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LIGHT GRAPHS IN PLANAR GRAPHS OF LARGE GIRTH

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Abstract

A graph H is defined to be light in a graph family $\mathcal{G}$ if there exist finite numbers $\varphi(H,\mathcal{G})$ and $w(H,\mathcal{G})$ such that each $G \in \mathcal{G}$ which contains H as a subgraph, also contains its isomorphic copy K with $\Delta_G(K) \leq \varphi(H,\mathcal{G})$ and $\sum_{x\in V(K)} \deg_G(x) \leq w(H,\mathcal{G})$. In this paper, we investigate light graphs in families of plane graphs of minimum degree 2 with prescribed girth and no adjacent 2-vertices, specifying several necessary conditions for their lightness and providing sharp bounds on φ and w for light $K_{1,3}$ and C_{10}.

Keywords: planar graph, girth, light graph.

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1. Introduction

Throughout this paper, we consider connected plane graphs without loops or multiple edges. We use the standard graph terminology by [12]. By C_k and S_k we denote a k-vertex cycle (also called k-cycle) and a k-star $K_{1,k}$, respectively. The *facial walk* of a face α is the shortest closed walk containing all edges incident with α. A k-vertex is a vertex of degree k, and a k^+-vertex (k^--vertex) is a vertex of degree at least (at most) k; similarly, a k-face (or k^+-face) is a face having a facial walk of length k (or at least k). A $(k_1, k_2, \dots, k_n)$-path is an n-vertex path consisting of k_1-vertex, k_2-vertex, $\dots$, k_n-vertex (in this order from an endvertex). The *weight* of a subgraph H in a graph G is the sum $\sum_{x\in V(H)} \deg_G(x)$.

The research of the structure of plane graphs has a long and fruitful history tracking, in one direction, the use of the obtained knowledge in various graph colourings (the most prominent example being the proof of the Four Colour Theorem, see [1] or, more recently, [11]) and, in another direction, the interest coming from pure theoretical motivation (originated, in systematic way, from the classical paper by Lebesgue [10]). Various known results of the latter approach (see, for example [2, 9]) were subsequently described in a unified way by the following formalization.

Let $\mathcal{G}$ be a family of graphs and let H be a connected graph contained, as a subgraph, in infinitely many members of $\mathcal{G}$. Let $w(H,\mathcal{G})$ and $\varphi(H,\mathcal{G})$ be the smallest integers with the property that each graph $G \in \mathcal{G}$ which contains H as a subgraph, contains also its isomorphic copy K such that $\sum_{x\in V(K)} \deg_G(x) \le w(H,\mathcal{G})$ and, for each vertex $x \in V(K)$, $\deg_G(x) \le \varphi(H,\mathcal{G})$ holds; if such finite integers do not exist (note that $w(H,\mathcal{G}) \le |V(H)| \cdot \varphi(H,\mathcal{G})$ and $\varphi(H,\mathcal{G}) \le w(H,\mathcal{G}) - \delta(\mathcal{G}) \cdot (|V(H)| - 1)$ where $\delta(\mathcal{G}) = \min\{\delta(G) : G \in \mathcal{G}\}$; thus, these numbers are both finite or both infinite), we set $\varphi(H,\mathcal{G}) = w(H,\mathcal{G}) := +\infty$ and we say that H is *heavy* in $\mathcal{G}$, and *light* in $\mathcal{G}$ otherwise.

