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VERTICES CONTAINED IN ALL OR IN NO MINIMUM SEMITOTAL DOMINATING SET OF A TREE

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Abstract

Let G be a graph with no isolated vertex. In this paper, we study a parameter that is squeezed between arguably the two most important domination parameters; namely, the domination number, $\gamma(G)$, and the total domination number, $\gamma_t(G)$. A set S of vertices in a graph G is a semitotal dominating set of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal domination number, $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of G. We observe that $\gamma(G) \le \gamma_{t2}(G) \le \gamma_t(G)$. We characterize the set of vertices that are contained in all, or in no minimum semitotal dominating set of a tree.

Keywords: domination, semitotal domination, trees.

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1. Introduction

In this paper, we continue the study of a parameter, called the semitotal domination number, that is squeezed between arguably the two most important domination parameters; namely, the domination number and the total domination

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number. A *dominating set* in a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A *total dominating set*, abbreviated a TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the so-called domination book [4]. Total domination is now well studied in graph theory. For a recent book on the topic, see [9]. A survey of total domination in graphs can also be found in [5].

The concept of semitotal domination in graphs was introduced and studied by Goddard, Henning and McPillan [3], and studied further in [6, 7] and elsewhere. A set S of vertices in a graph G with no isolated vertices is a *semitotal dominating set*, abbreviated semi-TD-set, of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The *semitotal domination number*, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-TD-set of G. A semi-TD-set of G of cardinality $\gamma_{t2}(G)$ is called a $\gamma_{t2}(G)$*-set*. Since every TD-set is a semi-TD-set, and since every semi-TD-set is a dominating set, we have the following observation first observed in [3]. For every graph G with no isolated vertex, $\gamma(G) \le \gamma_{t2}(G) \le \gamma_t(G)$.

Mynhardt [10] characterized all the vertices that are in all, or in no minimum dominating set. Moreover, the same type of results were established by Cockayne, Henning and Mynhardt in [2] for total domination, Henning and Plummer [8] for paired domination and Blidia, Chellali and Khelifi [1] for double domination. Motivated by these results, we aim to characterize all the vertices that are in all, or in no minimum semitotal dominating set in a rooted tree T.

1.1. Terminology and Notation

For notation and graph theory terminology that are not defined herein, we refer the reader to [9]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n = |V|$ and edge set $E = E(G)$ of size $m = |E|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. A *leaf* of G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex with at least two leaf-neighbors. We define a *branch vertex* as a vertex of degree at least 3. A *star* is a tree with at most one vertex that is not a leaf.

For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A *cycle* and *path* on n vertices are denoted by C_n and P_n, respectively. For two vertices u and v in a connected graph G, the *distance* $d_G(u, v)$ between u and v is the length of a shortest (u, v)-path in G. The distance $d_G(v, S)$ between a vertex

v and a set S of vertices in a graph G is the minimum distance from v to a vertex of S in G. The maximum distance among all pairs of vertices of G is the *diameter* of a graph G which is denoted by diam(G). The *open neighborhood* of a vertex v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set

$$N_G(S) = \bigcup\nolimits_{v \in S} N_G(v),$$

and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we omit it in the above expressions. For example, we write $d(u)$, $d(u, v)$, $N(v)$ and $N[v]$ rather than $d_G(u)$, $d_G(u, v)$, $N_G(v)$ and $N_G[v]$, respectively.

Let X and Y be subsets of vertices in G. If $Y \subseteq N[X]$, then we say the set X *dominates* the set Y in G and that the set Y is *dominated* by X. Furthermore, if $Y = \{y\}$, then we simply say that y is dominated by X rather than $\{y\}$ is dominated by X. Thus, if a vertex v is dominated by X, then $N[v] \cap X \neq \emptyset$. We note that if X dominates V, then X is a dominating set in G. Hence, if X is a dominating set in G, then $N[X] = V$. Additionally, we say that X *semitotally dominates* the set Y in G if each vertex in X lies within distance 2 of another vertex in X, and in turn the set Y is said to be *semitotally dominated* by X.

For a graph G, we define the sets $\mathcal{A}_{t2}(G)$ and $\mathcal{N}_{t2}(G)$ as follows:

$$\mathcal{A}_{t2}(G) = \{v \in V(G) \mid v \text{ is in every } \gamma_{t2}(G)\text{-set}\},$$

and

$$\mathcal{N}_{t2}(G) = \{v \in V(G) \mid v \text{ is in no } \gamma_{t2}(G)\text{-set}\}.$$

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. We denote all the children of a vertex v by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. Thus, every child of v is a descendant of v. A *grandchild* of v is a descendant of v at distance 2 from v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The set of leaves in T is denoted by $L(T)$ and the set of support vertices is denoted by $S(T)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. The set of leaves in T_v distinct from v we denote by $L(v)$; that is, $L(v) = D(v) \cap L(T)$. The set of branch vertices of T is denoted by $B(T)$. For $j \in \{0, 1, 2, 3, 4\}$, we define

$$L^j(v) = \{u \in L(v) \mid d(u, v) \equiv j \pmod 5\}.$$

Furthermore, let

$$L^1_1(v) = \{x \in L^1(v) \mid d(v, x) = 1\} \quad \text{and} \quad L^1_2(v) = L^1(v) \setminus L^1_1(v).$$

We sometimes write $L^j_T(v)$ to emphasize the tree (or subtree) concerned. Additionally, we define the path from v to a leaf in $L^j(v)$ to be a $L^j(v)$-*path*. Given a vertex x of a tree T, we say we *attach a path* of length q to x if we add a vertex-disjoint path P_q on q vertices and join x to a leaf of the path P_q. In this case, we simply write that we *attach* P_q to x. We next define an essential support vertex in a tree.

Definition 1. A vertex v in a tree T is an *essential support vertex* in T if and only if v has exactly one leaf-neighbor, $v \in \mathcal{A}_{t2}(T)$ and $N(v) \subseteq \mathcal{N}_{t2}(T)$.

We note that if v is an essential support vertex in a tree T, then v has exactly one leaf-neighbor and $N[v] \cap D = \{v\}$ for every $\gamma_{t2}(T)$-set D.

2. Tree Pruning

In this paper, we use a method called *tree pruning* to characterize the sets $\mathcal{A}_{t2}(T)$ and $\mathcal{N}_{t2}(T)$ for an arbitrary tree T. Let T be a tree rooted at a vertex v. Suppose that T is not a star. We let $C^{(4)}(v)$ denote the set of children of v that belong to P_4's that are attached to v. Furthermore, we let the descendants at distance 2 from v along P_5's that are attached to v be denoted by $\mathrm{Gr}(v)$ and we call them *special grandchildren* of v. The pruning of T is performed with respect to its root, v. If $d(u) \le 2$ for each $u \in V(T_v) \setminus \{v\}$, then let $\overline{T}_v = T$. Otherwise, let u be a branch vertex at maximum distance from v (we note that $|C(u)| \ge 2$ and $d(x) \le 2$ for each $x \in D(u)$). We identify the following types of branch vertices:

(T.1) $|L^3(u)| \ge 1$.
(T.2) $L^3(u) = \emptyset$, $|L^1(u)| \ge 1$ and $|L^0(u) \cup L^2(u) \cup L^4(u)| \ge 1$.
(T.3) $L^3(u) = L^0(u) = L^2(u) = L^4(u) = \emptyset$ and $|L^1(u)| \ge 2$.
(T.4) $L^3(u) = L^1(u) = \emptyset$ and $|L^4(u)| \ge 1$.
(T.5) $L^3(u) = L^1(u) = L^4(u) = \emptyset$, $|L^2(u)| = 1$ and $|L^0(u)| \ge 1$.
(T.6) $L^3(u) = L^1(u) = L^4(u) = \emptyset$ and $|L^2(u)| \ge 2$.
(T.7) $L^3(u) = L^1(u) = L^4(u) = L^2(u) = \emptyset$.

We now apply the following pruning process.
(a) If u is type **(T.1)** or **(T.2)**, then delete $D(u)$ and attach a P_3 to u.
(b) If u is type **(T.3)**, then delete $D(u)$ and attach a P_1 to u.
(c) If u is type **(T.4)** or **(T.6)**, then delete $D(u)$ and attach a P_4 to u.
(d) If u is type **(T.5)**, then delete $D(u)$ and attach a P_2 to u.
(e) If u is type **(T.7)**, then delete $D(u)$ and attach a P_5 to u.

This step of the pruning process, where all the descendants of u are deleted and a path of length 1, 2, 3, 4 or 5 is attached to u to give a tree in which u has degree 2, is called a *pruning of* T_v *at* u. Repeat the above process until a tree

$\overline{T}_v$ is obtained with $d(u) \leq 2$ for each $u \in V(\overline{T}_v) \setminus \{v\}$. The tree $\overline{T}_v$ is called the *pruning of* T_v. To simplify notation, we write $\overline{L}^j(v)$ instead of $L^j_{\overline{T}_v}(v)$.

