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# $\gamma$-CYCLES IN ARC-COLORED DIGRAPHS

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**Abstract**

We call a digraph $D$ an $m$-colored digraph if the arcs of $D$ are colored with $m$ colors. A directed path (or a directed cycle) is called monochromatic if all of its arcs are colored alike. A subdigraph $H$ in $D$ is called rainbow if all of its arcs have different colors. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of $D$ if it satisfies the two following conditions:

(i) for every pair of different vertices $u, v \in N$ there is no monochromatic path in $D$ between them, and

(ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an $xy$-monochromatic path in $D$.

A $\gamma$-cycle in $D$ is a sequence of different vertices $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ such that for every $i \in \{0, 1, \ldots, n\}$:

(i) there is a $u_i u_{i+1}$-monochromatic path, and

(ii) there is no $u_{i+1} u_i$-monochromatic path.

The addition over the indices of the vertices of $\gamma$ is taken modulo $(n+1)$. If $D$ is an $m$-colored digraph, then the closure of $D$, denoted by $\mathfrak{C}(D)$, is the $m$-colored multidigraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$, $A(\mathfrak{C}(D)) =$

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$A(D) \cup \{(u, v)$ with color $i \mid$ there exists a $uv$-monochromatic path colored $i$ contained in $D\}$.

In this work, we prove the following result. Let $D$ be a finite $m$-colored digraph which satisfies that there is a partition $C = C_1 \cup C_2$ of the set of colors of $D$ such that:

(1) $D[\widehat{C}_i]$ (the subdigraph spanned by the arcs with colors in $C_i$) contains no $\gamma$-cycles for $i \in \{1, 2\}$;

(2) If $\mathfrak{C}(D)$ contains a rainbow $C_3 = (x_0, z, w, x_0)$ involving colors of $C_1$ and $C_2$, then $(x_0, w) \in A(\mathfrak{C}(D))$ or $(z, x_0) \in A(\mathfrak{C}(D))$;

(3) If $\mathfrak{C}(D)$ contains a rainbow $P_3 = (u, z, w, x_0)$ involving colors of $C_1$ and $C_2$, then at least one of the following pairs of vertices is an arc in $\mathfrak{C}(D)$: $(u, w)$, $(w, u)$, $(x_0, u)$, $(u, x_0)$, $(x_0, w)$, $(z, u)$, $(z, x_0)$.

Then $D$ has a kernel by monochromatic paths.

This theorem can be applied to all those digraphs that contain no $\gamma$-cycles. Generalizations of many previous results are obtained as a direct consequence of this theorem.

**Keywords:** digraph, kernel, kernel by monochromatic paths, $\gamma$-cycle.

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## 1. Introduction

For general concepts we refer the reader to [1]. Let $D$ be a digraph, and let $V(D)$ and $A(D)$ denote the sets of vertices and arcs of $D$, respectively. We recall that a subdigraph $D_1$ of $D$ is a *spanning subdigraph* if $V(D_1) = V(D)$. If $S$ is a nonempty subset of $V(D)$, then the *subdigraph induced* by $S$, denoted by $D[S]$, is the digraph having vertex set $S$, and whose arcs are all those arcs of $D$ joining vertices of $S$. An arc $u_1u_2$ of $D$ will be called an $S_1S_2$-*arc* of $D$ whenever $u_1 \in S_1$ and $u_2 \in S_2$.

A set $I \subseteq V(D)$ is *independent* if $A(D[I]) = \emptyset$. A *kernel* $N$ of $D$ is an independent set of vertices such that for each $z \in V(D) - N$ there exists a $zN$-arc in $D$, that is, an arc from $z$ toward some vertex in $N$. A digraph $D$ is a *kernel-prefect digraph* when every induced subdigraph of $D$ has a kernel. Sufficient conditions for the existence of kernels in digraphs have been investigated by several authors, von Neumann and Morgenstern [17]; Richardson [18, 19]; Duchet and Meyniel [4]; Duchet [2, 3]; Galeana-Sánchez and Neumann-Lara [9, 10]. The concept of kernel is very useful in applications.

