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ON AN EXTREMAL PROBLEM IN THE CLASS OF BIPARTITE 1-PLANAR GRAPHS

Július Czap

Department of Applied Mathematics and Business Informatics
Faculty of Economics, Technical University of Košice
Němcovej 32, 040 01 Košice, Slovakia

e-mail: julius.czap@tuke.sk

Jakub Przybyło[1]

AGH University of Science and Technology
Faculty of Applied Mathematics
al. A. Mickiewicza 30, 30–059 Kraków, Poland

e-mail: jakubprz@agh.edu.pl

and

Erika Škrabuľáková[2]

Institute of Control and Informatization of Production Processes
Faculty of Mining, Ecology, Process Control and Geotechnology
Technical University of Košice, Košice, Slovakia

e-mail: erika.skrabulakova@tuke.sk

Abstract

A graph $G = (V, E)$ is called 1-planar if it admits a drawing in the plane such that each edge is crossed at most once. In this paper, we study bipartite 1-planar graphs with prescribed numbers of vertices in partite sets. Bipartite 1-planar graphs are known to have at most $3n - 8$ edges, where n denotes the order of a graph. We show that maximal-size bipartite 1-planar graphs which are almost balanced have not significantly fewer edges than indicated by this upper bound, while the same is not true for unbalanced

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ones. We prove that the maximal possible size of bipartite 1-planar graphs whose one partite set is much smaller than the other one tends towards $2n$ rather than $3n$. In particular, we prove that if the size of the smaller partite set is sublinear in n, then $|E| = (2 + o(1))n$, while the same is not true otherwise.

Keywords: 1-planar graph, bipartite graph, graph size.

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1. Introduction

One of the general questions in extremal graph theory can be formulated in the following way: Given a family $\mathcal{G}$ of graphs, what is the maximum number of edges of an n-vertex graph $G \in \mathcal{G}$? One of the fundamental results in this area is the theorem of Turán, which states that if $\mathcal{G}$ is the family of k-clique-free graphs, then the maximum number of edges of an n-vertex graph $G \in \mathcal{G}$ is at most $\frac{(k-2)n^2}{2(k-1)}$. Turán's theorem was rediscovered many times and has many corollaries. For $k = 3$ we obtain Mantel's theorem: the maximum number of edges of an n-vertex bipartite graph is at most $\frac{n^2}{4}$.

By prescribing the family $\mathcal{G}$ we can study different classes of graphs. If $\mathcal{G}$ is a family of planar graphs, then from the Euler's formula we obtain that any n-vertex planar graph ($n \geq 3$) contains at most $3n - 6$ edges. More strongly, any n-vertex planar graph can be extended to an n-vertex planar graph with $3n - 6$ edges. Similar proposition holds for bipartite planar graphs: any n-vertex bipartite planar graph ($n \geq 3$) contains at most $2n - 4$ edges, moreover, every n-vertex bipartite planar graph can be extended to an n-vertex bipartite planar graph with $2n - 4$ edges.

If a graph is not planar, then each of its drawing in the plane contains some crossings of its edges. If a graph G can be drawn in the plane so that each of its edges is crossed by at most one other edge, then it is 1-*planar*. It is known [5, 7, 8] that any n-vertex 1-planar graph ($n \geq 3$) has at most $4n - 8$ edges, but not every n-vertex 1-planar graph can be extended to an n-vertex 1-planar graph with $4n - 8$ edges, see [1].

In this paper we deal with the family of bipartite 1-planar graphs. We consider the problem of finding a bipartite 1-planar graph with given sizes of partite sets which has the largest number of edges among all such graphs. It is known [6] that any n-vertex bipartite 1-planar graph has at most $3n-8$ edges for even $n \neq 6$ and at most $3n - 9$ edges for odd n and for $n = 6$. At the end of Section 4 (and in Lemma 6) we give a construction confirming that this upper bound is sharp. The maximal possible number of edges in such a graph keeps also relatively close to $3n$ when the cardinalities of its partite sets are almost equal (see Lemma 6).