The state of the art of results on light graphs in various families of plane graphs is summarized in the survey article [8]. Most of them concern the families of polyhedral graphs or plane graphs constrained by lower bound on the minimum degree or minimum face size, which is usually assumed to be at least 3. Indeed, in the family of 2-connected graphs (or, more generally, graphs of minimum degree at least 2), no graph except K_1 is light. This can be seen from the following construction: given a plane connected graph $H \ne K_1$ and a large positive integer r, construct a plane graph G by taking $|E(H)|$ copies of $K_{2,r}$ and, for each edge of H, identify its endvertices with two "big" vertices of a copy of $K_{2,r}$ (the original edges of H are preserved); in G, every isomorphic copy of H contains a vertex of degree at least $r + 1$. On the other hand, observe that the "basic brick" for this construction — the graph $K_{2,r}$ — has girth 4; this motivated further research of the structure of plane graphs with higher girths in hope of obtaining some larger light graphs. In particular, in [3], it was proved that each plane graph of girth at least 7 contains a light edge of weight at most 5, which was further extended for smaller girths in [4, 5] and [6] along with description of types of 3-vertex paths in these graphs (which yields that P_3 is light in the family of plane graphs of girth at least 7). Note that the condition of sufficiently high girth itself is not sufficient for the existence of light graphs other than paths, as can be seen by a modification of the above described construction where $K_{2,r}$ is replaced by a theta-graph formed by identification of the endvertices of r disjoint long paths. However, this example features adjacent 2-vertices, thus, one can hope that a combination of prescribed girth with forbidding adjacency of 2-vertices would enforce the existence of non-trivial light graphs in plane graphs. Hence, in the following, we will investigate the structure of graphs from the family $\mathcal{P}^\bullet(g)$ of plane graphs of girth at least g, with minimum degree at least 2 and without adjacent 2-vertices (note that the latter condition implies, using the well-known fact that each plane graph of minimum degree at least 2 and girth at least $5k+1$ contains a k-path of 2-vertices, that the girth of plane graphs with these restrictions is at most 10).

Our results are summarized in the following theorems.

Theorem 1. *S_3 is light in $\mathcal{P}^\bullet(g)$ if and only if $g \geq 7$; in particular,*

(i) $\varphi(S_3, \mathcal{P}^\bullet(7)) = 5$;
(ii) $\varphi(S_3, \mathcal{P}^\bullet(g)) = 3$ *and* $w(S_3, \mathcal{P}^\bullet(g)) = 10$ *for* $g \in \{8, 9\}$;
(iii) $\varphi(S_3, \mathcal{P}^\bullet(10)) = 3$ *and* $w(S_3, \mathcal{P}^\bullet(10)) = 9$.

Theorem 2. (i) *Let $g \leq 10$ and H be a connected plane graph of girth at least g and maximum degree at least 4. Then H is heavy in $\mathcal{P}^\bullet(g)$.*
(ii) *For $g \in \{5, 6, 7, 8, 9\}$ and $k \geq g$, C_k is heavy in $\mathcal{P}^\bullet(g)$.*

Theorem 3. *For $k \geq 11, k \neq 28$, C_k is heavy in $\mathcal{P}^\bullet(10)$ while C_{10} is light in $\mathcal{P}^\bullet(10)$; in particular, $\varphi(C_{10}, \mathcal{P}^\bullet(10)) = 5$ and $w(C_{10}, \mathcal{P}^\bullet(10)) = 27$.*

2. PROOFS

***Proof of Theorem* 1.** To show that $g \geq 7$ is a necessary condition for the lightness of S_3 in $\mathcal{P}^\bullet(g)$ it suffices to consider the graph from the family $\mathcal{P}^\bullet(6)$ presented in Figure 1 in which any 3-star contains a vertex of large degree. The proof of sufficiency is divided into several cases.

Case (i). By contradiction. Assume that there exists a counterexample $\widehat{G}$ of girth at least 7 containing no 3-star with all vertices of degree at most 5 (in this proof called light 3-star). Now, in $\widehat{G}$, subdivide each edge uv of $\widehat{G}$ where u, v are vertices of degree 3, 4 or 5. It is easy to see that the resulting plane graph $G = (V, E, F)$ is also a counterexample (note that, in G, no two vertices of degrees 3, 4 or 5 are adjacent).