We call the digraph $D$ an *m-colored digraph* if the arcs of $D$ are colored with $m$ colors. Along this paper, all the paths and cycles will be directed paths and directed cycles. A path is *monochromatic* if all of its arcs are colored alike. A cycle is called a *quasi-monochromatic cycle* if with at most one exception all of its arcs are colored alike. A subdigraph $H$ of $D$ is *rainbow* if all its arcs have distinct colors. A set $N$ of vertices of $D$ is a *kernel by monochromatic paths* if for every pair of vertices of $N$ there is no monochromatic path between them and for every vertex $v$ not in $N$ there is a monochromatic path from $v$ to some vertex in $N$. If $D$ is an $m$-colored digraph, then the *closure* of $D$, denoted by $\mathfrak{C}(D)$, is the $m$-colored multidigraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$, $A(\mathfrak{C}(D)) = A(D) \cup \{(u,v)$ with color $i \mid$ there exists a $uv$-monochromatic path colored $i$ contained in $D\}$. Notice that for any digraph $D$, $\mathfrak{C}(\mathfrak{C}(D)) \cong \mathfrak{C}(D)$, and $D$ has a kernel by monochromatic paths if and only if $\mathfrak{C}(D)$ has a kernel.

In [22] Sands, Sauer and Woodrow proved that any 2-colored digraph $D$ has a set $S$ of vertices such that: (i) for any $x, y \in S$, there is no monochromatic path between them, and (ii) for every vertex $x \notin S$, there is a monochromatic path from $x$ to a vertex of S (i.e., $D$ has a kernel by monochromatic paths, a concept that was introduced later by Galeana-Sánchez [5]). In particular, they proved that any 2-colored tournament $T$ has a kernel by monochromatic paths. They also raised the following problem: Let $T$ be a 3-colored tournament such that every cycle of length 3 is a quasi-monochromatic cycle; must $T$ have a kernel by monochromatic paths? This question still remains open. In [21] Shen Minggang proved that if $T$ is an $m$-colored tournament such that every cycle of length 3 is a quasi-monochromatic cycle, and every transitive tournament of order 3 is quasi-monochromatic, then $T$ has a kernel by monochromatic paths. He also proved that this result is the best possible for $m$-colored tournaments with $m \geq 5$. In fact, he proved that for each $m \geq 5$ there exists an $m$-colored tournament $T$ such that every cycle of length 3 is quasi-monochromatic and $T$ has no kernel by monochromatic paths. Also for every $m \geq 3$ there exists an $m$-colored tournament $T'$ such that every transitive tournament of order 3 is quasi-monochromatic and $T'$ has no kernel by monochromatic paths. In 2004 [11] Galeana-Sánchez and Rojas-Monroy presented a 4-colored tournament $T$ such that every cycle of order 3 is quasi-monochromatic, but $T$ has no kernel by monochromatic paths. The known sufficient conditions for the existence of kernel by monochromatic paths in $m$-colored ($m \geq 3$) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of certain subdigraphs. More information on $m$-colored digraphs can be found in [5, 6, 7, 23, 24].

If $\mathcal{C} = (z_0, z_1, \ldots, z_n, z_0)$ is a cycle, we will denote by $\ell(\mathcal{C})$ its length, and if $z_i, z_j \in V(\mathcal{C})$ with $i \leq j$, then we denote by $(z_i, \mathcal{C}, z_j)$ the $z_i z_j$-path contained in $\mathcal{C}$. A sequence of different vertices $\gamma = (u_0, \ldots, u_n, u_0)$ is a *$\gamma$-cycle* if for every $i \in \{0, 1, \ldots, n\}$ there is a $u_i u_{i+1}$-monochromatic path, and there is no $u_{i+1} u_i$-monochromatic path. The addition over the indices of the vertices of $\gamma$ is taken modulo $(n+1)$.

In this paper we prove that if $D$ is a finite $m$-colored digraph, and if there exists a partition $C = C_1 \cup C_2$ of the set of colors of $D$ such that:

(1) $D[\widehat{C}_i]$ contains no $\gamma$-cycles for $i \in \{1, 2\}$, ($\widehat{C}_i$ denotes the set of arcs of $D$ with colors in $C_i$);

(2) If $\mathfrak{C}(D)$ contains a rainbow $C_3 = (x_0, z, w, x_0)$ involving colors of $C_1$ and $C_2$, then $(x_0, w) \in A(\mathfrak{C}(D))$ or $(z, x_0) \in A(\mathfrak{C}(D))$;

(3) If $\mathfrak{C}(D)$ contains a rainbow $P_3 = (u, z, w, x_0)$ involving colors of $C_1$ and $C_2$, then at least one of the following pairs of vertices is an arc in $\mathfrak{C}(D)$: $(u, w)$, $(w, u)$, $(x_0, u)$, $(u, x_0)$, $(x_0, w)$, $(z, u)$, $(z, x_0)$.

Then $D$ has a kernel by monochromatic paths.

We will need the following results.