On the other hand we notice that as graphs investigated get more unbalanced (i.e., one partite set becomes much smaller than the other) then this value drops (see Corollary 2) and tends towards the double of the order. Investigating this process more thoroughly, due to Corollary 2 and Lemma 4 we are in fact able to precisely describe for what proportions of the sizes of the partite sets we may observe this phenomenon, see comments in the concluding section.

Our results also partially answer the question of Sopena [9]: How many edges we have to remove from the complete bipartite graph with given sizes of the partite sets to obtain a 1-planar graph? Observe that this question is equivalent to our problem.

2. Notation

In this paper we consider simple graphs. We use the standard graph theory terminology of [4]. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph G, respectively. The *degree* of a vertex v is denoted by $\deg(v)$. A vertex of degree k is called a *k-vertex*. Similarly, a face (of a plane graph) of size k is called a *k-face*.

We will use the following notation introduced in [5]. Let G be a 1-planar graph and let $D = D(G)$ be a 1-*planar drawing* of G (that is, a drawing of G in the plane in which every edge is crossed at most once; we will also assume that no edge is self-crossing and adjacent edges do not cross). Given two non-adjacent edges pq, $rs \in E(G)$, the crossing of pq, rs is the common point of two arcs $\widehat{pq}$, $\widehat{rs} \in D$ (corresponding to edges pq, rs). Denote by $C = C(D)$ the set of all crossings in D and by E_0 the set of all non-crossed edges in D. The associated plane graph $D^\times = D^\times(G)$ of D is the plane graph such that $V(D^\times) = V(D) \cup C$ and $E(D^\times) = E_0 \cup \{xz, yz | xy \in E(D) - E_0, z \in C, z \in xy\}$. Thus, in $D^\times$, the crossings of D become new vertices of degree 4; we call these vertices *false*. Vertices of $D^\times$ which are also vertices of D are called *true*. Similarly, the edges and faces of $D^\times$ are called false, if they are incident with a false vertex, and true otherwise.

Note that a 1-planar graph may have different 1-planar drawings, which lead to non-isomorphic associated plane graphs.

3. Unbalanced Bipartite 1-Planar Graphs

Let G be a bipartite 1-planar graph such that the partite sets of G have sizes x and y. If $x = y$, then G is called *balanced*. In this part of the paper we show that if x is small compared to y, then the maximal number of edges in a corresponding bipartite 1-planar graph G shall tend towards $2|V(G)|$ rather than staying close to $3|V(G)|$.

3.1. An upper bound for the number of edges

The following assertion improves the result of [2] (stating that any 1-planar drawing of an n-vertex 1-planar graph has at most $n-2$ crossings) when x is small compared to y.

Lemma 1. *Let G be a bipartite 1-planar graph such that the partite sets of G have sizes x and y, $2 \le x \le y$. Then G has a 1-planar drawing with at most $6x - 12$ crossings.*

Proof. Color the vertices of G in the smaller partite set with black and the rest of the vertices with white. Among all possible 1-planar drawings of G, we denote by D a drawing that has the minimum number of crossings and by $D^{\times}$ its associated plane graph. Color the false vertices with red.

Now we extend $D^{\times}$ in the following way. Let v be a false vertex incident with black vertices v_1 and v_2. We draw a new noncrossing edge v_1v_2 as shown in Figure 1.

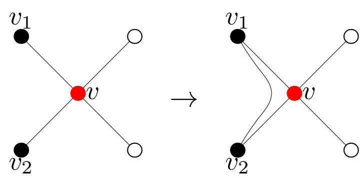

Figure 1. The extension of $D^{\times}$.

For every false vertex v we draw a new edge v_1v_2 as described. Note that the new drawing might contain parallel edges. Denote the new (multi)graph by H.