Euler's formula $|V| - |E| + |F| = 2$ applied to G yields

$$-14|V| + 10|E| + 4|E| - 14|F| = -28, \tag{1}$$

$$\sum\nolimits_{v \in V}(5\deg(v) - 14) + \sum\nolimits_{f \in F}(2\deg(f) - 14) = -28. \tag{2}$$

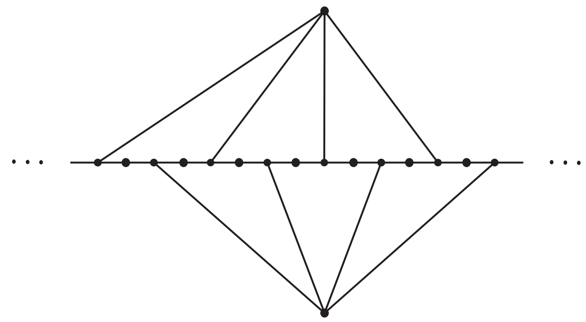

Figure 1. The graph of $\mathcal{P}^{\bullet}(6)$ with no light 3-star.

Define the initial charge $c: V \cup F \to \mathbb{Z}$ of vertices and faces of G in the following way:

$c(v) := 5\deg(v) - 14$ for every $v \in V$, $c(f) := 2\deg(f) - 14$ for every $f \in F$.

According to (2), the total sum of all initial charges is negative. We will redistribute these charges according to specific discharging rules in a way that the sum of the charges always remains the same; then, we show that, after the completion of discharging process, the final charge c^* of all elements of $V \cup F$ is non-negative, which is a contradiction with (2).

The discharging rules are the following:

R1: If v is a 2-vertex and

(a) v is incident with two 3-vertices or two 4^+-vertices, then v sends the charge -2 to both its neighbours,

(b) v is incident with a 3-vertex x and a 4^+-vertex y, then v sends the charge $-\frac{5}{3}$ to x and the charge $-\frac{7}{3}$ to y.

R2: Each 6^+-vertex sends $\frac{8}{3}$ to every adjacent 3-vertex.

R3: If f is an 8^+-face incident with $p \geq 1$ vertices of degree 3, then f sends $\frac{2\deg(f)-14}{p}$ to each of them.

A 7-face f is called *bad* if its facial walk contains, as a subpath, a $(3, 2, 3, 2, 3)$-path; the central 3-vertex of this subpath will be called *bad vertex of f*.

R4: (a) Each 6^+-vertex sends $\frac{1}{3}$ to every incident bad face.
(b) Each bad vertex of face f sends $-\frac{1}{3}$ to face f.

Now we show that the final charge c^* of every element of $V \cup F$ is non-negative. The initial charge of each 7-face α is 0 and, after applying Rule R4, its final charge is 0 or $\frac{1}{3}$ (either α is a bad face — then at least one vertex incident with α is a 6^+vertex because in G there are no adjacent 3-, 4- and 5-vertices; or

$c^*(\alpha) = c(\alpha)$). Similarly, for an 8^+-face f incident with p vertices of degree 3, its final charge is $2\deg(f) - 14 - p \cdot \frac{2\deg(f)-14}{p} = 0$ if $p \geq 1$ or $2\deg(f) - 14 > 0$ if $p = 0$.

To analyze the final charge of vertices of G, we consider several cases.

1. $\deg(v) = 2$. The initial charge of v is -4 and, according to Rule R1, $c^*(v) = 0$.

2. $\deg(v) = 3$. As every copy of S_3 in G contains a 6^+-vertex, v is adjacent to at least one 6^+-vertex. If we apply Rule R1(a) to v at most once, then $c^*(v) \geq 1 - 2 - \frac{5}{3} + \frac{8}{3} = 0$ according to Rules R1 and R2 (and possibly to R3). Suppose that the Rule R1(a) is applied to v twice. If v is incident with an 8^+-face then, according to Rules R1(a), R2, R3 and the fact that 3-vertices are not adjacent in G (hence every k-face is incident with at most $\lfloor \frac{k}{2} \rfloor$ 3-vertices), it holds $c^*(v) \geq 1 - 2 \cdot 2 + \frac{8}{3} + \frac{2 \cdot 8 - 14}{4} > 0$. If v is incident only with 7-faces, then the Rule R4(b) is applied (at least once) to v and its final charge is at least $1 - 2 \cdot 2 + \frac{8}{3} - (-\frac{1}{3}) = 0$.