3. MAIN RESULTS

In this paper, we aim to establish a characterization of the set of vertices contained in all or none of the minimum semi-TD-sets in a tree T of order $n \geq 2$.

In the trivial case when $T = P_2$, we note that $\mathcal{A}_{t2}(T) = V(T)$, while if $T = P_3$, then $\mathcal{A}_{t2}(T) = \mathcal{N}_{t2}(T) = \emptyset$. If T is a star $K_{1,n-1}$ with central vertex v and $n \geq 4$, then $\mathcal{A}_{t2}(T) = \{v\}$ and $\mathcal{N}_{t2}(T) = \emptyset$. Hence in what follows we restrict our attention to the more interesting case when $n \geq 4$ and T is not a star. We shall prove the following main results.[3]

Theorem 1. *Let T be a tree with order at least 4 that is not a star and is rooted at a vertex v such that $d(u) \leq 2$ for each $u \in V(T) \setminus \{v\}$. Then,*

(a) *$v \in \mathcal{A}_{t2}(T)$ if and only if one of the following hold:*
  - (i) $|L^3(v)| \geq 1$ *and* $|L^1(v) \cup L^3(v)| \geq 2$.
  - (ii) $L^3(v) = \emptyset$ *and* $|L^1(v)| \geq 3$.
  - (iii) $L^3(v) = \emptyset$ *and* $|L^1_1(v)| = 2$.
  - (iv) $L^3(v) = \emptyset$, $|L^1_1(v)| \leq 1$, $|L^1(v)| = 2$ *and* $|L^0(v) \cup L^2(v) \cup L^4(v)| \geq 1$.
  - (v) $L^2(v) = L^3(v) = L^4(v) = \emptyset$, $|L^1(v)| = |L^1_1(v)| = 1$ *and* $|L^0(v)| \geq 1$.

(b) *$v \in \mathcal{N}_{t2}(T)$ if and only if one of the following hold:*
  - (i) $L^1(v) = L^3(v) = \emptyset$ *and* $|L^4(v)| \geq 1$, *or*
  - (ii) $L^1(v) = L^3(v) = L^4(v) = \emptyset$ *and* $|L^2(v)| \geq 2$.

Theorem 2. *Let v be a vertex of a tree T with order at least 4 that is not a star. Then,*

(a) *$v \in \mathcal{A}_{t2}(T)$ if and only if one of the following hold:*
  - (i) $|\overline{L}^3(v)| \geq 1$ *and* $|\overline{L}^1(v) \cup \overline{L}^3(v)| \geq 2$.
  - (ii) $\overline{L}^3(v) = \emptyset$ *and* $|\overline{L}^1(v)| \geq 3$.
  - (iii) $\overline{L}^3(v) = \emptyset$ *and* $|\overline{L}^1_1(v)| = 2$.
  - (iv) $\overline{L}^3(v) = \emptyset$, $|\overline{L}^1_1(v)| \leq 1$, $|\overline{L}^1(v)| = 2$ *and* $|\overline{L}^0(v) \cup \overline{L}^2(v) \cup \overline{L}^4(v)| \geq 1$.
  - (v) $\overline{L}^2(v) = \overline{L}^3(v) = \overline{L}^4(v) = \emptyset$, $|\overline{L}^1(v)| = |\overline{L}^1_1(v)| = 1$ *and* $|\overline{L}^0(v)| \geq 1$.

(b) *$v \in \mathcal{N}_{t2}(T)$ if and only if one of the following hold:*
  - (i) $\overline{L}^1(v) = \overline{L}^3(v) = \emptyset$ *and* $|\overline{L}^4(v)| \geq 1$, *or*
  - (ii) $\overline{L}^1(v) = \overline{L}^3(v) = \overline{L}^4(v) = \emptyset$ *and* $|\overline{L}^2(v)| \geq 2$.

[3] An example to illustrate Theorem 2 is presented in the appendix.

4. Preliminary Results

The semitotal domination number of a path and a cycle is determined in [3].

Lemma 3 [3]**.** *For $n \geq 3$, $\gamma_{t2}(P_n) = \gamma_{t2}(C_n) = \lceil \frac{2n}{5} \rceil$.*

Lemma 3 immediately infers that every path P_n where $n \equiv 0 \pmod 5$ has a unique $\gamma_{t2}(P_n)$-set. That is, if we number the vertices in $V(P_n)$ consecutively starting at 1, then the $\gamma_{t2}(P_n)$-set is the set of all vertices with numbers congruent to $2 \pmod 5$ and $4 \pmod 5$. Additionally, the paths P_2 and P_7 also have unique minimum semi-TD-sets. We state this formally as follows.

Observation 4. *The paths P_2, P_7 and P_n, where $n \equiv 0 \pmod 5$, all have unique minimum semi-TD-sets.*

We shall need the following result first observed in [6].

Observation 5. *If G is a connected graph that is not a star, then there is a $\gamma_{t2}(G)$-set that contains no leaf of G.*

We proceed with the following two lemmas that will be useful when proving our main results. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

Lemma 6. *Let T be a tree of order at least 3. Let t be a support vertex in T and let u' be a leaf-neighbor of t. If T' is the tree obtained from T by attaching a path of length 5 to u', then $\gamma_{t2}(T') = \gamma_{t2}(T) + 2$.*

Proof. Suppose T' is obtained from T by adding to u' the path $uwxyz$ together with the edge uu'. Every $\gamma_{t2}(T)$-set can be extended to a semi-TD-set of T' by adding to it the vertices w and y, and so $\gamma_{t2}(T') \leq \gamma_{t2}(T) + 2$. Let D' be a $\gamma_{t2}(T')$-set. If $z \in D'$, then we can replace z in D' by y. Hence we may choose D' so that $D' \cap \{y, z\} = \{y\}$. In order to semitotally dominate the vertex y, we note that x or w belong to D'. If $x \in D'$, then we can replace x in D' by w. Hence we may choose D' so that $D' \cap \{x, w\} = \{w\}$. If $u \in D'$, then we can replace u in D' by u'. Hence we may choose D' so that $u \notin D'$. If $t \in D'$, then we can replace u' in D' with a neighbor of t different from u'. If $t \notin D'$ and $|D' \cap N(t)| \geq 2$, then we can replace u' in D' with the vertex t. If $t \notin D'$ and $D' \cap N[t] = \{u'\}$, then in order to dominate the neighbors of t different from u', the set D' contains at least one vertex at distance 2 from t in T, implying once again that we can replace u' in D' with the vertex t. Hence, we may choose D' so that $u' \notin D'$. In order to dominate the vertex u', we note that $t \in D'$. Since D' is a semi-TD-set of T', the set $D' \setminus \{w, y\}$ is necessarily a semi-TD-set of T, implying that $\gamma_{t2}(T) \leq |D'| - 2 = \gamma_{t2}(T') - 2$. Consequently, $\gamma_{t2}(T') = \gamma_{t2}(T) + 2$. ■

Lemma 7. *Let T be a tree with order at least 3. Let t be a support vertex in T and let u' be a leaf-neighbor of t. Let T' be the tree obtained from T by attaching a path of length 5 to u'. If t is an essential support vertex in T, let $v \in V(T)\setminus\{u', t\}$. If t is not an essential support vertex in T, let $v \in V(T)$. Then the following hold.*

(a) $v \in \mathcal{A}_{t2}(T)$ *if and only if* $v \in \mathcal{A}_{t2}(T')$.

(b) $v \in \mathcal{N}_{t2}(T)$ *if and only if* $v \in \mathcal{N}_{t2}(T')$.

Proof. Suppose T' is obtained from T by adding to u' the path $uwxyz$ together with the edge uu'.

(a) Suppose that $v \notin \mathcal{A}_{t2}(T)$. Let D be a $\gamma_{t2}(T)$-set that does not contain v. Then, $D\cup\{w, y\}$ is a semi-TD-set of T' of cardinality $|D|+2 = \gamma_{t2}(T)+2 = \gamma_{t2}(T')$ by Lemma 6. Consequently, $D \cup \{w, y\}$ is a $\gamma_{t2}(T')$-set that does not contain v, implying that $v \notin \mathcal{A}_{t2}(T')$. Therefore, by contraposition, if $v \in \mathcal{A}_{t2}(T')$, then $v \in \mathcal{A}_{t2}(T)$.

Conversely, suppose that $v \in \mathcal{A}_{t2}(T)$. Suppose to the contrary that $v \notin \mathcal{A}_{t2}(T')$. Let D' be a $\gamma_{t2}(T')$-set that does not contain the vertex v, and let $D = D' \cap V(T)$. If $v = u'$, then by Observation 5, there exists a $\gamma_{t2}(T)$-set that does not contain v, contradicting our assumption that $v \in \mathcal{A}_{t2}(T)$. Hence, $v \neq u'$. Proceeding as in the proof of Lemma 6, we can choose D' so that $D' \cap \{w, x, y, z, u\} = \{w, y\}$. Thus, $D = D' \setminus \{w, y\}$ and, by Lemma 6, $|D| = |D'| - 2 = \gamma_{t2}(T') - 2 = \gamma_{t2}(T)$. If $v \neq t$, then proceeding as in the proof of Lemma 6, we can additionally choose D' so that $D' \cap \{u', t\} = \{t\}$, implying that the set D is a $\gamma_{t2}(T)$-set that does not contain v, a contradiction. Hence, $v = t$. By supposition, $v \notin D'$, and so neither neighbor of u' in T' belongs to D', implying that $u' \in D'$.