**Assertion 1.1.** *Let $D$ be a finite or infinite digraph and $u, v \in V(D)$. Every $uv$-walk in $D$ contains a $uv$-path.*

**Assertion 1.2.** *Let $D$ be a finite or infinite digraph. Every closed walk in $D$ contains a cycle.*

**Assertion 1.3.** *Let $D$ be a finite digraph. If every vertex $v \in V(D)$ fulfills that $\delta_D^-(v) \geq 1$ ($\delta_D^+(v) \geq 1$), then $D$ contains a cycle.*

**Theorem 1.4** (Duchet [2]). *If $D$ is a finite digraph such that every cycle of $D$ has at least one symmetrical arc, then $D$ has a kernel.*

**Theorem 1.5** (Rojas-Monroy, Villarreal-Valdés [20]). *Let $D$ be a finite or infinite digraph. If every cycle and every infinite outward path has a symmetrical arc, then there exists $x \in V(D)$ which satisfies $(x, u) \in A(D)$ implies $(u, x) \in A(D)$.*

The following lemma has been important to obtain many results on the existence of kernels by monochromatic paths in finite $m$-colored digraphs [5, 6, 8, 12, 13, 14, 15, 16].

**Lemma 1.6.** *Let $D$ be a finite or infinite $m$-colored digraph and $\mathfrak{C}(D)$ its closure. Then $D$ contains no $\gamma$-cycles if and only if every cycle in $\mathfrak{C}(D)$ has at least one symmetrical arc.*

It follows from Lemma 1.6 and Theorem 1.5 that if $D$ is a finite $m$-colored digraph which contains no $\gamma$-cycles, then $D$ has a kernel by monochromatic paths.

## 2. $\gamma$-Cycles and Monochromatic Paths in Arc-Colored Digraphs

The following three lemmas are about $m$-colored digraphs containing no $\gamma$-cycles, and they are useful to prove our main result.

**Lemma 2.1.** *Let $D$ be a finite $m$-colored digraph, and suppose that $D$ contains no $\gamma$-cycles. There exists $x_0 \in V(D)$ such that for every $z \in V(D) - \{x_0\}$ if there exists an $x_0z$-monochromatic path contained in $D$, then there exists a $zx_0$-monochromatic path contained in $D$.*

***Proof.*** Assume, for a contradiction, that $D$ is a digraph as in the hypothesis of the Lemma 2.1, and that there is no vertex $x_0$ satisfying the assertion of Lemma 2.1.

Let $x_0 \in V(D)$, it follows from our assumptions that there is $x_1 \in V(D) - \{x_0\}$ such that there is an $x_0x_1$-monochromatic path contained in $D$ and there is no $x_1x_0$-monochromatic path contained in $D$. Again from our assumptions there is $x_2 \in V(D) - \{x_1\}$ such that there is an $x_1x_2$-monochromatic path contained in $D$ and there is no $x_2x_1$-monochromatic path contained in $D$. Once chosen $x_0, x_1, \ldots, x_n$; given our supposition we can choose $x_{n+1} \in V(D) - \{x_n\}$ in such a way that there is an $x_nx_{n+1}$-monochromatic path in $D$ and there is no $x_{n+1}x_n$-monochromatic path in $D$. Thus, we obtain a sequence of vertices $(x_0, x_1, x_2, x_3, \ldots)$ such that for every $i \in \{0, 1, 2, \ldots\}$ there is an $x_ix_{i+1}$-monochromatic path contained in $D$ and there is no $x_{i+1}x_i$-monochromatic path contained in $D$. Since $D$ is a finite digraph, there is $\{i, j\} \subseteq \mathbb{N} \cup \{0\}$ with $i < j$ such that $x_j = x_i$. Let $j_0 = \min\{j \mid x_j = x_i \text{ for some } i < j\}$, and let $i_0 \in \{0, 1, \ldots, j_0 - 1\}$ such that $x_{i_0} = x_{j_0}$ (notice that $i_0$ is unique because of the definition of $j_0$). Without loss of generality suppose that $i_0 = 0$ and $j_0 = n$. Thus, $C = (x_0, x_1, \ldots, x_{n-1}, x_n = x_0)$ is a sequence of $n$ different vertices such that for every $i \in \{0, \ldots, n-1\}$ there is an $x_ix_{i+1}$-monochromatic path contained in $D$ and there is no $x_{i+1}x_i$-monochromatic path contained in $D$ (the indices of the vertices will be taken modulo $n$). Therefore, $C = (x_0, x_1, \ldots, x_{n-1}, x_n = x_0)$ is a $\gamma$-cycle, which contradicts the hypothesis. ∎

Let $D$ be an $m$-colored digraph and let $H$ be a subdigraph of $D$. We will say that $S \subseteq V(D)$ is a *semikernel by monochromatic paths* modulo $H$ of $D$ if $S$ is independent by monochromatic paths in $D$ and for every $z \in V(D) - S$, if there is a $Sz$-monochromatic path contained in $D - H$, then there is a $zS$-monochromatic path contained in $D$.