Let H' be a subgraph of H (with the same embedding) induced by the black and red vertices. First we show that if the (multi)graph H' has a separating 2-cycle (i.e., a 2-cycle whose interior and exterior contain a vertex), then its interior and also exterior contain at least one black vertex each. To see that, assume, for a contradiction, that H' contains a separating 2-cycle which has only red vertices in the interior. This 2-cycle is a separating cycle also in H since it consists of two edges which join black vertices. The red vertices correspond to crossings, therefore there are some white vertices in the interior of this separating 2-cycle in H. No black vertex is in the interior of this cycle, hence all edges which join white vertices from this interior with black vertices could have been drawn without edge crossing, contrary to the minimality of the number of crossings in the considered 1-planar drawing D.

If for every 2-cycle of H' with an empty interior or exterior we remove one edge of the 2-cycle incident with it, then we obtain the graph H''. We say that an edge of H'' is black if both its endvertices are black. Observe that every red vertex is incident with a 3-face in H'' (similarly as in H'). Moreover, every such

3-face is incident with one black edge. Therefore, the number of red vertices in H'' is at most the double number of black edges in H''.

Now consider the subgraph of H'' induced by the black vertices. This (multi) graph can be extended to a triangulation by inserting new additional edges (without inserting new vertices) because even if it contains a 2-cycle, then its interior and exterior contain a vertex. This triangulation has at most $3x - 6$ edges (because it has x vertices). From this it follows that the graph H'' has at most $2(3x - 6)$ red vertices. Consequently, the number of crossings in D is at most $6x - 12$. ∎

Corollary 2. *If G is a bipartite 1-planar graph such that the partite sets of G have sizes x and y, $2 \le x \le y$, then $|E(G)| \le 2|V(G)| + 6x - 16$.*

Proof. Lemma 1 implies that by removing at most $6x - 12$ edges from G we can get a planar graph. This planar graph is also bipartite. Thus, it has at most $2|V(G)| - 4$ edges. Consequently, the number of edges of G is at most $2|V(G)| + 6x - 16$. ∎

Note that the bound in Corollary 2 is tight for $x = 2$ (in this case G is a bipartite planar graph). For $x = 3$ we can obtain a tight upper bound by a different approach.

Lemma 3. *If G is a bipartite 1-planar graph such that the partite sets of G have sizes 3 and $y \ge 3$, then $|E(G)| \le 2|V(G)|$. Moreover, this bound is tight.*

Proof. Let V_1 and V_2 be the partite sets of G, where $|V_1| = 3$. In [3] it is proved that the complete bipartite graph $K_{3,7}$ is not 1-planar. Therefore, there are at most six vertices of degree three in V_2 (and the remaining vertices have degree at most two). Consequently, $|E(G)| = \sum_{v \in V_2} \deg(v) \le 6 \cdot 3 + (|V(G)| - 9) \cdot 2 = 2|V(G)|$.

$K_{3,6}$ is the smallest bipartite 1-planar graph for which the upper bound is attained. By adding 2-vertices to the larger partite set we can obtain more such graphs, see Figure 2. ∎

3.2. Lower bound for the number of edges

Lemma 4. *Let x, y be integers such that $x \ge 3$ and $y \ge 6x - 12$. Then there exists a bipartite 1-planar graph G such that the partite sets of G have sizes x and y, and $|E(G)| \ge 2|V(G)| + 4x - 12$.*

Proof. First assume that $y = 6x - 12$.

Let T be a triangulation on x vertices. From the Euler's formula it follows that every triangulation on x vertices has $2x - 4$ faces. Let T' be a graph obtained from T by inserting a configuration W_3 depicted in Figure 3 into each of its faces.