If D is a semi-TD-set in T, then D is a $\gamma_{t2}(T)$-set that does not contain the vertex v, contradicting our supposition that $v \in \mathcal{A}_{t2}(T)$. Hence, D is not a semi-TD-set in T, implying that no vertex in D is at distance 1 or 2 from u'. Thus, $D \cap N[v] = \{u'\}$. In particular, we note that u' is the only leaf-neighbor of v in T.

We show next that for every $\gamma_{t2}(T)$-set S, $N[v] \cap S = \{v\}$. For notational convenience, let T be rooted at the vertex v and let $N(v) \setminus \{u'\} = \{v_1, \ldots, v_k\}$. For $i \in [k]$, let T_i denote the maximal subtree of T rooted at v_i (and so, $T_i = T_{v_i}$) and let $D_i = D \cap V(T_i)$. We note that $v_i \notin D_i$ and that the set D_i is a semi-TD-set in T_i for all $i \in [k]$. Suppose that there exists a $\gamma_{t2}(T)$-set, S, such that $|N[v] \cap S| \geq 2$. Since $v \in \mathcal{A}_{t2}(T)$, we note that $v \in S$. If $u' \in S$, we can simply replace u' in S with a neighbor of v that is not a leaf. Renaming the children of v if necessary, we may therefore assume that $v_1 \in S$. Let $S_1 = S \cap V(T_1)$. Since the set D_1 contains a vertex at distance 2 from v in T, we note that the set $(S \setminus S_1) \cup D_1$ is a semi-TD-set of T, implying that $|S| = \gamma_{t2}(T) \leq |S| - |S_1| + |D_1|$, or, equivalently, $|S_1| \le |D_1|$. We now consider the set $S^* = (D \setminus D_1) \cup S_1$. Since u' and v_1 are at distance 2 apart in T, the set S^* is a semi-TD-set of T, implying that $\gamma_{t2}(T) \le |S^*| \le |D| - |D_1| + |S_1| \le |D| = \gamma_{t2}(T)$. Consequently, $|S^*| = \gamma_{t2}(T)$ and S^* is a $\gamma_{t2}(T)$-set that does not contain the vertex v, a contradiction. Therefore, for every $\gamma_{t2}(T)$-set S, we have $N[v] \cap S = \{v\}$. Moreover, this result together with our earlier observation that u' is the only leaf-neighbor of v in T imply that v is an essential support vertex in T, a contradiction (recalling that here $v = t$). Hence, $v \in \mathcal{A}_{t2}(T')$. This completes the proof of part (a).

(b) Suppose that $v \in \mathcal{N}_{t2}(T')$. We show that $v \in \mathcal{N}_{t2}(T)$. Suppose to the contrary that there exists a $\gamma_{t2}(T)$-set, D, that contains the vertex v. Then, $D \cup \{w, y\}$ is a semi-TD-set of T' of cardinality $|D| + 2 = \gamma_{t2}(T) + 2 = \gamma_{t2}(T')$. Consequently, $D \cup \{w, y\}$ is a $\gamma_{t2}(T')$-set that contains v, a contradiction. Therefore, $v \in \mathcal{N}_{t2}(T)$.

Conversely, suppose that $v \in \mathcal{N}_{t2}(T)$. We show that $v \in \mathcal{N}_{t2}(T')$. Suppose to the contrary that there exists a $\gamma_{t2}(T')$-set, D', that contains the vertex v. Let $D = D' \cap V(T)$. Proceeding as in the proof of Lemma 6, we can choose D' so that $D' \cap \{w, x, y, z, u\} = \{w, y\}$. Thus, $D = D' \setminus \{w, y\}$. If $v \ne u'$, then proceeding as in the proof of Lemma 6, we can further choose D' so that $D' \cap \{u', t\} = \{t\}$, implying that the set D is a $\gamma_{t2}(T)$-set containing v, a contradiction. Hence, $v = u'$. If D is a semi-TD-set in T, then the set D is a $\gamma_{t2}(T)$-set containing v, a contradiction. Hence, D is not a semi-TD-set in T, implying that no vertex in D is at distance 1 or 2 from u'. Thus, $D \cap N[t] = \{u'\}$. In particular, this implies that u' is the only leaf-neighbor of t in T. An analogous proof to that employed in the proof of part (a) shows the vertex t is an essential support vertex in T, contradicting the fact that in this case $v = u'$. Therefore, $v \in \mathcal{N}_{t2}(T')$. ■

5. Proof of Theorem 1

Proof. Let T be a tree with order at least 4 that is not a star and is rooted at a vertex v such that $d(u) \le 2$ for each $u \in V(T) \setminus \{v\}$. For each $w \in L(v)$ such that $d_T(v, w) \ge 6$, let T' be the tree obtained by replacing the (v, w)-path in T with a (v, w)-path of length j, $j \in \{5, 6, 2, 3, 4\}$ if $w \in L^i(v)$, $i \in \{0, 1, 2, 3, 4\}$, respectively. By repeated applications of Lemma 7, $v \in \mathcal{A}_{t2}(T)$ ($\mathcal{N}_{t2}(T)$, respectively) if and only if $v \in \mathcal{A}_{t2}(T')$ ($\mathcal{N}_{t2}(T')$, respectively). Hence, in what follows, we assume $T = T'$. If v is a leaf of T, then by our earlier assumptions, T is a path P_n where $n \in \{4, 5, 6, 7\}$. If $n \in \{4, 6\}$, then $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$. If $n \in \{5, 7\}$, then by Observation 4, $v \in \mathcal{N}_{t2}(T)$. Hence, we may assume that v is not a leaf in T. Let D be an arbitrary $\gamma_{t2}(T)$-set and let W be the set of vertices at distance 3 from a leaf of some $L_2^1(v)$-path. We proceed further with a series of claims.

Claim A. *If* $|L_1^1(v)| \geq 2$, *then* $v \in \mathcal{A}_{t2}(T)$.

Proof. Suppose $|L_1^1(v)| \geq 2$. Thus, v is a strong support vertex in T and therefore has at least two leaf-neighbors. Moreover, $|L^0(v) \cup L_2^1(v) \cup L^2(v) \cup L^3(v) \cup L^4(v)| \geq 1$ since T is not a star. Let w be a neighbor of v that is not a leaf. Suppose, to the contrary, that $v \notin \mathcal{A}_{t2}(T)$. Let S be a $\gamma_{t2}(T)$-set that does not contain the vertex v. The set S contains all leaf-neighbors of v. Since $N[w] \cap S \neq \emptyset$, we note that v is within distance 2 from at least one vertex in $N[w] \cap S$. Further, no vertex in $N[w] \cap S$ is a leaf-neighbor of v. Replacing the leaf-neighbors of v in S with the vertex v produces a semi-TD-set in T of cardinality less than $|S| = \gamma_{t2}(T)$, a contradiction. Hence, $v \in \mathcal{A}_{t2}(T)$. □

By Claim A, we may assume that $|L_1^1(v)| \leq 1$.

Claim B. *If* $L(v) = L^0(v)$, *then* $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$.

Proof. Suppose $L(v) = L^0(v)$. Then, $L^1(v) \cup L^2(v) \cup L^3(v) \cup L^4(v) = \emptyset$. Let $S = \mathrm{Gr}(v) \cup S(T) \cup \{v\}$. The set S is a semi-TD-set of T, and so $\gamma_{t2}(T) \leq |S| = 2|L^0(v)| + 1$. Recall that D is an arbitrary $\gamma_{t2}(T)$-set. If $vv_1v_2v_3v_4v_5$ is a path emanating from v in T, then v_5 is a leaf in T and $|D \cap \{v_2, v_3, v_4, v_5\}| \geq 2$, implying that the set D contains at least two vertices from each path of order 5 attached to v and at least one vertex in $N[v]$. Thus, $\gamma_{t2}(T) = |D| \geq 2|L^0(v)|+1 = |S| \geq \gamma_{t2}(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $|S| = \gamma_{t2}(T) = 2|L^0(v)| + 1$ and S is a $\gamma_{t2}(T)$-set. Replacing v in S with an arbitrary neighbor of v produces a $\gamma_{t2}(T)$-set not containing v. Hence, $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$. □

By Claim B, we may assume that $L(v) \neq L^0(v)$.