3. $4 \leq \deg(v) \leq 5$. Then v is adjacent to at most two 2-vertices (otherwise G would contain a 3-star with all vertices of degree at most 5), so, according to Rule R1, $c^*(v) \geq 5\deg(v) - 14 - 2 \cdot \frac{7}{3} = 5\deg(v) - \frac{56}{3} > 0$.

4. $\deg(v) \geq 6$. Let t denote the number of 2-vertices adjacent to v, s denote the number of 6^+-vertices adjacent to v and b the number of bad faces incident with v. It is easy to check that an edge incident with a 2-vertex and a 6^+-vertex can be incident with at most one bad face, so $b \leq t + 2s$. Now using Rules R1, R2 and R4(a), we get

$$\begin{aligned} c^*(v) &\geq 5\deg(v) - 14 - \tfrac{7}{3}t - \tfrac{8}{3}(\deg(v) - t - s) - \tfrac{1}{3}b = \tfrac{7}{3}\deg(v) + \tfrac{1}{3}t + \tfrac{8}{3}s - \tfrac{1}{3}b - 14 \\ &\geq \tfrac{7}{3}\deg(v) + \tfrac{1}{3}t + \tfrac{8}{3}s - \tfrac{1}{3}(t + 2s) - 14 = \tfrac{7}{3}\deg(v) + 2s - 14 \geq 0. \end{aligned}$$

Hence, all elements of G have non-negative final charges, which completes the proof.

In the plane graph from the family $\mathcal{P}^\bullet(7)$ presented in Figure 2, each 3-star contains a 5-vertex, which proves that the upper bound 5 on $\varphi(S_3, \mathcal{P}^\bullet(7))$ is best possible.

Case (ii). By contradiction. Let $G = (V, E, F)$ be a counterexample of girth at least 8 such that the central vertex of every 3-star in G is adjacent with a 4^+-vertex or at least two 3-vertices. Again, Euler's formula $|V| - |E| + |F| = 2$ applied to G yields

$$-8|V| + 6|E| + 2|E| - 8|F| \;=\; -16, \tag{3}$$

$$\sum\nolimits_{v \in V} (3\deg(v) - 8) + \sum\nolimits_{f \in F} (\deg(f) - 8) \;=\; -16. \tag{4}$$

Define the initial charge $c:\; V \cup F \to \mathbb{Z}$ of vertices and faces of G in the following way:

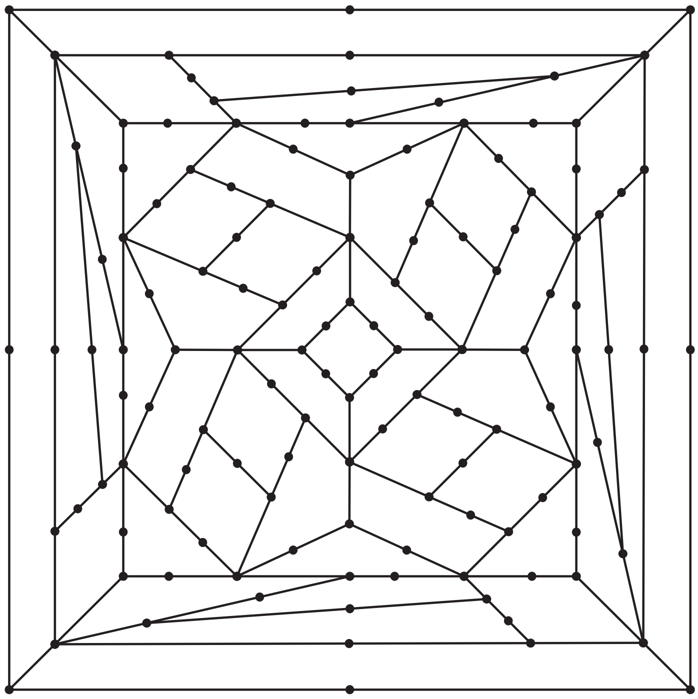

Figure 2. The graph of $\mathcal{P}^{\bullet}(7)$ whose all 3-stars have a 5-vertex.