**Lemma 2.2.** *Let $D$ be a finite $m$-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of $D$ such that $D[\widehat{C}_1]$ contains no $\gamma$-cycles. Then there exists $x_0 \in V(D)$ such that $\{x_0\}$ is a semikernel by monochromatic paths* (mod $D[\widehat{C}_2]$) *of $D$.*

***Proof.*** It follows by applying Lemma 2.1 to $D - \widehat{C}_2$. ∎

Let $D$ be a finite $m$-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of $D$ and $D[\widehat{C}_1]$ contains no $\gamma$-cycles.

Denote by

$\mathcal{S} = \{S \mid S \neq \emptyset$ and $S$ is a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$) of $D\}$.

Notice that by Lemma 2.2, there exists a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$) of $D$, and thus $\mathcal{S} \neq \emptyset$.

Whenever $\mathcal{S} \neq \emptyset$, we will denote by $D_{\mathcal{S}}$ the loopless digraph defined as follows:

(1) $V(D_{\mathcal{S}}) = \mathcal{S}$ (i.e, for every element of $\mathcal{S}$ we put a vertex in $D_{\mathcal{S}}$), and

(2) $(S_1, S_2) \in A(D_{\mathcal{S}})$ if and only if for every $s_1 \in S_1$ there exists $s_2 \in S_2$ such that $s_1 = s_2$ or there exists an $s_1 s_2$-monochromatic path contained in $D[\widehat{C}_2]$ and there is no $s_2 S_1$-monochromatic path contained in $D$.

**Lemma 2.3.** *Let $D$ be a finite $m$-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of $D$ and $D[\widehat{C}_i]$ contains no $\gamma$-cycles for $i \in \{1, 2\}$. Then $D_{\mathcal{S}}$ is an acyclic digraph.*

***Proof.*** Observe that by Lemma 2.2, there exists a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$) of $D$ and therefore $\mathcal{S} \neq \emptyset$. Thus, we can consider the digraph $D_{\mathcal{S}}$. Suppose, for a contradiction, that the digraph $D_{\mathcal{S}}$ contains some cycle, say $\mathcal{C} = (S_0, S_1, \ldots, S_{n-1}, S_0)$ of length $n \geq 2$. Since $\mathcal{C}$ is a cycle in $D_{\mathcal{S}}$, we have that $S_i \neq S_j$ whenever $i \neq j$.

**Claim 1.** *There exists $i_0 \in \{0, 1, 2, \ldots, n-1\}$ such that for some $z \in S_{i_0}$, $z \notin S_{i_0+1}$ (mod $n$).*

***Proof.*** Otherwise, for every $i \in \{0, 1, \ldots, n-1\}$ and every $z \in S_i$ we have that $z \in S_{i+1}$, and then $S_i = S_j$ for all $i, j \in \{0, 1, \ldots, n-1\}$. So, $\mathcal{C} = (S_0)$, which is a contradiction, since the digraph is loopless. □

**Claim 2.** *If there exists $i_0 \in \{0, 1, \ldots, n-1\}$ such that for some $z \in S_{i_0}$ and some $w \in S_{i_0+1}$ (mod $n$) there exists a $zw$-monochromatic path, then there exists $j_0 \neq i_0$, $j_0 \in \{0, 1, \ldots, n-1\}$, such that $w \in S_{j_0}$ and $w \notin S_{j_0+1}$(mod $n$).*

***Proof.*** Suppose without loss of generality that $i_0 = 0$. First, observe that $w \notin S_n = S_0$, since otherwise we have a $zw$-monochromatic path with $\{z, w\} \subseteq S_0$, contradicting that $S_0$ is independent by monochromatic paths. Since $w \in S_1$, let $j_0 = \max\{i \in \{0, 1, \ldots, n-1\} \mid w \in S_i\}$ (notice that for both previous observations $j_0$ is well defined). So, $w \in S_{j_0}$ and $w \notin S_{j_0+1}$. □