Claim C. *If* $L(v) = L^0(v) \cup L_1^1(v)$ *where* $|L_1^1(v)| = 1$ *and* $|L^0(v)| \geq 1$, *then* v *is an essential support vertex in* T. *In particular,* $v \in \mathcal{A}_{t2}(T)$.

Proof. Suppose $L(v) = L^0(v) \cup L_1^1(v)$ where $|L_1^1(v)| = 1$ and $|L^0(v)| = k \geq 1$. In this case, $L_2^1(v) \cup L^2(v) \cup L^3(v) \cup L^4(v) = \emptyset$. Let $L_1^1(v) = \{u\}$. We note that u is the only leaf-neighbor of v in T. We show that $v \in \mathcal{A}_{t2}(T)$ and $N(v) \subseteq \mathcal{N}_{t2}(T)$, implying that v is an essential support vertex of T. Let $S = \mathrm{Gr}(v) \cup S(T) \cup \{v\}$. The set S is a semi-TD-set of T, and so $\gamma_{t2}(T) \leq |S| = 2k + 1$. If $vv_1v_2v_3v_4v_5$ is a path emanating from v in T, then v_5 is a leaf in T and $|D \cap \{v_2, v_3, v_4, v_5\}| \geq 2$. In particular, the set D contains at least two vertices from each path of order 5 attached to v. Further, D contains at least one of u and v. Thus, $\gamma_{t2}(T) = |D| \geq 2k + 1 = |S| \geq \gamma_{t2}(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $|S| = \gamma_{t2}(T) = 2k + 1$, implying that S is a $\gamma_{t2}(T)$-set.

Suppose that there exists a $\gamma_{t2}(T)$-set, D', that does not contain v. In this case, $u \in D'$. Further, in order to semitotally dominate u, we note that $|(D' \setminus \{u\}) \cap N(v)| \geq 1$. This, however, implies that along one of P_5's attached to v in T, at least three of its vertices belong to D', which in turn implies that $|D'| \geq 2k+2 > |S|$, a contradiction. Hence, $v \in \mathcal{A}_{t2}(T)$. As observed earlier, if $vv_1v_2v_3v_4v_5$ is a path emanating from v in T, then $|D \cap \{v_2, v_3, v_4, v_5\}| \geq 2$. Further, since $v \in \mathcal{A}_{t2}(T)$, we note that $v \in D$. Thus if $|D \cap N(v)| \geq 1$, then $\gamma_{t2}(T) = |D| \geq 2k+2$, a contradiction. Therefore, $N(v) \cap D = \emptyset$, implying that $N(v) \subseteq \mathcal{N}_{t2}(T)$. Thus, v is an essential support vertex in T. □

By our earlier assumptions, $|L_1^1(v)| \leq 1$ and $L(v) \neq L^0(v)$. By Claim C, we may assume that $L(v) \neq L^0(v) \cup L_1^1(v)$.

Claim D. *Suppose $|L^3(v)| \geq 1$. Then the following hold.*

(a) *If $|L^3(v)| \geq 2$, then $v \in \mathcal{A}_{t2}(T)$.*

(b) *If $|L^3(v)| = 1$ and $|L^1(v)| \geq 1$, then $v \in \mathcal{A}_{t2}(T)$.*

(c) *If $|L^3(v)| = 1$, $L^1(v) = \emptyset$ and $|L^0(v) \cup L^2(v) \cup L^4(v)| \geq 1$, then $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$.*

Proof. (a) Suppose $|L^3(v)| \geq 2$. Let $\{u_3, v_3\} \subseteq L^3(v)$ and let $vu_1u_2u_3$ and $vv_1v_2v_3$ be the (v, u_3)-path and the (v, v_3)-path. By our earlier assumptions, the vertex v has at most one leaf-neighbor. Further, we remark that there may exist leaves at distance 2, 4, 5 and 6 from v in T. The set $S(T) \cup C^{(4)}(v) \cup \mathrm{Gr}(v) \cup W \cup \{v\}$ is a semi-TD-set of cardinality $2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + |L^3(v)| + 1$, and so $\gamma_{t2}(T) \leq 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + |L^3(v)| + 1$.

Suppose D does not contain v. Then, D will contain at least two vertices from each $L^0(v)$-path, at least three vertices from each $L_2^1(v)$-path, at least one vertex from each $L^2(v)$-path, at least two vertices from each $L^3(v)$-path, and at least two vertices from each $L^4(v)$-path. Further, if $|L_1^1(v)| = 1$, then D contains the leaf-neighbor of v. If $u_3 \in D$, we can replace u_3 in D with u_2. Hence, we may choose D so that $D \cap \{u_1, u_2, u_3\} = \{u_1, u_2\}$. This implies that $\gamma_{t2}(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + 3|L_2^1(v)| + |L^2(v)| + 2|L^3(v)| + |L_1^1(v)| \geq 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 2|L^3(v)| > 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + |L^3(v)| + 1$, a contradiction. Hence, $v \in D$. Since D is an arbitrary $\gamma_{t2}(T)$-set, we deduce that $v \in \mathcal{A}_{t2}(T)$.

(b) Suppose that $|L^3(v)| = 1$ and $|L^1(v)| \geq 1$. Let $L^3(v) = \{u_3\}$ and let $vu_1u_2u_3$ be the (v, u_3)-path. Suppose firstly that $L_1^1(v) = \emptyset$, and so $L^1(v) = L_2^1(v)$. In this case, the set $S(T) \cup C^{(4)}(v) \cup \mathrm{Gr}(v) \cup W \cup \{v\}$ is a semi-TD-set of cardinality $2(|L^0(v)| + |L^1(v)| + |L^4(v)|) + |L^2(v)| + 2$, and so $\gamma_{t2}(T) \leq 2(|L^0(v)| + |L^1(v)| + |L^4(v)|) + |L^2(v)| + 2$. Suppose D does not contain v. Then, D contains at least two vertices on the path $u_1u_2u_3$ and at least three vertices from each $L_2^1(v)$-path. Further, D contains at least two vertices from each $L^0(v)$-path, two vertices from each $L^4(v)$-path and one vertex from each $L^2(v)$-path. However, this implies that $\gamma_{t2}(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + 3|L^1(v)| + |L^2(v)| + 2 > 2(|L^0(v)| + |L^1(v)| + |L^4(v)|) + |L^2(v)| + 2$, a contradiction. Hence, $v \in D$, and since D is an arbitrary $\gamma_{t2}(T)$-set, $v \in \mathcal{A}_{t2}(T)$.

Suppose secondly that $|L_1^1(v)| = 1$. Let $L_1^1(v) = \{u\}$. In this case, the set $S(T) \cup C^{(4)}(v) \cup \mathrm{Gr}(v) \cup W \cup \{v\}$ is a semi-TD-set of cardinality $2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 2$, and so $\gamma_{t2}(T) \leq 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 2$. Suppose D does not contain v. Then, $u \in D$ and D contains at least two vertices on the path $u_1u_2u_3$ and at least three vertices from each $L_2^1(v)$-path. The number of vertices needed from each $L^0(v)$-path, $L^2(v)$-path and $L^4(v)$-path remains unchanged. However, this implies that $\gamma_{t2}(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + 3|L_2^1(v)| + |L^2(v)| + 2|L^3(v)| + |L_1^1(v)| = 2(|L^0(v)| + |L^4(v)|) + 3|L_2^1(v)| + |L^2(v)| + 3 > 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 2$, a contradiction. Hence, $v \in D$, and since D is an arbitrary $\gamma_{t2}(T)$-set, $v \in \mathcal{A}_{t2}(T)$.

(c) Suppose that $|L^3(v)| = 1$, $L^1(v) = \emptyset$ and $|L^0(v) \cup L^2(v) \cup L^4(v)| \geq 1$. Let $L^3(v) = \{u_3\}$ and let $vu_1u_2u_3$ be the (v, u_3)-path. Every leaf of T, different from u_3, is at distance 2, 4 or 5 from v, and so $L(v) \setminus \{u_3\} = L^0(v) \cup L^2(v) \cup L^4(v)$. By Observation 5, there is a $\gamma_{t2}(T)$-set, say D', that contains no leaf of T, implying that $S(T) \subseteq D'$. The set D' contains at least two vertices from each $L^0(v)$-path and at least two vertices from each $L^4(v)$-path. Further, D' contains at least one vertex from each $L^2(v)$-path and at least two vertices from the (v, u_3)-path. This implies that $\gamma_{t2}(T) \geq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 2$. On the other hand, the set of children of v that do not belong to any $L^0(v)$-path, together with the set $S(T) \cup \mathrm{Gr}(v)$ form a semi-TD-set, say S, of T of cardinality $2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 2$, implying that $\gamma_{t2}(T) \leq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 2$. Consequently, $\gamma_{t2}(T) = 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 2$. Moreover, S and $(S \setminus \{u_1\}) \cup \{v\}$ are $\gamma_{t2}(T)$-sets, implying that $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$. □

By Claim D, we may assume that $L^3(v) = \emptyset$.