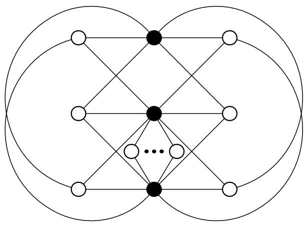

Figure 2. A bipartite 1-planar graph G with $3+y$ vertices and $2|V(G)|$ edges.

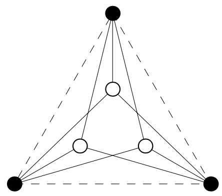

Figure 3. The configuration W_3.

Let G be a graph obtained from T' by removing the original edges of T. Clearly, G is a bipartite 1-planar graph (the original vertices of T form the first partite set and the added vertices form the second partite set). It is a routine matter to check that $|V(G)| = 7x - 12$ and $|E(G)| = 18x - 36 = 2|V(G)| + 4x - 12$.

Now suppose that $y = 6x - 12 + k$ for some $k \geq 1$. In this case we take the 1-planar drawing of G (as it is defined above), next we add k vertices to a 4-face of $G^{\times}$ and finally we join (without edge crossings) each of them with the two true vertices of this face. In such a way we obtain a new bipartite 1-planar graph which has $7x - 12 + k$ vertices and $18x - 36 + 2k$ edges. Hence, $|E(H)| = 2|V(H)| + 4x - 12$. ■

4. Almost Balanced Bipartite 1-Planar Graphs

Lemma 5. *Let x, y be integers such that $x \geq 3$, $y \geq 6$ and $x \leq y \leq 6x - 12$. Then there exists a bipartite 1-planar graph G such that the partite sets of G have sizes x, y and $|E(G)| \geq \frac{5}{2}|V(G)| + \frac{x}{2} - \frac{17}{2}$.*

Proof. First assume that $y = 6r$ for some integer $r \geq 1$. Let T be a triangulation on $\frac{6r}{6}+2 = \frac{y}{6}+2$ vertices. Color the vertices of T with black. Let $x = \frac{y}{6}+2+3s+t$, where $s \geq 0$ and $t \in \{0, 1, 2\}$ ($x \geq \frac{y}{6} + 2$ since $y \leq 6x - 12$). Let T' be a graph obtained from T by inserting a configuration B_3 depicted in Figure 4 into s faces, a configuration B_2 into one face if $t = 2$, a configuration B_1 into one face if $t = 1$ and the configuration B_0 to the other faces of T and removing the original edges of T. This modification is possible if and only if T has at least $s+1$ faces (or s faces if $t=0$). The number of faces of T is $2(\frac{y}{6}+2)-4=\frac{y}{3}$, so we need to show that $s+1 \leq \frac{y}{3}$. From $x=\frac{y}{6}+2+3s+t$ and $x \leq y$ we obtain $\frac{4}{5}+\frac{6}{5}s+\frac{2}{5}t \leq \frac{y}{3}$. The inequality $s+1 \leq \frac{4}{5}+\frac{6}{5}s+\frac{2}{5}t$, equivalently $1 \leq s+2t$, does not hold if and only if $t=s=0$. But in this case the inequality $s+1 \leq \frac{y}{3}$ trivially holds. Observe that T' has $(\frac{y}{6}+2)+3s+t=x$ black vertices and $3 \cdot \frac{y}{3}=y$ white vertices, moreover it has $3(x+y-(\frac{y}{6}+2))=\frac{5}{2}(x+y)+\frac{x}{2}-6$ edges.

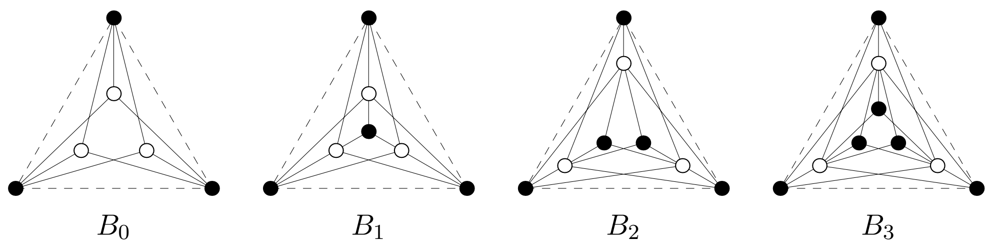

Figure 4. The configurations B_0, B_1, B_2 and B_3.

