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UNIQUE-MAXIMUM COLORING OF PLANE GRAPHS

Igor Fabrici [1]

Institute of Mathematics
P.J. Šafárik University in Košice
Slovakia

and

Frank Göring

Fakultät für Mathematik
Technische Universität Chemnitz
Germany

Abstract

A unique-maximum k-coloring with respect to faces of a plane graph G is a coloring with colors $1, \ldots, k$ so that, for each face α of G, the maximum color occurs exactly once on the vertices of α. We prove that any plane graph is unique-maximum 3-colorable and has a proper unique-maximum coloring with 6 colors.

Keywords: plane graph, weak-parity coloring, unique-maximum coloring.

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1. Introduction

Graphs considered in this paper are simple, finite, and undirected. We use the notation and terminology of Bondy and Murty [1]. A k-(*vertex-*)*coloring* of a graph G is a mapping $\varphi : V(G) \to \{1, \ldots, k\}$. A coloring φ of G is *proper* if, for any two adjacent vertices x and y, $\varphi(x) \neq \varphi(y)$ holds. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a generalization of a graph, its (hyper-)edges are subsets of V of arbitrary (positive) size. A (vertex) coloring of hypergraphs can be defined in many ways, so that restricting the definition to simple graphs coincides with proper graph coloring.

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1.1. Conflict-free coloring

A coloring of a hypergraph $\mathcal{H}$ is *conflict-free* (*CF*) if, for every edge $e \in \mathcal{E}(\mathcal{H})$, there is a color that occurs exactly once on the vertices of e. The *CF chromatic number* of $\mathcal{H}$ is the minimum k for which $\mathcal{H}$ has a CF k-coloring. The CF coloring of hypergraphs was introduced (in a geometric setting) by Even *et al.* [8] in connection with frequency assignment problems for cellular networks. For simple graphs, Cheilaris [3] studied the *CF coloring with respect to neighborhoods*, i.e., the coloring in which, for every vertex x, there is a color that occurs exactly once in the neighborhood $N(x)$ or in the closed neighborhood $N[x] = N(x) \cup \{x\}$, respectively, and proved the upper bound $2\sqrt{n}$ for the CF chromatic number with respect to neighborhoods of a graph of order n. For closed neighbourhood, this bound was improved by Pach and Tardos [14] to $O(\log^{2+\varepsilon} n)$, for any $\varepsilon > 0$. Cheilaris and Tóth [6] and Cheilaris, Pecker, and Zachos [5] studied the *CF coloring* of a graph G *with respect to paths*, i.e., the coloring in which, for every path P of G, there is a color that occurs exactly once on the vertices of P. Note that the mentioned CF colorings of graphs are special cases of CF coloring of hypergraphs. For more results on CF coloring see, e.g., [6, 10, 12] and for another applications of CF coloring see [16].

1.2. Unique-maximum coloring

A coloring of a hypergraph $\mathcal{H}$ is *unique-maximum* (*UM*) if, for every edge $e \in \mathcal{E}(\mathcal{H})$, the maximum color on the vertices of e is unique. The *UM chromatic number* of $\mathcal{H}$ is the minimum k for which $\mathcal{H}$ has a UM k-coloring. UM coloring of hypergraphs and its relation to CF coloring was investigated by Cheilaris, Keszegh, and Pálvölgyi [4]. For simple graphs, the *UM coloring with respect to paths* requires that the maximum color on the vertices of any path is unique and it is alternatively known as *ordered coloring* or *vertex ranking*, in which for every path with equally colored end-vertices there is an internal vertex with higher color. Katchalski, McCuaig, and Seager [11] found the upper bound $3(\sqrt{6}+2)\sqrt{n}$ for UM chromatic number with respect to paths of plane graph of order n. For more results on UM coloring see, e.g., [5, 6].

1.3. Weak-parity coloring

A coloring of a hypergraph $\mathcal{H}$ is *weak-parity* (*WP*) if, for every edge $e \in \mathcal{E}(\mathcal{H})$, there is a color c with the odd number of vertices of e colored c. The *WP chromatic number* of $\mathcal{H}$ is the minimum k for which $\mathcal{H}$ has a WP k-coloring. The WP coloring of hypergraphs was introduced (under the notion *odd coloring*) by Cheilaris, Keszegh, and Pálvölgyi [4] as a generalization of the *WP coloring* of graphs *with respect to paths* defined originally by Bunde *et al.* [2].