Claim E. *If* $|L^1(v)| \geq 3$, *then* $v \in \mathcal{A}_{t2}(T)$.

Proof. Suppose, firstly, that $L^0(v) \cup L^2(v) \cup L^4(v) \neq \emptyset$. The vertex set $S(T) \cup C^{(4)}(v) \cup \mathrm{Gr}(v) \cup W \cup \{v\}$ is a semi-TD-set of cardinality $2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$, and so $\gamma_{t2}(T) \leq 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$. Suppose D does not contain v. If $L_1^1(v) = \emptyset$, then every leaf is at distance 2, 4, 5 or 6 from v in T and $L^1(v) = L_2^1(v)$. In this case, D contains at least three vertices from each $L_2^1(v)$-path, two vertices from each $L^0(v)$-path, two vertices from each $L^4(v)$-path and one vertex from each $L^2(v)$-path. Hence, $\gamma_{t2}(T) = |D| > 3|L^1(v)| + 2(|L^0(v) + |L^4(v)|) + |L^2(v)| > 2(|L^0(v)| + |L^1(v)| + |L^4(v)|) + |L^2(v)| + 1$, a contradiction. Therefore, $L_1^1(v) \neq \emptyset$. Let $L_1^1(v) = \{u\}$. Every leaf is at distance 1, 2, 4, 5 or 6 from v in T. In this case, D contains the leaf u, implying that $\gamma_{t2}(T) = |D| > 3|L_2^1(v)| + 2(|L^0(v) + |L^4(v)|) + |L^2(v)| + |L_1^1(v)| > 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$, a contradiction. Hence, $v \in D$, and since D is an arbitrary $\gamma_{t2}(T)$-set, $v \in \mathcal{A}_{t2}(T)$.

Suppose, secondly, that $L^0(v) \cup L^2(v) \cup L^4(v) = \emptyset$. Thus, $L(v) = L^1(v)$. Let $u_6 \in L^1(v)$ and let $vu_1u_2u_3u_4u_5u_6$ be the (v, u_6)-path. The vertex set $S(T) \cup W \cup \{u_1, v\}$ is a semi-TD-set of cardinality $2|L_2^1(v)| + 2$, and so $\gamma_{t2}(T) \le 2|L_2^1(v)| + 2$. Suppose D does not contain v. If $L_1^1(v) = \emptyset$, then every leaf is at distance 6 from v in T and $L(v) = L^1(v) = L_2^1(v)$. In this case, D contains at least three vertices from each $L^1(v)$-path. Hence, $\gamma_{t2}(T) = |D| \ge 3|L^1(v)| > 2|L^1(v)| + 2$, a contradiction. If $L_1^1(v) \ne \emptyset$, then letting $L_1^1(v) = \{u\}$, every leaf in $L^1(v) \setminus \{u\}$ is at distance 6 from v in T. In this case, D contains at least three vertices from each $L_2^1(v)$-path and the leaf u. Hence, $\gamma_{t2}(T) = |D| \ge 3|L_2^1(v)| + 1 > 2|L_2^1(v)| + 2$, a contradiction. Hence, $v \in D$, and since D is an arbitrary $\gamma_{t2}(T)$-set, $v \in \mathcal{A}_{t2}(T)$. □

By Claim E, we may assume that $|L^1(v)| \le 2$.

Claim F. *Suppose $|L^1(v)| = 2$. Then the following hold.*

(a) *If $|L^0(v) \cup L^2(v) \cup L^4(v)| \ge 1$, then $v \in \mathcal{A}_{t2}(T)$.*

(b) *If $L^0(v) = L^2(v) = L^4(v) = \emptyset$, then $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$.*

Proof. (a) Suppose $|L^0(v) \cup L^2(v) \cup L^4(v)| \ge 1$. The vertex set $S(T) \cup C^{(4)}(v) \cup \mathrm{Gr}(v) \cup W \cup \{v\}$ is a semi-TD-set of cardinality $2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$, and so $\gamma_{t2}(T) \le 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$. Suppose D does not contain v. If $L_1^1(v) = \emptyset$, then $L^1(v) = L_2^1(v)$ and $|L_2^1(v)| = 2$. In this case, D contains at least three vertices from each $L_2^1(v)$-path, two vertices from each $L^0(v)$-path, two vertices from each $L^4(v)$-path and one vertex from each $L^2(v)$-path. Hence, $\gamma_{t2}(T) = |D| \ge 2(|L^0(v) + |L^4(v)|) + |L^2(v)| + 6 > 2(|L^0(v) + |L^4(v)|) + |L^2(v)| + 5 = 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$, a contradiction. Therefore, $L_1^1(v) \ne \emptyset$. Let $L_1^1(v) = \{u\}$ and let $L_2^1(v) = \{u_6\}$. Additionally, let $vu_1u_2u_3u_4u_5u_6$ be the (v, u_6)-path. In this case, D contains the leaf u and at least three vertices from the (u_1, u_6)-path, at least one vertex from each $L^2(v)$-path and at least two vertices from each $L^0(v)$-path and $L^4(v)$-path, implying that $\gamma_{t2}(T) = |D| \ge 2(|L^0(v) + |L^4(v)|) + |L^2(v)| + 4 > 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 3 = 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$, a contradiction. Hence, $v \in D$, and since D is an arbitrary $\gamma_{t2}(T)$-set, $v \in \mathcal{A}_{t2}(T)$.

(b) Suppose $L^0(v) = L^2(v) = L^4(v) = \emptyset$. Let $u_6 \in L_2^1(v)$ and let the path $vu_1u_2u_3u_4u_5u_6$ be the (v, u_6)-path. Suppose firstly that $L_1^1(v) = \emptyset$. Then, $L^1(v) = L_2^1(v)$. Let $v_6 \in L_2^1(v) \setminus \{u_6\}$ and let $vv_1v_2v_3v_4v_5v_6$ be the (v, v_6)-path. In this case, $T = P_{13}$ and $\gamma_{t2}(T) = 6$. Further, the set $S = \{u_1, u_3, u_5, v_1, v_3, v_5\}$

is a $\gamma_{t2}(T)$-set not containing v, while $(S \setminus \{u_1\}) \cup \{v\}$ is a $\gamma_{t2}(T)$-set containing v. Hence, $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$. Suppose secondly that $L_1^1(v) \neq \emptyset$ and let $L_1^1(v) = \{u\}$. In this case, $T = P_8$ and $\gamma_{t2}(T) = 4$. Further, the set $S = \{u, u_1, u_3, u_5\}$ is a $\gamma_{t2}(T)$-set not containing v. Moreover, $(S \setminus \{u_1\}) \cup \{v\}$ is a $\gamma_{t2}(T)$-set containing v. Hence, once again $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$. □

By Claim F, we may assume that $|L^1(v)| \leq 1$.

Claim G. *If $|L^1(v)| = 1$, then $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$.*

Proof. Suppose firstly that $L^2(v) = L^4(v) = \emptyset$. By our earlier assumptions, the vertex v is not a leaf in T, $L^3(v) = \emptyset$ and $L(v) \neq L^0(v) \cup L_1^1(v)$, implying that $|L^0(v)| \geq 1$ and $L^1(v) = L_2^1(v)$. Let $L_2^1(v) = \{u_6\}$ and let $vu_1u_2u_3u_4u_5u_6$ be the (v, u_6)-path. Every semi-TD-set of T contains at least two vertices from each $L^0(v)$-path and at least three vertices from the (v, u_6)-path, and so $\gamma_{t2}(T) \geq 2|L^0(v)| + 3$. However, the set $S = S(T) \cup \mathrm{Gr}(v) \cup \{v, u_3\}$ is a semi-TD-set of T of cardinality $2|L^0(v)| + 3$, and so $\gamma_{t2}(T) \leq |S| = 2|L^0(v)| + 3$. Consequently, $\gamma_{t2}(T) = 2|L^0(v)| + 3$ and S is a $\gamma_{t2}(T)$-set containing v. Moreover, $S' = (S \setminus \{v\}) \cup \{u_1\}$ is $\gamma_{t2}(T)$-set containing v. Hence, $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$.