It follows from Claim 1 that there exist $i_0 \in \{0, \ldots, n-1\}$ and $t_0 \in S_{i_0}$ such that $t_0 \notin S_{i_0+1}$. It follows from the fact that $(S_{i_0}, S_{i_0+1}) \in A(D_{\mathcal{S}})$ that there exists $t_1 \in S_{i_0+1}$ such that there exists a $t_0 t_1$-monochromatic path contained in $D[\widehat{C}_2]$ and there is no $t_1 S_{i_0}$-monochromatic path contained in $D$. From Claim 2, it follows that there exists an index $i_1 \in \{0,\ldots,n-1\}$ such that $t_1 \in S_{i_1}$ and $t_1 \notin S_{i_1+1}$. Since $(S_{i_1}, S_{i_1+1}) \in A(D_{\mathcal{S}})$ it follows that there exists $t_2 \in S_{i_1+1}$ such that there is a $t_1t_2$-monochromatic path contained in $D[\widehat{C}_2]$ and there is no $t_2S_{i_1}$-monochromatic path contained in $D$. Since $D$ is finite, we obtain a sequence of vertices $(t_0, t_1, t_2, \ldots, t_{m-1}, t_0)$ such that there exists a $t_it_{i+1}$-monochromatic path contained in $D[\widehat{C}_2]$ and there is no $t_{i+1}t_i$-monochromatic path contained in $D$ for each $i \in \{0,1,2,\ldots,m-1\}$ (mod $m$). But this contradicts that $D[\widehat{C}_2]$ contains no $\gamma$-cycles. ∎

## 3. The Main Result

The idea of the proof of our main theorem is to select $S \in V(D_{\mathcal{S}})$ such that $\delta^+_{D_{\mathcal{S}}}(S) = 0$ (such $S$ exists since $D_{\mathcal{S}}$ is acyclic) and prove that $S$ is a kernel by monochromatic paths of $D$.

**Theorem 3.1.** *Let $D$ be a finite $m$-colored digraph. If there exists a partition $C = C_1 \cup C_2$ of the set of colors of $D$ such that:*

(1) *$D[\widehat{C}_i]$ contains no $\gamma$-cycles for $i \in \{1,2\}$;*

(2) *If $\mathfrak{C}(D)$ contains a rainbow $C_3 = (x_0, z, w, x_0)$ involving colors of $C_1$ and $C_2$, then $(x_0, w) \in A(\mathfrak{C}(D))$ or $(z, x_0) \in A(\mathfrak{C}(D))$;*

(3) *If $\mathfrak{C}(D)$ contains a rainbow $P_3 = (u, z, w, x_0)$ involving colors of $C_1$ and $C_2$, then at least one of the following pairs of vertices is an arc in $\mathfrak{C}(D)$: $(u,w)$, $(w,u)$, $(x_0,u)$, $(u,x_0)$, $(x_0,w)$, $(z,u)$, $(z,x_0)$.*

*Then $D$ has a kernel by monochromatic paths.*

***Proof.*** Consider the digraph $D_{\mathcal{S}}$ of the digraph $D$. Since $D_{\mathcal{S}}$ is a finite digraph, and from Lemma 2.3 it contains no cycles, it follows that $D_{\mathcal{S}}$ has at least one vertex of zero outdegree. Let $S \in V(D_{\mathcal{S}})$ be such that $\delta^+_{D_{\mathcal{S}}}(S) = 0$. We will prove that $S$ is a kernel by monochromatic paths of $D$.

Suppose, for a contradiction, that $S$ is not a kernel by monochromatic paths of $D$. Since $S \in V(D_{\mathcal{S}})$, we have that $S$ is independent by monochromatic paths.

Let $X = \{z \in V(D) \mid$ there is no $zS$-monochromatic path in $D\}$. It follows from our assumption that $X \neq \emptyset$. Consider $D - \widehat{C}_2$ and its closure $\mathfrak{C}(D - \widehat{C}_2)$. Note that $D[\widehat{C}_1]$ is a subdigraph of $D - \widehat{C}_2$ which satisfies $A(D[\widehat{C}_1]) = A(D - \widehat{C}_2)$. Since $D[\widehat{C}_1]$ contains no $\gamma$-cycles, we have that $D - \widehat{C}_2$ contains no $\gamma$-cycles either. Lemma 1.6 implies that every cycle in $\mathfrak{C}(D - \widehat{C}_2)$ has at least one symmetrical arc. Let $H = \mathfrak{C}(D - \widehat{C}_2)[X]$ be the subdigraph of $\mathfrak{C}(D - \widehat{C}_2)$ induced by $X$. We have that $H$ also satisfies that every cycle has at least one symmetrical arc, by Theorem 1.5 there is a vertex $x_0$ which satisfies that $(x_0, u) \in A(H)$ implies $(u, x_0) \in A(H)$.