2. Colorings of Plane Graphs with Respect to Faces

Let G be a plane graph with the face set $F(G)$. For a face $\alpha \in F(G)$, $V(\alpha)$ denotes the set of vertices incident with α. The *face-hypergraph* $\mathcal{H}(G)$ of G is the hypergraph with the vertex set $V(G)$ and the edge set $\{V(\alpha) : \alpha \in F(G)\}$, i.e., every face of G corresponds to an edge of $\mathcal{H}(G)$ consisting of the vertices incident with this face. Kündgen and Ramamurthi [13] and Ramamurthi and West [15] considered a coloring of face-hypergraphs as the corresponding face-constrained coloring of plane graphs. Motivated by these papers, we can define the following *colorings* of plane graphs *with respect to faces* as special cases of mentioned colorings for hypergraphs.

- A *WP k-coloring* of G is a colouring such that, for every face $\alpha \in F(G)$, there is a color c with the odd number of vertices of α colored c. The minimum k for which G has a (proper) WP k-coloring is denoted by $\chi_{\rm wp}(G)$ ($\chi_{\rm pwp}(G)$).
- A *CF k-coloring* of G is a coloring such that, for each face $\alpha \in F(G)$, there is a color that occurs exactly once on the vertices of α. The minimum k for which G has a (proper) CF k-coloring is denoted by $\chi_{\rm cf}(G)$ ($\chi_{\rm pcf}(G)$).
- A *UM k-coloring* of G is a coloring such that, for each face α of G, the maximum color (of α) occurs exactly once on the vertices of α. The minimum k for which G has a (proper) UM k-coloring is denoted by $\chi_{\rm um}(G)$ ($\chi_{\rm pum}(G)$).

A graph G is *WP* (*CF*, *UM*) *k-colorable* if there is a WP (CF, UM) k-coloring of G.

A simple relation between the chromatic numbers we have defined is the following

Proposition 1. *If G is a plane graph, then*
(1) $\chi_{\rm wp}(G) \le \chi_{\rm cf}(G) \le \chi_{\rm um}(G)$,
(2) $\chi(G) \le \chi_{\rm pwp}(G) \le \chi_{\rm pcf}(G) \le \chi_{\rm pum}(G)$.

Czap and Jendroľ proved the following upper bound on WP chromatic number

Theorem 2 [7]**.** *If G is a connected plane graph, then $\chi_{\rm wp}(G) \le 4$.*

and conjectured that this upper bound can be improved.

Conjecture 3 [7]**.** *If G is a connected plane graph, then $\chi_{\rm wp}(G) \le 3$.*

Moreover, they proved this conjecture for 2-connected cubic plane graphs. For the proper WP (CF) coloring, the tight upper bounds are known.

Theorem 4 [7]. *If G is a 2-connected plane graph, then $\chi_{\text{pcf}}(G) \leq 4$; moreover, the bound 4 is tight.*

Corollary 5 [7]. *If G is a 2-connected plane graph, then $\chi_{\text{pwp}}(G) \leq 4$; moreover, the bound 4 is tight.*

For results on corresponding (WP, CF, UM) chromatic indices we refer to [9]. For a better overview, in the next theorem and in the following lemma we use the colors black = 1, blue = 2, and red = 3. We prove Conjecture 3 in the following stronger form:

Theorem 6. *Every plane graph has a 3-coloring with colors black, blue and red such that*

(1) *each face is incident with at most one red vertex, and*

(2) *each face that is not incident with a red vertex is incident with exactly one blue vertex.*

Note that the roles of colors red and blue in this theorem are slightly assymetric. While the number of red vertices incident with a face is bounded by one, the theorem gives no bound for the number of blue vertices incident with a face (incident with a red vertex). Indeed, for $n \geq 5$, the n-prism (i.e., the cartesian product $C_n \square K_2$) shows that the number of blue vertices incident with a face has to be unbounded. There are at most two red vertices in total because each vertex is incident with one of the two n-gons. Since each vertex is incident with only two quadrangles and each quadrangle is incident with at least one red or blue vertex, there are at least $\frac{n}{2}$ blue or red vertices in the considered coloring. Consequently, at least one of the n-gons is incident with at least $\frac{n}{4} - 1$ blue vertices.