Suppose secondly that $|L^2(v) \cup L^4(v)| \geq 1$. The vertex set $S = S(T) \cup C^{(4)}(v) \cup \mathrm{Gr}(v) \cup W \cup \{v\}$ is a semi-TD-set of cardinality $2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$, and so $\gamma_{t2}(T) \leq 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$. Suppose $L_1^1(v) = \emptyset$, and so $L^1(v) = L_2^1(v)$ and $|L_2^1(v)| = 1$. In this case, let $L^1(v) = \{u_6\}$ and let $vu_1u_2u_3u_4u_5u_6$ be the (v, u_6)-path. The set D contains at least three vertices from the (v, u_6)-path, at least one vertex from each $L^2(v)$-path and at least two vertices from each $L^0(v)$-path and $L^4(v)$-path, implying that $\gamma_{t2}(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 3 = 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$. Consequently, $\gamma_{t2}(T) = 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 3$ and S is a $\gamma_{t2}(T)$-set containing v. Moreover, the set $(S \setminus \{v\}) \cup \{u_1\}$ is a $\gamma_{t2}(T)$-set that does not contain v. Hence, $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$. Suppose next that $L^1(v) = L_1^1(v) = \{u\}$. In this case, $|L_2^1(v)| = 0$ and the set D contains at least one of u and v, at least one vertex from each $L^2(v)$-path and at least two vertices from each $L^0(v)$-path and $L^4(v)$-path, implying that $\gamma_{t2}(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 1 = 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1$. Consequently, $\gamma_{t2}(T) = 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 1$ and S is a $\gamma_{t2}(T)$-set containing v. Moreover, the set $(S \setminus \{v\}) \cup \{u\}$ is a $\gamma_{t2}(T)$-set that does not contain v. Hence, once again $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$. □

By Claim G, we may assume that $L^1(v) = \emptyset$.

Claim H. *Suppose $L^1(v) = \emptyset$. Then the following hold.*

(a) *If $|L^4(v)| \geq 1$, then $v \in \mathcal{N}_{t2}(T)$.*

(b) *If $|L^2(v)| = 1$ and $L^4(v) = \emptyset$, then $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$.*

(c) *If* $|L^2(v)| \geq 2$ *and* $L^4(v) = \emptyset$, *then* $v \in \mathcal{N}_{t2}(T)$.

Proof. (a) Suppose $|L^4(v)| \geq 1$. Every leaf is at distance 2, 4 or 5 from v in T. The set D contains at least one vertex from each $L^2(v)$-path and at least two vertices from each $L^0(v)$-path and each $L^4(v)$-path. Thus, $\gamma_{t2}(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)|$ with strict inequality if $v \in D$. The set $C^{(4)}(v) \cup S(T) \cup \mathrm{Gr}(v)$ is a semi-TD-set of T of cardinality $2(|L^0(v)| + |L^4(v)|) + |L^2(v)|$, and so $\gamma_{t2}(T) \leq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)|$. Consequently, $\gamma_{t2}(T) = |D| = 2(|L^0(v)| + |L^4(v)|) + |L^2(v)|$ and $v \notin D$. Since D is an arbitrary $\gamma_{t2}(T)$-set, $v \in \mathcal{N}_{t2}(T)$.

(b) Suppose $|L^2(v)| = 1$ and $L^4(v) = \emptyset$. Let $L^2(v) = \{u_2\}$ and let vu_1u_2 be the (v, u_2)-path. Then, $L^0(v) = L(v) \setminus \{u_2\}$ and $S = S(T) \cup \mathrm{Gr}(v) \cup \{v, u_1\}$ is a semi-TD-set of cardinality $2|L^0(v)| + 2$, and so $\gamma_{t2}(T) \leq |S| = 2|L^0(v)| + 2$. The set D contains at least two vertices from the set $N[v] \cup \{u_2\}$ and at least two vertices not in $N[v]$ from each $L^0(v)$-path. Thus, $\gamma_{t2}(T) = |D| \geq 2|L^0(v)| + 2$. Consequently, $\gamma_{t2}(T) = 2|L^0(v)| + 2$ and S is a $\gamma_{t2}(T)$-set that contains the vertex v. Moreover, the set $(S \setminus \{v\}) \cup \{u_2\}$ is a $\gamma_{t2}(T)$-set that does not contain v. Hence, $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$.

(c) Suppose that $|L^2(v)| \geq 2$ and $L^4(v) = \emptyset$. Every leaf is at distance 2 or 5 from v in T. The set D contains at least one vertex from each $L^2(v)$-path and at least two vertices from each $L^0(v)$-path. Thus, $\gamma_{t2}(T) = |D| \geq 2|L^0(v)| + |L^2(v)|$ with strict inequality if $v \in D$. The set $S(T) \cup \mathrm{Gr}(v)$ is a semi-TD-set of cardinality $2|L^0(v)| + |L^2(v)|$, and so $\gamma_{t2}(T) \leq 2|L^0(v)| + |L^2(v)|$. Consequently, $\gamma_{t2}(T) = |D| = 2|L^0(v)| + |L^2(v)|$ and $v \notin D$. Since D is an arbitrary $\gamma_{t2}(T)$-set, $v \in \mathcal{N}_{t2}(T)$. □

Theorem 1 now follows from Claims A, B, C, D, E, F, G and H. ■

6. Proof of Theorem 2

Let T be a rooted tree that is not a star with root v that contains at least one branch vertex different from v. We shall adopt the following notation. Let u be a branch vertex at maximum distance from v and let $k_0 = |L^0(u)|$, $k_1 = |L^1(u)|$, $k_2 = |L^2(u)|$, $k_3 = |L^3(u)|$ and $k_4 = |L^4(u)|$. Let w be the parent of u (possibly, $v = w$). Let T' be the tree obtained from T by applying the following operations.

$\mathcal{O}_1$: For $k_3 \geq 1$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a path P_3 to u.

$\mathcal{O}_2$: For $k_3 = 0$, $k_1 \geq 1$ and $k_0 + k_2 + k_4 \geq 1$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a path P_3 to u.

$\mathcal{O}_3$: For $k_0 = k_2 = k_3 = k_4 = 0$ and $k_1 \geq 2$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a path P_1 to u.

$\mathcal{O}_4$: For $k_1 = k_3 = 0$ and $k_4 \geq 1$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a path P_4 to u.

$\mathcal{O}_5$: For $k_1 = k_3 = k_4 = 0$, $k_2 = 1$ and $k_0 \geq 1$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a path P_2 to u.

$\mathcal{O}_6$: For $k_1 = k_3 = k_4 = 0$ and $k_2 \geq 2$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a path P_4 to u.

$\mathcal{O}_7$: For $k_1 = k_2 = k_3 = k_4 = 0$, let T' be the tree obtained from T by deleting $D(u)$ and attaching a path P_5 to u.

Our next result, namely Theorem 2, establishes a key result relating the semitotal domination numbers of the trees T and T'. Theorem 2 follows immediately from Theorem 1 and Theorem 8. We use the standard notation $[k] = \{1, 2, \ldots, k\}$ once again.

Theorem 8. *Let T be a tree with order at least 4 that is not a star and is rooted at a vertex v such that T contains at least one branch vertex u different from v and let T' be the tree defined immediately before the statement of the theorem. Let w be the parent of u (possibly, $w = v$). Suppose that T' is obtained from T by applying operation $\mathcal{O}_i$ for some $i \in [7]$. Then,*

$$\gamma_{t2}(T') = \begin{cases} \gamma_{t2}(T) - 2k_0 - k_2 - k_3 - 2k_4 + 1 & \text{for } i = 1, \\ \gamma_{t2}(T) - 2k_0 - k_2 - 2k_4 + 1 & \text{for } i = 2, \\ \gamma_{t2}(T) & \text{for } i = 3, \\ \gamma_{t2}(T) - 2k_0 - k_2 - 2k_4 + 2 & \text{for } i = 4, \\ \gamma_{t2}(T) - 2k_0 & \text{for } i = 5, \\ \gamma_{t2}(T) - 2k_0 - k_2 + 2 & \text{for } i = 6, \\ \gamma_{t2}(T) - 2k_0 + 2 & \text{for } i = 7. \end{cases}$$

Further, in all cases, the following properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold:

$P_{\mathcal{A}}$: *$v \in \mathcal{A}_{t2}(T)$ if and only if $v \in \mathcal{A}_{t2}(T')$.*

$P_{\mathcal{N}}$: *$v \in \mathcal{N}_{t2}(T)$ if and only if $v \in \mathcal{N}_{t2}(T')$.*

Proof. For each vertex $x \in L(u)$ replace the (u, x)-path in T with a (u, x)-path of length j, where $j \in \{5, 1, 2, 3, 4\}$ if $x \in L^i(u)$ when $i \in \{0, 1, 2, 3, 4\}$, respectively. Let T'' denote the resulting tree. By repeated applications of Lemma 7, we deduce that $v \in \mathcal{A}_{t2}(T)$ ($\mathcal{N}_{t2}(T)$, respectively) if and only if $v \in \mathcal{A}_{t2}(T'')$ ($\mathcal{N}_{t2}(T'')$, respectively). Hence, we assume $T = T''$. With this assumption, every leaf of T that is a descendant of u is within distance 5 from u. We proceed further with a series of five claims.