If $y=6r+u$, where $r \geq 1$ and $u \in \{1,2,3,4,5\}$, then we proceed similarly as above. In this case T is a triangulation on $\frac{6r+6}{6}+2$ vertices. Using this triangulation we obtain (by the same construction as previously) a bipartite 1-planar graph on $x+(6r+6)$ vertices and $3x+\frac{5}{2}\cdot(6r+6)-6$ edges. Note that if $x \neq 11$ or $y \neq 11$, then we must insert the configuration B_0 into at least two faces of T according to our construction, since otherwise the graph T' has at least $(\frac{6r+6}{6}+2)+3\cdot(\frac{6r+6}{3}-2)+1=7r+4$ black vertices. At the same time, the graph T' has $x \leq 6r+u$ black vertices, and therefore $7r+4 \leq 6r+u$, equivalently $r+4 \leq u$. This inequality in turn has only one solution, namely $r=1$ and $u=5$. This implies $x=y=11$.

If we remove $6-u$ white vertices of two configurations of type B_0, then we obtain a bipartite 1-planar graph with $x+y$ vertices and $3x+\frac{5}{2}\cdot(6r+6)-6-3(6-u)=\frac{5}{2}\cdot(x+6r+u)+\frac{x}{2}-9+\frac{u}{2} \geq \frac{5}{2}\cdot(x+y)+\frac{x}{2}-\frac{17}{2}$ edges.

If $x=y=11$, then there is a bipartite 1-planar graph G such that the partite sets of G have sizes x and $|E(G)|=3|V(G)|-8$, see [6]. ∎

The following result provides a lower-bound improvement in the case when G is very close to being balanced.

Lemma 6. *Let x,y,z be positive integers such that $x \geq 3$, $y=x+z$, $z \geq 0$. Then there exists a bipartite 1-planar graph G such that the partite sets of G have sizes x and y and $|E(G)|=3|V(G)|-8-z$.*

Proof. First we take a 1-planar drawing of a bipartite 1-planar graph G on $x+x$ vertices and $6x-8$ edges (given in [6]). The edges of this drawing divide the plane into some regions. We insert z vertices to the region which is incident with two vertices from the same partite set and join the added vertices with these two ones (without edge crossings) as shown in Figure 5.

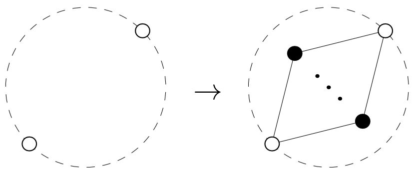

Figure 5. The extension of G.

Let H denote the obtained graph. Clearly, H is a bipartite 1-planar graph with $|V(H)| = |V(G)| + z = 2x + z$ and $|E(H)| = |E(G)| + 2z = 6x - 8 + 2z = 3|V(H)| - 8 - z$. ■

For the sake of completeness we describe a construction of a bipartite 1-planar graph G on $x + x$ vertices and $3(x + x) - 8$ edges.

Let $x = 2k$ for some positive integer k. If $k = 1$, then G is a cycle on four vertices. Let H be a graph consisting of $k \geq 2$ cycles $C_i = x_{1,i}y_{1,i}x_{2,i}y_{2,i}x_{1,i}$ on four vertices, $i = 1, \ldots, k$. Take an embedding of H such that the cycle C_i is in the inner part of C_j (i.e., inside the bounded part of the plane with boundaries determined by C_j) if $i < j$. Next we extend this drawing of H by adding the edges $x_{1,i}y_{1,i+1}$, $x_{1,i}y_{2,i+1}$, $x_{2,i}y_{1,i+1}$, $x_{2,i}y_{2,i+1}$, $x_{1,i+1}y_{1,i}$, $x_{1,i+1}y_{2,i}$, $x_{2,i+1}y_{1,i}$ and $x_{2,i+1}y_{2,i}$ for $i = 1, \ldots, k - 1$ so that the edge $x_{\ell,i+1}y_{j,i}$ crosses the edge $x_{\ell,i}y_{j,i+1}$ for $j, \ell \in \{1, 2\}$, $i = 1, \ldots, k - 1$ (see Figure 6).