To prepare the proof of this theorem, we provide the following lemma:

Lemma 7. *Let G be a plane graph, let $xy \in E(G)$ be a selected edge of G incident with the outer face, and let $c \in \{black, blue\}$. There is a 3-coloring of G with colors black, blue, and red such that*

(1) *vertex x has color c,*

(2) *vertex y is black,*

(3) *each edge is incident with at most one blue vertex,*

(4) *no vertex incident with the outer face is red,*

(5) *each inner face is incident with at most one red vertex, and*

(6) *each inner face that is not incident with a red vertex is incident with exactly one blue vertex.*

The proof of Theorem 6 using Lemma 7 is as follows.

***Proof of Theorem* 6.** Let G be a plane graph. Choose a vertex $z \in V(G)$ incident with the outer face and color it red. If $G - z$ is edgeless, then G is a forest (i.e., it has only one face) and we can color all other vertices black. Otherwise, choose an edge xy of the outer face of $G - z$, color x and y black, and apply Lemma 7 on $G - z$ (with the selected edge xy and the color $c = \text{black}$) to obtain colors of the remaining vertices. Any face α of G is either an inner face of $G - z$ and thus colored correctly by Lemma 7, or is incident with the red vertex z. Since the vertices of the outer face of $G - z$ are colored black or blue, there is no other red vertex on α. ■

***Proof of Lemma* 7.** The proof is by induction on the number of vertices. Let G be a plane graph, let $xy \in E(G)$ be a selected edge of G incident with the outer face ω, and let $c \in \{\text{blue}, \text{black}\}$.

Case 1. If ω is the only face of G (i.e., if G is a forest), the precoloring of x and y can be extended to the required coloring of G by coloring all other vertices black.

Case 2. If G is disconnected, denote G_1 the component of G containing xy and let $G_2 = G - G_1$. We apply the induction hypothesis to color G_1 (with the selected edge xy and the selected color c). For an arbitrary edge $x_2y_2 \in E(G_2)$ incident with the outer face of G_2 (and thus incident with ω as well) we color the graph G_2 (with the selected edge x_2y_2 and the color $c_2 = \text{black}$) by induction hypothesis, or we simply color all vertices of G_2 black, if G_2 is edgeless.

Hence, we may assume that G is connected and has at least two faces (i.e., G has a cycle and therefore it has at least three vertices and at least three edges).

Case 3. Let $U \neq \emptyset$ be the set of vertices incident with no inner face of G (note that, for $u \in U$, every edge incident with u is a bridge of G).

Case 3.1. If there exists $u \in U \setminus \{x, y\}$, we apply induction hypothesis to color $G - u$ and finally we color u black.

Case 3.2. If $x \in U$ and x is a pendant vertex of G (i.e a vertex of degree one) then y has degree at least 2. Let x' be a neighbor of y on ω which is different from x. Now we color $G - x$ (with the selected edge $x'y$ and the color $c' = \text{black}$) by the induction hypothesis. Together with the vertex x colored by c we have a required coloring. (We proceed analogously if $y \in U$ is pendant.)

In the next two cases, let both x and y have degree at least 2.

Case 3.3. If $y \in U$ then we apply the induction hypothesis to color $G - y$ (with a selected edge xy' incident with the outer face of $G - y$ and the color c) and finally we color y black.

Case 3.4. For $U = \{x\}$, let $y_1, \dots, y_k$ be the neighbors of x in G (note that y is one of them). Clearly, all these neighbors have degree at least 2. For $i \in \{1, \dots, k\}$, let G_i be the component of $G - x$ containing y_i, let $y_i x_i$ be an edge of G_i incident with the outer face of G_i (and thus incident with ω as well), and let $c_i =$ black. We apply the induction hypothesis to every graph G_i (with the selected edge $x_i y_i$ and the color c_i) and, together with the vertex x colored by c, we obtain a required coloring.

Hence, we may assume that each vertex of G is incident with an inner face of G.

Case 4. Let $B = G[V(\omega)]$ be the graph induced by the vertices incident with the outer face ω in G.