Let $T = \{z \in S \mid$ there is no $zx_0$-monochromatic path in $D[\widehat{C}_2]\}$. From the definition of $T$, we have that for every $z \in (S - T)$ there exists a $zx_0$-monochromatic path contained in $D[\widehat{C}_2]$.

**Claim 1.** *$T \cup \{x_0\}$ is independent by monochromatic paths.*

***Proof.*** Since $T \subseteq S$ with $S \in \mathcal{S}$ and $x_0 \in X$, it remains to prove that there is no $wx_0$-monochromatic path in $D[\widehat{C}_1]$ for $w \in T$. Suppose that such path there exists. Since $S$ is a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$), there is an $x_0S$-monochromatic path in $D$, but this is a contradiction with the definition of $X$. □

**Claim 2.** *For each $z \in V(D) - (T \cup \{x_0\})$, if there exists a $(T \cup \{x_0\})z$-monochromatic path contained in $D[\widehat{C}_1]$, then there exists a $z(T \cup \{x_0\})$-monochromatic path contained in $D$.*

***Proof.*** *Case* 1. There exists a $Tz$-monochromatic path contained in $D[\widehat{C}_1]$. Since $T \subseteq S$ and $S \in \mathcal{S}$, it follows that there exists a $zS$-monochromatic path contained in $D$. We may suppose that there exists a $z(S-T)$-monochromatic path contained in $D$ (otherwise we are done). Let $\alpha_1$ be a $uz$-monochromatic path contained in $D[\widehat{C}_1]$ with $u \in T$, and let $\alpha_2$ be a $zw$-monochromatic path with $w \in (S - T)$ contained in $D$. Since $w \in (S - T)$, it follows from the definition of $T$ that there exists a $wx_0$-monochromatic path $\alpha_3$ contained in $D[\widehat{C}_2]$.

Moreover, color$(\alpha_1) \neq$ color$(\alpha_2)$ (color$(\alpha)$ denotes the color used in the arcs of $\alpha$), otherwise there exists a $uw$-monochromatic path contained in $\alpha_1 \cup \alpha_2$, with $\{u, w\} \subseteq S$, in contradiction with the fact that $S$ is independent by monochromatic paths. In addition, we will suppose that color$(\alpha_2) \neq$ color$(\alpha_3)$, since when color$(\alpha_2) =$ color$(\alpha_3)$ we have $\alpha_2 \cup \alpha_3$ contains a $zx_0$-monochromatic path and Claim 2 is proved. Also color$(\alpha_1) \neq$ color$(\alpha_3)$ as color$(\alpha_1) \in C_1$ and color$(\alpha_3) \in C_2$.

So, we obtain that $(u, z, w, x_0)$ is a rainbow $P_3$ in $\mathfrak{C}(D)$ involving colors of both $C_1$ and $C_2$, and by the hypothesis there exists at least one of the following monochromatic paths in $D$: from $u$ to $w$; from $w$ to $u$; from $x_0$ to $u$; from $u$ to $x_0$; from $x_0$ to $w$; from $z$ to $u$; from $z$ to $x_0$. If there exists a $zu$-monochromatic path or a $zx_0$-monochromatic path in $D$, then Claim 2 is proved. So, we will demonstrate that is not possible the existence of the other paths.

(i) There is no $uw$-monochromatic path in $D$, since $\{u, w\} \subseteq S$ and $S$ is a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$) of $D$.

(ii) There is no $wu$-monochromatic path in $D$, (the same reason as in (i)).

(iii) There is no $x_0u$-monochromatic path in $D$ as $T \cup \{x_0\}$ is independent by monochromatic paths in $D$.

(iv) There is no $ux_0$-monochromatic path in $D$ (the same reason as in (iii)).

(v) There is no $x_0w$-monochromatic path in $D$, since $x_0 \in X$ and $w \in S$.