$$c(v) := 3\deg(v) - 8 \text{ for every } v \in V,\ c(f) := \deg(f) - 8 \text{ for every } f \in F.$$

We redistribute the charges according to the following discharging rules:

R1: Each 2-vertex sends -1 to both its neighbours.

R2: Each 4^{+}-vertex sends 1 to every adjacent 3-vertex.

Now we show that the final charge c^* of every element of G is non-negative, which will lead to a contradiction with (4). As the girth of G is at least 8, the final charge of every face is non-negative. To analyze the final charge of vertices of G, we consider several cases.

1. $\deg(v) = 2$. Then, according to Rule R1, $c^*(v) = 0$.

2. $\deg(v) = 3$. If v is adjacent to a 4^{+}-vertex, then $c^*(v) \geq 1 - 2 \cdot 1 + 1 = 0$. Otherwise, v is adjacent to at least two 3-vertices, thus Rule R1 applies at most once and $c^*(v) \geq 1 - 1 = 0$.

3. $\deg(v) \geq 4$. Then, according to Rules R1 and R2, it holds $c^*(v) \geq 3\deg(v) - 8 - \deg(v) \geq 2\deg(v) - 8 \geq 0$.

In the plane graph from the family $\mathcal{P}^{\bullet}(9)$ presented in Figure 3, each 3-star contains a 3-vertex (see thick edges), which proves that the upper bounds 3 and 10 on $\varphi(S_3, \mathcal{P}^\bullet(g))$ and $w(S_3, \mathcal{P}^\bullet(g))$ are best possible.

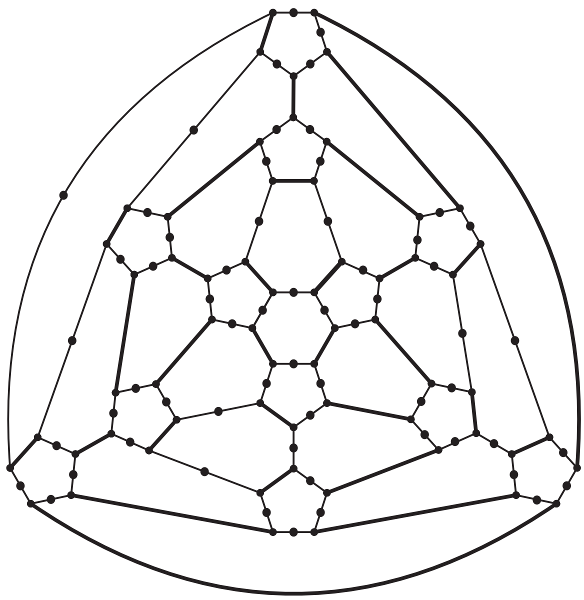

Figure 3. The graph of $\mathcal{P}^\bullet(9)$ whose all 3-stars have a 3-vertex and weight 10.

Case (iii). By contradiction. Let $G = (V, E, F)$ be a counterexample of girth 10 such that the central vertex of every 3-star in G is incident with at least one 3-vertex. Again, Euler's formula $|V| - |E| + |F| = 2$ applied to G yields

$$-10|V| + 8|E| + 2|E| - 10|F| \;=\; -20, \tag{5}$$

$$\sum\nolimits_{v\in V}(4\deg(v) - 10) + \sum\nolimits_{f\in F}(\deg(f) - 10) \;=\; -20. \tag{6}$$

Define the initial charge $c:\; V \cup F \to \mathbb{Z}$ of vertices and faces of G in the following way:

$c(v) := 4\deg(v) - 10$ for every $v \in V$, $c(f) := \deg(f) - 10$ for every $f \in F$.

