m)$ is a regular graph of degree r, where $r = 2m - 1$ when $s_m = n/2$, and $r = 2m$ otherwise. The circulant graph $C_n(s_1, \ldots, s_m)$ has $d = \gcd(s_1, \ldots, s_m, n)$ connected components (see [5]), which are isomorphic to $C_{n/d}(s_1/d, \ldots, s_m/d)$.

If n is odd, then $C_n(s_1)$, $C_n(s_2), \dots, C_n(s_m)$ are pairwise edge-disjoint 2-factors which form a decomposition of $C_n(s_1, \dots, s_m)$. Moreover, $C_n(s_i)$ is isomorphic to $dC_{n/d}$, where $d = \gcd(s_i, n)$ and n/d are odd integers. As odd number of copies of a cycle of odd order is an $(a, 1)$-antimagic graph (see [9]), the graph $C_n(s_i)$ is $(a, 1)$-antimagic.

Semaničová [12] proved that $C_{2k}(s_1, \dots, s_m, k)$ is not supermagic when k is even and that a 3-regular circulant graph $C_{2k}(s, k)$ is supermagic if and only if both of k and s are odd. Using Theorem 5 we get the following result.

Corollary 12. *Let m and $1 \le s_1 < \cdots < s_m$ be positive integers such that $|\{j \in [1, m] : s_j \equiv 0 \pmod 2\}| \ne 1$. Then $C_{2k}(s_1, \dots, s_m, k)$ is a supermagic graph for every odd integer $k > s_m$.*

Proof. Denote by v_0, $v_1, \dots, v_{2k-1}$ the vertices of $C_{2k}(s_1, \dots, s_m, k)$ and by u_0, $u_1, \dots, u_{k-1}$ the vertices of $C_k(1, 2, \dots, (k-1)/2)$. For every $i \in [1, m]$ put

$$(t_i, o_i) = \begin{cases} (s_i, 1) & \text{if } s_i < k/2, \\ (k - s_i, 2) & \text{if } s_i > k/2. \end{cases}$$

Evidently, the pairs (t_i, o_i), $i \in [1, m]$, are pairwise different.

For every $i \in [1, m]$ let H_i be a 2-factor of ${}^2C_k(1, 2, \dots, (k-1)/2)$ defined by $E(H_i) = \{(e, o_i) : e \in E(C_k(t_i)\}$. Clearly, H_i is isomorphic to $C_k(t_i)$ and so it is $(a, 1)$-antimagic. Let G be a $2m$-regular spanning subgraph of ${}^2C_k(1, 2, \dots, (k-1)/2)$ defined by $E(G) = \bigcup_{i=1}^m E(H_i)$. The graphs H_1, $H_2, \dots, H_m$ are pairwise edge-disjoint 2-factors which form a decomposition of G. According to Lemma 5, there exists an m-gradual $(a', 1)$-antimagic labelling f of G such that $F_m(f; i) = E(H_i)$ for each $i \in [1, m]$. Therefore, f respects the set $Z := \bigcup_{j \in S} E(H_j)$, where $S = \{j \in [1, m] : s_j \equiv 0 \pmod 2\}$. By Theorem 5, the graph $D(G; Z, V(G))$ is supermagic when $|S| > 1$. Similarly, by Theorem 4, the graph $D(G; Z, V(G))$ admits a $(2m+1)$-gradual supermagic labelling when $|S| = 0$.

Now consider the mapping φ from $\{v_0, v_1, \dots, v_{2k-1}\}$ onto $\bigcup_{i=0}^{k-1} \{u_i^0, u_i^1\}$ given by

$$\varphi(v_i) = \begin{cases} u_i^0 & \text{if } i < k \text{ and } i \equiv 0 \pmod 2, \\ u_i^1 & \text{if } i < k \text{ and } i \equiv 1 \pmod 2, \\ u_{i-k}^0 & \text{if } i \ge k \text{ and } i \equiv 0 \pmod 2, \\ u_{i-k}^1 & \text{if } i \ge k \text{ and } i \equiv 1 \pmod 2. \end{cases}$$

It is not difficult to check that φ is an isomorphism from $C_{2k}(s)$ to $D\big(C_k(s); \emptyset, \emptyset\big)$ when $s < k/2$ is an odd integer. Similarly, φ is an isomorphism from $C_{2k}(s)$ onto $D\big(C_k(k-s); \emptyset, \emptyset\big)$ when $s > k/2$ is an odd integer. If s is an even integer, then φ is an isomorphism from $C_{2k}(s)$ onto $D\big(C_k(s); E(C_k(s)), \emptyset\big)$ when $s < k/2$, or onto $D\big(C_k(k-s); E(C_k(k-s)), \emptyset\big)$ when $s > k/2$. Therefore, φ is an isomorphism from

$C_{2k}(s_1, \ldots, s_m, k)$ onto the supermagic graph $D(G; Z, V(G))$, which completes the proof. ∎

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