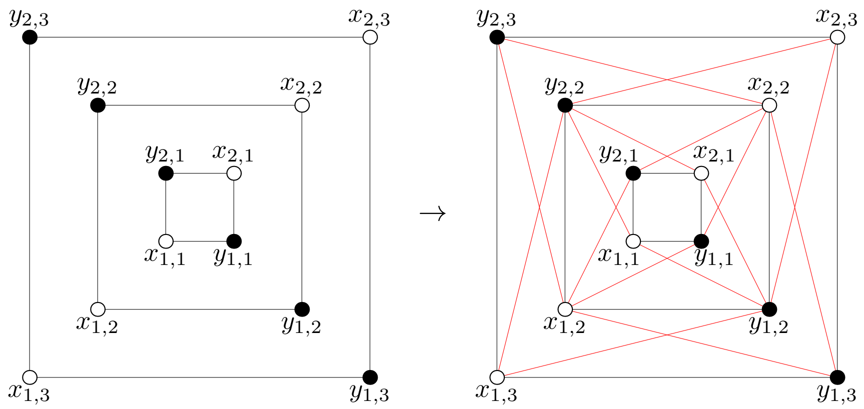

Figure 6. A construction of a bipartite 1-planar graph with $x + x$ vertices and $6x - 8$ edges for x even.

The new graph has $4k - 8$ vertices of degree six and eight vertices of degree four, therefore it has $12k - 8$ edges.

If $x = 2k+1$, then we modify the graph obtained for $x = 2k$ in the following way. First we remove the edges $x_{1,i}y_{1,i-1}$ for $i = 2, 3, \ldots, k$ and the edges $x_{1,i}y_{1,i}$ for $i = 2, 3, \ldots, k-1$. Thereafter we add the edges $x_{1,i}y_{1,i+2}$ for $i = 1, 2, \ldots, k-2$ and the edges $x_{1,i}y_{1,i+3}$ for $i = 1, 2, \ldots, k-3$. Finally, we add a vertex to the region which is incident with the vertices $x_{1,1}, y_{1,1}, y_{1,2}$ and join it with the vertices $y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}$; then add a vertex to the region which is incident with the vertices $x_{1,k-1}, x_{1,k}, y_{1,k}$ and join it with the vertices $x_{1,k-2}, x_{1,k-1}, x_{1,k}, x_{2,3}$, as shown in Figure 7.

We removed $2k-3$ edges and added two vertices and $2k+3$ edges. Therefore the obtained bipartite 1-planar graph has $4k+2$ vertices and $(12k-8) - (2k-3) + (2k+3) = 12k-2 = 3(4k+2) - 8$ edges.

5. Comments

For given integers x, y, $x \leq y$, let $G_{x,y}$ be a bipartite 1-planar graph with partite sets of sizes x and y with the maximal number of edges. Denote by $g_{x,y}$ the size of this graph. By [6], we always have $g_{x,y} \leq 3|V(G_{x,y})| - 8$. It follows from Lemma 6, that $g_{x,y}$ keeps close to $3|V(G_{x,y})|$ if $G_{x,y}$ is balanced enough, i.e., when x is not significantly smaller than y. The larger is the difference between x and y, the smaller the ratio $g_{x,y}/|V(G_{x,y})|$ is, as shown in Corollary 2. By Lemma 4 however the number of edges never drops under $2|V(G_{x,y})|$ if $x \geq 3$. This implies a natural question on how the ratio $g_{x,y}/|V(G_{x,y})|$ depends on the proportion of x and y, in particular, when this ratio gets closer to 2 rather than 3.