We redistribute the charges (keeping their total sum) according to the following discharging rule:

R: Each 2-vertex sends -1 to both its neighbours.

Now we show that the final charge c^* of every element of G is non-negative, which will lead to a contradiction with (6). As the girth of G is 10, the final charge of every face is non-negative. To analyze the final charge of vertices of G, we consider several cases.

1. $\deg(v) = 2$. Then, according to Rule R, $c^*(v) = 0$.

2. $\deg(v) = 3$. From the assumption of theorem, v is adjacent to at least one 3-vertex, thus Rule R applies at most twice and $c^*(v) \geq 2 - 2 = 0$.

3. $\deg(v) \geq 4$. Then, according to Rule R, it holds $c^*(v) \geq 4\deg(v) - 10 - \deg(v) \geq 3\deg(v) - 10 > 0$.

In the plane graph of girth 10 obtained from the dodecahedron graph by subdividing each its edge by a 2-vertex, each 3-vertex has only neighbours of degree 2, which proves the optimality of the weight 9. ■

***Proof of Theorem* 2.** The particular proofs for a graph H with $\Delta(H) \geq 4$ or for a cycle C_k follow the same general strategy: given a large integer n, we first construct a graph H_n from the particular family $\mathcal{P}^\bullet(g)$ such that H_n contains an n-vertex and the subgraph of H_n induced by vertices of degree less than n does not contain an isomorphic copy of H (or C_k). Next, we take two disjoint copies $H_n^{(1)}, H_n^{(2)}$ of H_n drawn in the plane in such a way that, for $i \in \{1, 2\}$, the facial walk of the outerface β_i of $H_n^{(i)}$ contains an n-vertex u_i; let v_i be a vertex on the facial walk of β_i which is most distant from u_i (that is, the shortest (u_i, v_i)-path which is a part of the facial walk of β_i has the maximum possible length). Then, we add a new edge v_1v_2 thereby obtaining the plane graph $D_n \in \mathcal{P}^\bullet(g)$ containing two n-vertices in its outerface such that their distance is at least $g-1$ (note that, in each particular proof, H_n is designed in such a way that the subgraph of D_n induced by vertices of degree less than n still does not contain an isomorphic copy of H, resp. C_k as well).

To complete the proof, we take the plane drawing of H (or C_k) and transform it to a plane graph G_n in such a way that, for each edge xy of H, resp. C_k, we identify the vertices x and y with two n-vertices from the outerface of a copy of D_n (we preserve edge xy). Due to the above mentioned properties of H_n and D_n, we obtain that $G_n \in \mathcal{P}^\bullet(g)$ and contains H (or C_k) as a subgraph, but each isomorphic copy of H, resp. C_k, in G_n must contain an n^+-vertex, which proves that H and C_k are not light in $\mathcal{P}^\bullet(g)$.

(i) The graph H_n is constructed in the following way: take two $3n$-cycles $C_1 = a_1b_1c_1a_2b_2c_2 \cdots a_nb_nc_n$, $C_2 = p_1q_1r_1p_2q_2r_2 \cdots p_nq_nr_n$ and two new vertices x, y; for each $i = 1, \ldots, n$, add new edges $b_ip_i, c_ir_i, xa_i, yq_i$. This can be done in such a way that the obtained graph H_n' is plane (note that H_n' is cubic and has girth 5). Then H_n is obtained from H_n' by subdividing each edge with a new 2-vertex (see Figure 4).