Claim I. *Suppose $k_3 \geq 1$. Then, T' is obtained from T by operation $\mathcal{O}_1$ and $\gamma_{t2}(T') = \gamma_{t2}(T) - 2k_0 - k_2 - k_3 - 2k_4 + 1$ and properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold.*

Proof. Suppose $k_3 \geq 1$. Thus, T' is obtained from T by operation $\mathcal{O}_1$. Let $u_3 \in L^3(u)$ and let $uu_1u_2u_3$ be the (u, u_3)-path. Renaming vertices, if necessary, we may assume that $T' = T - (D(u) \setminus \{u_1, u_2, u_3\})$. Let $H = T[D(u) \setminus \{u_1, u_2, u_3\}]$ and let $X_H = (S(T) \cup \text{Gr}(u) \cup C^{(4)}(u)) \cap V(H)$. We note that $|X_H| = 2k_0 + k_2 + k_3 + 2k_4 - 1$. By Observation 5, there exists a $\gamma_{t2}(T')$-set S that contains the vertex u_2. If $u_1 \in S$, then we can replace u_1 in S with u. Thus, we may assume $S \cap \{u, u_1, u_2, u_3\} = \{u, u_2\}$. The set S can be extended to a semi-TD-set of T by adding to it the set X_H, implying that $\gamma_{t2}(T) \leq |S| + |X_H| = \gamma_{t2}(T') + |X_H|$.

Conversely, let D be a $\gamma_{t2}(T)$-set and let $D_u = D \cap D(u)$. The set D contains at least two vertices from each $L^0(u)$-path and $L^4(u)$-path, and at least one vertex from each $L^2(u)$-path and $L^3(u)$-path, implying that $|D_u| \geq 2k_0 + k_2 + k_3 + 2k_4 = |X_H| + 1$. By Observation 5, we can choose D so that $S(T) \subseteq D$. In particular, $u_2 \in D$. If $u_1 \in D$, then we can replace u_1 in D with u. Hence, we may assume that $D \cap \{u, u_1, u_2, u_3\} = \{u, u_2\}$, implying that $D \cap V(T') = (D \setminus D_u) \cup \{u_2\}$ is a semi-TD-set of T'. Therefore, $\gamma_{t2}(T') \leq |D| - |D_u| + 1 \leq |D| - (|X_H| + 1) + 1 = |D| - |X_H| = \gamma_{t2}(T) - |X_H|$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T') + |X_H| = \gamma_{t2}(T') + 2k_0 + k_2 + k_3 + 2k_4 - 1$.

Suppose $v \notin \mathcal{A}_{t2}(T')$ and let S' be a $\gamma_{t2}(T')$-set that does not contain the vertex v. If $u_3 \in S'$, then we can replace u_3 in S' by u_2. Hence, we may assume that $u_2 \in S'$. If $u_1 \in S'$, then we can replace u_1 in S' by u. Hence, we may assume that $S' \cap \{u, u_1, u_2, u_3\} = \{u, u_2\}$. With these assumptions, the set $S' \cup X_H$ is a semi-TD-set of T of cardinality $|S'| + |X_H| = \gamma_{t2}(T') + |X_H| = \gamma_{t2}(T)$. Hence, $S' \cup X_H$ is a $\gamma_{t2}(T)$-set not containing v, implying that $v \notin \mathcal{A}_{t2}(T)$. Therefore, by contraposition, if $v \in \mathcal{A}_{t2}(T)$, then $v \in \mathcal{A}_{t2}(T')$.

Conversely, suppose $v \in \mathcal{A}_{t2}(T')$. Suppose to the contrary that $v \notin \mathcal{A}_{t2}(T)$. Let D be a $\gamma_{t2}(T)$-set that does not contain v. Analogous to our earlier arguments, we can choose such a set D so that $D \cap D[u] = X_H \cup \{u, u_2\}$. Therefore, $D \cap V(T')$ is a $\gamma_{t2}(T')$-set that does not contain v, a contradiction. Hence, if $v \in \mathcal{A}_{t2}(T')$, then $v \in \mathcal{A}_{t2}(T)$. Thus, property $P_{\mathcal{A}}$ holds. Analogous arguments show that property $P_{\mathcal{N}}$ holds. □

By Claim I, we may assume that $k_3 = 0$, for otherwise the desired result follows.

Claim J. *Suppose $k_1 \geq 1$. Then, T' is obtained from T by operation $\mathcal{O}_i$ for some $i \in \{2, 3\}$ and*

$$\gamma_{t2}(T') = \begin{cases} \gamma_{t2}(T) - 2k_0 - k_2 - 2k_4 + 1 & \text{for } i = 2, \\ \gamma_{t2}(T) & \text{for } i = 3. \end{cases}$$

Further, the properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold in both cases.

Proof. Suppose $k_1 \geq 1$. Let u' be a leaf-neighbor of u. We proceed further with a series of two subclaims.

Claim J.1. *If $k_0 + k_2 + k_4 \geq 1$, then $\gamma_{t2}(T') = \gamma_{t2}(T) - 2k_0 - k_2 - 2k_4 + 1$ and properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold.*

Proof. Suppose $k_0 + k_2 + k_4 \geq 1$. Thus, T' is obtained from T by operation $\mathcal{O}_2$. Let $P\colon u_1u_2u_3$ be the path P_3 added to $T - D(u)$ when constructing T', where u is adjacent to u_1. Let $H = T[D(u)]$ and let $X_H = (S(T) \cup \mathrm{Gr}(u) \cup C^{(4)}(u)) \cap V(H)$. We note that $|X_H| = 2k_0 + k_2 + 2k_4$. By Observation 5 there exists a $\gamma_{t2}(T')$-set, S, such that $u_2 \in S$. If $u_1 \in S$, then we can replace u_1 in D with u. Hence, we may assume that $S \cap \{u, u_1, u_2, u_3\} = \{u, u_2\}$. Since $k_0 + k_2 + k_4 \geq 1$, the set $S \setminus \{u_2\}$ can be extended to a semi-TD-set of T by adding to it the set X_H, implying that $\gamma_{t2}(T) \leq |S \setminus \{u_2\}| + |X_H| = \gamma_{t2}(T') + |X_H| - 1$.

Conversely, let D be a $\gamma_{t2}(T)$-set and let $D_u = D \cap D(u)$. The set D contains at least two vertices from each $L^0(u)$-path and $L^4(u)$-path, and at least one vertex from each $L^2(u)$-path, implying that $|D_u| \geq 2k_0 + k_2 + 2k_4 = |X_H|$. By Observation 5, we can choose D so that $S(T) \subseteq D$. In particular, $u \in D$, implying that $(D \setminus D_u) \cup \{u_2\}$ is a semi-TD-set of T', and so $\gamma_{t2}(T') \leq |D| - |D_u| + 1$. If $|D_u| > |X_H|$, then $(D \setminus D_u) \cup X_H$ is a semi-TD-set of T of cardinality less than $|D|$, a contradiction. Hence, $|D_u| = |X_H|$ and $\gamma_{t2}(T') \leq |D| - |D_u| + 1 = \gamma_{t2}(T) - |X_H| + 1$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T') + |X_H| - 1 = \gamma_{t2}(T') + 2k_0 + k_2 + 2k_4 - 1$.

Suppose $v \notin \mathcal{A}_{t2}(T')$ and let S' be a $\gamma_{t2}(T')$-set that does not contain the vertex v. If $u_3 \in S'$, then we can replace u_3 in S' by u_2. Hence, we may assume that $u_2 \in S'$. If $u_1 \in S'$, then we can replace u_1 in S' by u. Hence, we may assume that $S' \cap \{u, u_1, u_2, u_3\} = \{u, u_2\}$. With these assumptions, the set $S = (S' \setminus \{u_2\}) \cup X_H$ is a semi-TD-set of T of cardinality $|S'| + |X_H| - 1 = \gamma_{t2}(T') + |X_H| - 1 = \gamma_{t2}(T)$. Hence, S is a $\gamma_{t2}(T)$-set not containing v, implying that $v \notin \mathcal{A}_{t2}(T)$. Therefore, by contraposition, if $v \in \mathcal{A}_{t2}(T)$, then $v \in \mathcal{A}_{t2}(T')$.

Conversely, suppose $v \in \mathcal{A}_{t2}(T')$. Suppose to the contrary that $v \notin \mathcal{A}_{t2}(T)$. Let D be a $\gamma_{t2}(T)$-set that does not contain v. Analogous to our earlier arguments, we can choose such a set D so that $D \cap D[u] = X_H \cup \{u\}$. Therefore, $(D \cap V(T')) \cup \{u_2\}$ is a $\gamma_{t2}(T')$-set that does not contain v, a contradiction. Hence, if $v \in \mathcal{A}_{t2}(T')$, then $v \in \mathcal{A}_{t2}(T)$. Thus, property $P_{\mathcal{A}}$ holds. Analogous arguments show that property $P_{\mathcal{N}}$ holds. □

Claim J.2. *If $k_0 + k_2 + k_4 = 0$, then $\gamma_{t2}(T') = \gamma_{t2}(T)$ and properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold.*

Proof. Since $k_0 + k_2 + k_4 = 0$, we have $k_1 \geq 2$. Thus, T' is obtained from T by operation $\mathcal{O}_3$. Renaming vertices if necessary, $T' = T - (D(u) \setminus \{u'\})$. By assumption, the tree T is not a star, implying that the tree T' is not a star. By Observation 5, there exists a $\gamma_{t2}(T')$-set S that contains the vertex u and no leaf in T'. Thus, we assume $u \in S$ and that no leaf of T' is contained in S. Thus, the set S is a semi-TD-set of T, implying that $\gamma_{t2}(T) \le |S| = \gamma_{t2}(T')$. Conversely, let D be a $\gamma_{t2}(T)$-set. By Observation 5, we can choose D so that $S(T) \subseteq D$. In particular, $u \in D$ and no leaf-neighbor of u belongs to D, implying that D is a semi-TD-set of T', and so $\gamma_{t2}(T') \le |D| = \gamma_{t2}(T)$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T')$.