*Case* 2. There exists an $x_0z$-monochromatic path contained in $D[\widehat{C}_1]$. Let $\alpha_1$ be such a path. Suppose that $z \in X$, then $(x_0, z)$ is an arc in $H$ (recall $H = \mathfrak{C}(D - \widehat{C}_2)[X]$). The choice of $x_0$ implies that $(z, x_0) \in A(H)$. By the definition of the closure of an $m$-colored digraph and the fact that $H$ is an induced subdigraph of $\mathfrak{C}(D - \widehat{C}_2)$ we conclude that there is a $zx_0$-monochromatic path in $D - \widehat{C}_2$, and this path is a $zx_0$-monochromatic path in $D$. Now, assume that $z \notin X$. It follows from the definition of $X$ that there exists some $zS$-monochromatic path contained in $D$, let $\alpha_2$ be such a path, say that $\alpha_2$ ends in $w$. We will suppose that $w \in (S - T)$. Since $w \in (S - T)$, by the definition of $T$, we have that there exists a $wx_0$-monochromatic path contained in $D[\widehat{C}_2]$, let $\alpha_3$ be such a path.

Again, we have that color$(\alpha_1) \neq$ color$(\alpha_2)$, otherwise there exists an $x_0w$-monochromatic path contained in $D$, contradicting that $x_0 \in X$ and $w \in S$. In addition, we may suppose that color$(\alpha_2) \neq$ color$(\alpha_3)$, since if color$(\alpha_2) =$ color$(\alpha_3)$, then $D$ contains a $zx_0$-monochromatic path and Claim 2 is proved. Also color$(\alpha_1) \neq$ color$(\alpha_3)$, since $\alpha_1 \subseteq D[\widehat{C}_1]$ and $\alpha_3 \subseteq D[\widehat{C}_2]$.

Then $(x_0, z, w, x_0)$ is a rainbow $C_3$ in $\mathfrak{C}(D)$ which involves colors of both $C_1$ and $C_2$, and from hypothesis there exist an $x_0w$-monochromatic path or a $zx_0$-monochromatic path in $D$. Since $x_0 \in X$ and $w \in S$, it follows directly from the definitions of $X$ and $S$ that there is no $x_0w$-monochromatic path in $D$. Then there is a $zx_0$-monochromatic path in $D$, and Claim 2 is proved. □

We conclude from Claims 1 and 2 that $T \cup \{x_0\} \in \mathcal{S}$ and therefore $T \cup \{x_0\} \in V(D_{\mathcal{S}})$. We have that $(S, T \cup \{x_0\}) \in A(D_{\mathcal{S}})$, since $T \subsetneq T \cup \{x_0\}$, and for each $s \in S - T$ there exists an $sx_0$-monochromatic path contained in $D[\widehat{C}_2]$, and there is no $x_0S$-monochromatic path contained in $D$. But this contradicts the fact that $\delta^+_{D_{\mathcal{S}}}(S) = 0$. Therefore $S$ is a kernel by monochromatic paths in $D$ and Theorem 3.1 is proved. ∎

**Remark 3.2.** Theorem 3.1 can be applied to all those digraphs that contain no $\gamma$-cycles. Generalizations of many previous results are obtained as a direct consequence of this theorem.

Now, we give some definitions and next we give a list of digraphs that contains no $\gamma$-cycles.

**Definition.** A digraph $D$ is *n-quasitransitive* if for every $\{u, v\} \subseteq V(D)$ such that there is a $uv$-directed path of length $n$, we have $(u, v) \in A(D)$ or $(v, u) \in A(D)$.

**Definition.** We denote by $A^+(u)$ the set of arcs of $D$ that have $u$ as the initial end-point, and $A^+(u)$ is *monochromatic* if all of its elements have the same color.

**Definition.** We denote by $T_4$ the digraph such that $V(T_4) = \{u, v, w, x\}$ and $A(T_4) = \{(u, v), (v, x), (x, w), (u, w)\}$, see Figure 1.

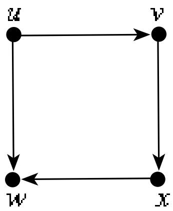


Figure 1. $T_4$.

**Definition.** A digraph $D$ is called a *bipartite tournament* if its set of vertices can be partitioned into two sets $V_1$ and $V_2$ such that: (i) every arc of $D$ has an end-point in $V_1$, and the other end-point in $V_2$, and (ii) for every $x_1 \in V_1$ and every $x_2 \in V_2$, we have $|\{(x_1, x_2), (x_2, x_1)\} \cap A(D)| = 1$.

**Definition.** $\widetilde{T}_6$ is the bipartite tournament defined as follows:

1. $V(\widetilde{T}_6) = \{u, v, w, x, y, z\}$,
2. $A(\widetilde{T}_6) = \{(u, w), (v, w), (w, x), (w, z), (x, y), (y, u), (y, v), (z, y)\}$,

with $\{(u, w), (w, x), (y, u), (z, y)\}$ coloured 1 and $\{(v, w), (w, z), (x, y), (y, v)\}$ colored 2, see Figure 2.