Case 4.1. If B contains a cut-vertex x_2, then we split the graph G on x_2 into two subgraphs G_1 and G_2 so that $xy \in E(G_1)$. More formally, let M be the component of $G - x_2$ containing x or y (note that either x and y belong to the same component of $G - x_2$ or $x_2 \in \{x, y\}$), let $G_2 = G[V(G) \setminus V(M)]$, and let $G_1 = G[V(M) \cup \{x_2\}]$. Moreover, let y_2 be a neighbor of x_2 on the outerface of G_2. There is a required 3-coloring φ_1 of G_1 (with the selected edge xy and the color c) and a required 3-coloring of G_2 (with the selected edge $x_2 y_2$ and the color $c_2 = \varphi_1(x_2) \in \{\text{black}, \text{blue}\}$, as x_2 is incident with the outer face of G_1), both by induction hypothesis.

Case 4.2. If B contains an inner edge $x_2 y_2$ (i.e., an edge not incident with ω—in this case, $\{x_2, y_2\}$ is a 2-vertex-cut of G), then we split the graph G on $x_2 y_2$ into two subgraphs G_1 and G_2 so that $xy \in E(G_1)$. More formally, let M be the component of $G - x_2 - y_2$ containing x or y, let $G_2 = G[V(G) \setminus V(M)]$, and let $G_1 = G[V(M) \cup \{x_2, y_2\}]$. There is a required 3-coloring φ_1 of G_1 (with the selected edge xy and the color c) and thereafter a required 3-coloring of G_2 (with the selected edge $x_2 y_2$ and the color $c_2 =$ black, if $\varphi_1(x_2) = \varphi_1(y_2) =$ black, or $c_2 =$ blue, if $\varphi_1(x_2) =$ blue or $\varphi_1(y_2) =$ blue, respectively), both by induction hypothesis.

Hence, we may assume that B is a cycle and y has a neighbor v on B that is different from x.

Case 4.3. If $G = B$ then we color vertex x by c, vertex v black or blue, but different from x, and all other vertices (inclusively y) black.

Case 4.4. If $G \neq B$, let α be the inner face of G incident with yv. Because $G[V(\alpha)] \neq B$, α has a vertex $u \notin V(B)$ (i.e., not incident with ω). We apply induction hypothesis (with the selected edge xy and the color c) on $G - u \setminus yv$ obtained from G by deleting the vertex u and the edge yv and finally we color u red to obtain a required coloring. The vertices of the outer face of $G - u \setminus yv$ are exactly the vertices incident with ω (in G) together with the vertices incident with the faces containing vertex u (in G). Obviously, none of them is colored red and therefore ω is incident with no red vertex. Any inner face of G is either an inner face of $G - u \setminus yv$ and thus colored correctly by induction hypothesis, or it is incident with the red vertex u (which is its unique red vertex). Moreover, there is no edge in G incident with two blue vertices. Namely, every edge of G is either an edge of $G - u \setminus yv$ and thus colored correctly by induction hypothesis, or it is incident with the red vertex u, or it is the edge yv, where y is black. ∎

With the fact that, for odd n, the n-prism is not WP 2-colorable (because for any WP 2-coloring it holds: from the pair of opposite edges of every quadrangle, one edge is monochromatic and the other one is bichromatic—a contradiction, see [7]), the following theorem is a direct consequence of Theorem 6.

Theorem 8. *If G is a plane graph, then $\chi_{\text{wp}}(G) \leq \chi_{\text{cf}}(G) \leq \chi_{\text{um}}(G) \leq 3$; moreover, the bound 3 is tight for all three chromatic numbers.*

With the help of the Four Color Theorem, we use Theorem 6 to prove the following upper bound on proper UM coloring.

Theorem 9. *If G is a plane graph, then $\chi_{\text{pum}}(G) \leq 6$.*

Proof. Let φ' be a UM 3-coloring of G with colors black = 1, blue = 5, and red = 6, and let φ'' be a proper 4-coloring of G with colors 1, 2, 3, 4. The coloring φ defined by $\varphi(x) = \max\{\varphi'(x), \varphi''(x)\}$, for $x \in V(G)$, is a proper UM 6-coloring of G. ∎

We believe that the following strengthening of the Four Color Theorem holds.

Conjecture 10. *If G is a plane graph, then $\chi_{\text{pum}}(G) \leq 4$.*

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