(ii) Suppose first that $g \in \{5, 6, 7, 8\}$ and $k \geq g$. Then the graph H_n is constructed from the dual graph of an n-antiprism (that is, the plane quadrangulation consisting of two nonadjacent n-vertices and $2n$ 3-vertices) by subdividing each its edge

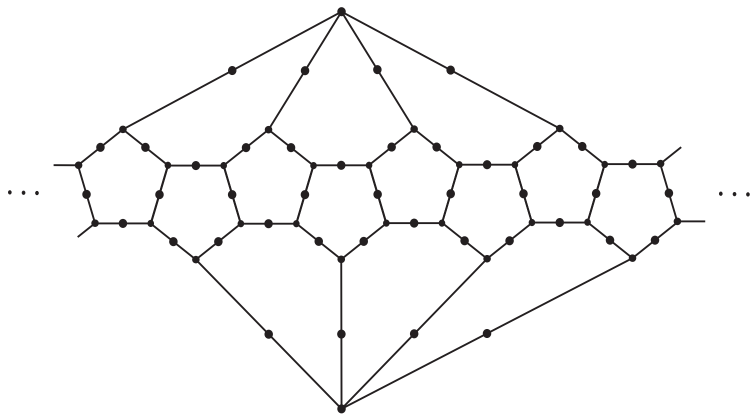

Figure 4. The graph H_n for H with $\Delta(H) \geq 4$.

with a new 2-vertex. Observe that the subgraph of H_n induced by its 3^--vertices contains a single cycle of length $4n$.

Now, let $g = 9$; we distinguish several cases.

(a) For $k \geq 26$ or $k \in \{10, 11, 12, 13, 22, 23\}$, H_n is the graph presented in Figure 5; observe that the subgraph of H_n induced by 3^--vertices is essentially (not regarding its pendant vertices) formed by interconnecting many copies of the same graph on 31 vertices, and that this graph contains neither cycles of length 10, 11, 12, 13, 22, 23, 26 – 31 (which can be checked by hand or, with less effort, by Wolfram Mathematica computer algebra system using the procedure `FindCycle[]`) nor any other cycles of length at least 26 provided n is much larger than k.

(b) For $k \in \{9, 14, 15, 17, 18, 19, 20, 21, 24, 25\}$, H_n is the same as in the proof of (i); it is easy to check that the subgraph of H_n induced on 3^--vertices contains only cycles of lengths $10 + 6\ell, \ell \geq 0$, or at least n.

(c) Finally, for $k = 16$, H_n is the graph presented in Figure 6; again, the specific sizes of its faces formed by 3^--vertices and the way they are adjacent yields that H_n itself contains no 16-cycle which passes only through 3^--vertices. ■

***Proof of Theorem* 3.** Let $G \in \mathcal{P}^\bullet(10)$. Consider the graph G' obtained from G by contracting all its 2-vertices. Then G' is a simple plane graph with $\delta(G') = 3$ and $g(G') = 5$. By the dual version of Lebesgue theorem, we obtain that G' contains a 5-cycle $\mathcal{C}$ incident with four 3-vertices and a 5^--vertex. As $g(G) = 10$, this implies that the cycle of G which was transformed to $\mathcal{C}$ in G' is a 10-cycle consisting of five 2-vertices, four 3-vertices and a 5^-vertex, which proves the claim.