Our research was thus motivated by the wish to reveal a kind of threshold for x, given by a function of y under which $g_{x,y}/|V(G_{x,y})|$ actually converges to 2 as y tends to infinity. The results of this paper imply the following solution of this problem. Suppose $x = f(y)$ is any fixed linear function of y (e.g., $x = 0.1y$), then by Corollary 2 and Lemma 4 there exist constants c_1 and c_2 such that

$$(2 + c_1)|V(G_{x,y})| \leq g_{x,y} \leq (2 + c_2)|V(G_{x,y})|$$

(for y large enough). If on the other hand, x is expressed by any sublinear function of y, then $g_{x,y}/|V(G_{x,y})| = 2 + o(1)$, cf. Corollary 2.

Note also that if $x \geq \frac{1}{6}y + 2$, then by Lemma 5, $g_{x,y}$ exceeds $\frac{5}{2}|V(G_{x,y})|$. On the other hand, we believe that for $x \leq \frac{1}{6}y + 2$, our construction from Lemma 4 is optimal and thus conclude by posing the following conjecture.

Conjecture 7. *For any integers x, y such that $x \geq 3$ and $y \geq 6x - 12$, every bipartite 1-planar graph G with partite sets of sizes x and y has at most $2|V(G)| + 4x - 12$ edges.*

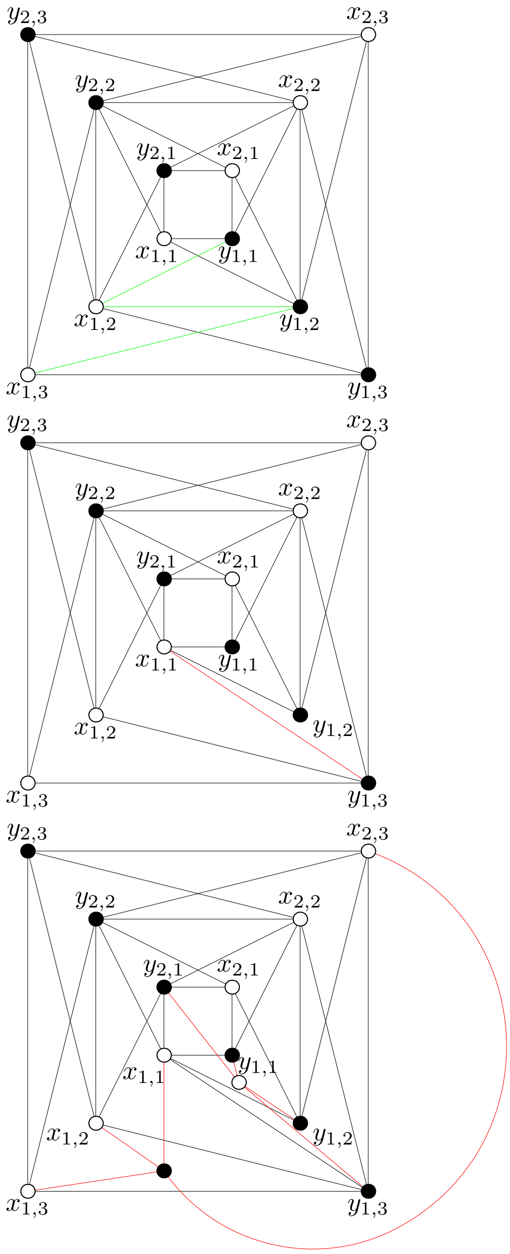

Figure 7. A construction of a bipartite 1-planar graph with $x + x$ vertices and $6x - 8$ edges for x odd.

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