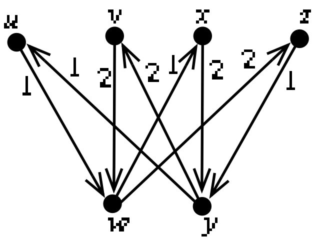


Figure 2. $\widetilde{T}_6$.

**Definition.** If $v$ is a vertex of an $m$-coloured tournament $T$, we denote by $\xi(v)$ the set of colours assigned to the arcs with $v$ as an end-point.

**Definition.** $\widetilde{T}_8$ is the digraph defined as follows:

1. $V(\widetilde{T}_8) = \{s, t, u, v, w, x, y, z\}$,
2. $A(\widetilde{T}_8) = \{(s, t), (s, x), (t, u), (t, y), (u, v), (u, z), (v, w), (v, s), (w, x), (w, t), (x, y), (x, u), (y, z), (y, v), (z, s), (z, w)\}$,

and each other arc in $\widetilde{T}_8$ colored 2, see Figure 3.

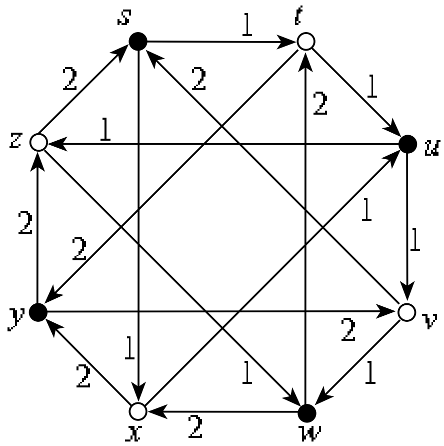


Figure 3. $\widetilde{T_8}$.

A list of theorems proving the existence of digraphs without $\gamma$-cycles.

**Theorem 3.3** (Galeana-Sánchez, Gaytán-Gómez, Rojas-Monroy [8]). *Let $D$ be a finite $m$-colored digraph such that every cycle in $D$ is monochromatic. Then $D$ contains no $\gamma$-cycle.*

**Theorem 3.4** ( Galeana-Sánchez, Rojas-Monroy, Zavala [16]). *Let $D$ be a finite $m$-colored 3-quasitransitive digraph such that for every vertex $u$ of $D$, $A^+(u)$ is monochromatic. If every $C_3$, $C_4$ and $T_4$ contained in $D$ is quasi-monochromatic, then there is no $\gamma$-cycles in $D$.*

**Theorem 3.5** ( Galeana-Sánchez [5]). *Let $T$ be a finite $m$-colored tournament. If each directed cycle contained in $T$ and of length at most 4 is a quasi-monochromatic cycle, then there is no $\gamma$-cycles in $T$.*

**Theorem 3.6** ( Galeana-Sánchez [6]). *Let $D$ be a finite $m$-colored digraph resulting from the deletion of a single arc $(x, y)$ of some $m$-colored tournament $T$ (i.e., $D \cong T - (x, y)$). If every directed cycle contained in $D$ of length at most 4 is quasi-monochromatic, then there is no $\gamma$-cycles in $D$.*

**Theorem 3.7** (Galeana-Sánchez and Rojas-Monroy [14]). *Let $T$ be a finite $m$-colored bipartite tournament. Assume that every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic, and $T$ has no subtournament isomorphic to $\widetilde{T_6}$. Then there is no $\gamma$-cycles in $T$.*

**Theorem 3.8** (Galeana-Sánchez and Rojas-Monroy [12]). *Let $T$ be a finite $m$-colored bipartite tournament. If every directed cycle of length 4 in $T$ is monochromatic, then there is no $\gamma$-cycles in $T$.*

**Theorem 3.9** (Galeana-Sánchez and Rojas-Monroy [13]). *Let $T$ be a finite 3-colored tournament such that every directed cycle of length 3 is quasi-monochromatic, and for each $v \in V(T)$ we have $|\xi(v)| \leq 2$, then there is no $\gamma$-cycles in $T$.*

**Theorem 3.10** (Galeana-Sánchez and Rojas-Monroy [15]). *Let $T$ be a finite $m$-colored bipartite tournament such that, every $C_4$ is quasi-monochromatic, every $T_4$ is quasi-monochromatic, and $T$ has no induced subdigraph isomorphic to $\widetilde{T}_8$. Then $T$ has no $\gamma$-cycles.*

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