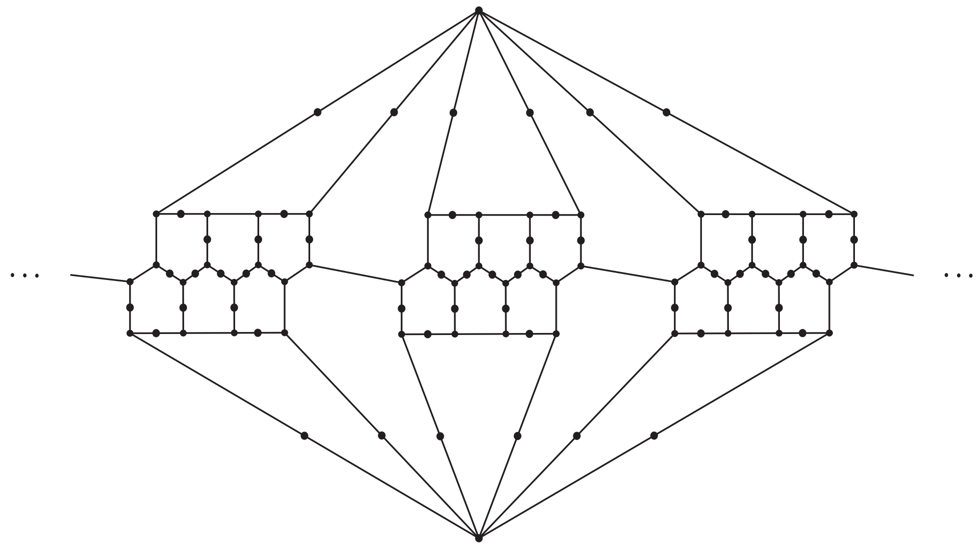

Figure 5. The graph H_n for $g = 9$ and $k \geq 26$ or $k \in \{10, 11, 12, 13, 22, 23\}$.

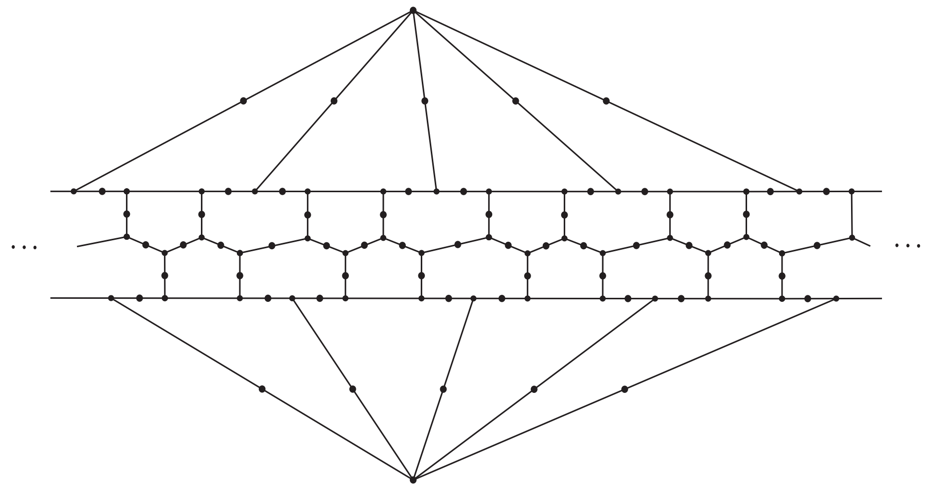

Figure 6. The graph H_n for $g = 9$ and $k = 16$.

To show that both the bounds 5 and 27 are best possible, consider the dual of the snub dodecahedron graph and subdivide each edge with a 2-vertex; this plane graph has girth 10 and every its 10-cycle contains a 5-vertex, and has weight 27.

By using the construction described in the proof of Theorem 2 with H being an odd cycle of length at least 11 and H_n being the graph presented in Figure 4, we obtain that no odd cycle is light in $\mathcal{P}^{\bullet}(10)$. Now, let n and $k \geq 6, k \neq 14$ be positive integers. As C_k is not light in a family of plane graphs with minimum vertex degree 3 and girth 5 (see [7]), there exists a plane graph G'_n with $\delta(G'_n) = 3$, $g(G'_n) = 5$ such that every its k-cycle contains a vertex of degree at least n. Let G_n be a graph obtained from G'_n by subdividing every its edge with a new 2-vertex. Thus $G \in \mathcal{P}^\bullet(10)$ and every $2k$-cycle in G_n contains a vertex of degree at least n, which gives that C_{2k} is heavy in $\mathcal{P}^\bullet(10)$. ■

Let us note that it is not known whether C_{14} is light or heavy in the family of plane graphs with minimum vertex degree 3 and girth 5. If C_{14} is heavy in that family, then C_{28} is also heavy in $\mathcal{P}^\bullet(10)$. However, if C_{14} is light in the family of plane graphs with minimum vertex degree 3 and girth 5 then we cannot conclude that C_{28} is light in $\mathcal{P}^\bullet(10)$.

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