Suppose $v \notin \mathcal{A}_{t2}(T')$ and let S' be a $\gamma_{t2}(T')$-set that does not contain the vertex v. If $u' \in S'$, then if $u \in S'$ we replace u' in S with a vertex from $x \in N[w] \setminus \{u\}$ such that $x \neq v$, else we replace u' in S with u. Hence we may assume that $u' \notin S'$ (which is possible since T' is not a star). Thus the set S' is a $\gamma_{t2}(T)$-set not containing v, implying that $v \notin \mathcal{A}_{t2}(T)$. Therefore, by contraposition, if $v \in \mathcal{A}_{t2}(T)$, then $v \in \mathcal{A}_{t2}(T')$.

Conversely, suppose $v \in \mathcal{A}_{t2}(T')$. Suppose to the contrary that $v \notin \mathcal{A}_{t2}(T)$. Let D be a $\gamma_{t2}(T)$-set that does not contain v. If D contains a leaf-neighbor z of u, then if $u \in D$ we can replace z in D with a vertex from $x \in N[w] \setminus \{u\}$ such that $x \neq v$ else we replace z in D with u to produce a new $\gamma_{t2}(T)$-set that does not contain v. Hence, we may choose the set D so that $D \cap D[u] = \{u\}$. Therefore, D is a $\gamma_{t2}(T')$-set that does not contain v, a contradiction. Hence, if $v \in \mathcal{A}_{t2}(T')$, then $v \in \mathcal{A}_{t2}(T)$. Thus, property $P_{\mathcal{A}}$ holds. Analogous arguments show that property $P_{\mathcal{N}}$ holds. $\square$

Claim J follows immediately from Claim J.1 and Claim J.2. $\square$

By Claim J, we may assume that $k_1 = 0$, for otherwise the desired result follows.

Claim K. *Suppose $k_4 \ge 1$. Then, T' is obtained from T by operation $\mathcal{O}_4$ and $\gamma_{t2}(T') = \gamma_{t2}(T) - 2k_0 - k_2 - 2k_4 + 2$ and properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold.*

Proof. Suppose $k_4 \ge 1$. Thus, T' is obtained from T by operation $\mathcal{O}_4$. By our earlier assumptions, $k_1 = k_3 = 0$. Let $u_4 \in L^4(u)$ and let $uu_1u_2u_3u_4$ be the (u, u_4)-path. Renaming vertices if necessary, we may assume that $T' = T - (D(u) \setminus \{u_1, u_2, u_3, u_4\})$. Let $H = T[D(u) \setminus \{u_1, u_2, u_3, u_4\}]$ and let $X_H = (S(T) \cup C^{(4)}(u) \cup \mathrm{Gr}(u)) \cap V(H)$. We note that $|X_H| = 2k_0 + k_2 + 2(k_4 - 1)$. By Observation 5, there exists a $\gamma_{t2}(T')$-set S that contains the vertex u_3. If $u_2 \in S$, then we can replace u_2 in S with u_1. Thus, we may assume $S \cap \{u_1, u_2, u_3, u_4\} = \{u_1, u_3\}$. Then the set S can therefore be extended to a semi-TD-set of T by adding to it the set X_H, implying that $\gamma_{t2}(T) \le |S| + |X_H| = \gamma_{t2}(T') + |X_H|$.

Conversely, let D be a $\gamma_{t2}(T)$-set and let $D_u = D \cap D(u)$. The set D contains at least two vertices from each $L^0(u)$-path and $L^4(u)$-path, and at least one vertex from each $L^2(u)$-path, implying that $|D_u| \ge 2k_0 + k_2 + 2k_4 = |X_H| + 2$. On the other hand, the set $(D \setminus D_u) \cup \{u_1, u_3\}$ is a semi-TD-set of T', and so

$\gamma_{t2}(T') \le \gamma_{t2}(T) - |D_u| + 2 \le \gamma_{t2}(T) - |X_H|$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T') + |X_H| = \gamma_{t2}(T') + 2k_0 + k_2 + 2k_4 - 2$.

Suppose $v \notin \mathcal{A}_{t2}(T')$ and let S' be a $\gamma_{t2}(T')$-set that does not contain the vertex v. If $u_4 \in S'$, then we can replace u_4 in S' with u_3. Hence we may choose S' so that $u_3 \in S'$. If $u_2 \in S'$, then we can replace u_2 in S' with u_1. Thus, we may assume $S' \cap \{u_1, u_2, u_3, u_4\} = \{u_1, u_3\}$. The set $S' \cup X_H$ is therefore a semi-TD-set of T of cardinality $|S'| + |X_H| = \gamma_{t2}(T') + |X_H| = \gamma_{t2}(T)$. Thus, $S' \cup X_H$ is a $\gamma_{t2}(T)$-set not containing the vertex v, implying that $v \notin \mathcal{A}_{t2}(T)$. Therefore, by contraposition, if $v \in \mathcal{A}_{t2}(T)$, then $v \in \mathcal{A}_{t2}(T')$.

Conversely, suppose $v \in \mathcal{A}_{t2}(T')$. Suppose to the contrary that $v \notin \mathcal{A}_{t2}(T)$. Let D be a $\gamma_{t2}(T)$-set that does not contain v and let $D_u = D \cap D(u)$. If $|D_u| > |X_H|+2$, then the set $(D \setminus D_u) \cup (X_H \cup \{u_1, u_3\})$ is a semi-TD-set of T of cardinality less than $|D|$, a contradiction. Hence, $|D_u| \le |X_H| + 2$. Analogous to our earlier arguments, $|D_u| \ge |X_H|+2$. Consequently, $|D_u| = |X_H|+2$ and $(D \setminus D_u) \cup \{u_1, u_3\}$ is a semi-TD-set of T' of cardinality $|D| - |D_u| + 2 = \gamma_{t2}(T) - |X_H| = \gamma_{t2}(T')$. Thus, $(D \setminus D_u) \cup \{u_1, u_3\}$ is a $\gamma_{t2}(T')$-set that does not contain v, a contradiction. Hence, if $v \in \mathcal{A}_{t2}(T')$, then $v \in \mathcal{A}_{t2}(T)$. Thus, property $P_{\mathcal{A}}$ holds. Analogous arguments show that property $P_{\mathcal{N}}$ holds. □

By Claim K, we may assume that $k_4 = 0$, for otherwise the desired result follows.

Claim L. *Suppose $k_2 \ge 1$. Then, T' is obtained from T by operation $\mathcal{O}_i$ for some $i \in \{5, 6\}$ and*

$$\gamma_{t2}(T') = \begin{cases} \gamma_{t2}(T) - 2k_0 & \text{for } i = 5, \\ \gamma_{t2}(T) - 2k_0 - k_2 + 2 & \text{for } i = 6. \end{cases}$$

Further, the properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold in both cases.

Proof. Suppose $k_2 \ge 1$. Let $u_2 \in L^2(u)$ and let uu_1u_2 be the (u, u_2)-path in T. By our earlier assumptions, $k_1 = k_3 = k_4 = 0$. We proceed further with a series of two subclaims.

Claim L.1. *If $k_2 = 1$, then $\gamma_{t2}(T') = \gamma_{t2}(T) - 2k_0$ and properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold.*

Proof. Suppose that $k_2 = 1$ and hence, $k_0 \ge 1$ and $L^2(u) = \{u_2\}$. Thus, T' is obtained from T by operation $\mathcal{O}_5$. Let uu_1u_2 be the (u, u_2)-path. Renaming vertices if necessary, $T' = T - (D(u) \setminus \{u_1, u_2\})$. Let $H = T[D(u) \setminus \{u_1, u_2\}]$ and let $X_H = (S(T) \cup \mathrm{Gr}(u)) \cap V(H)$. We note that $|X_H| = 2k_0$. Every $\gamma_{t2}(T')$-set S can be extended to a semi-TD-set of T by adding to it the set X_H, implying that $\gamma_{t2}(T) \le |S| + |X_H| = \gamma_{t2}(T') + |X_H|$.

Conversely, let D be an $\gamma_{t2}(T)$-set and let $D_u = D \cap D(u)$. The set D_u contains at least two vertices from each $L^0(u)$-path and one of the vertices u_1 or u_2, implying that $|D_u| \ge 2k_0 + 1 = |X_H| + 1$. The set $(D \setminus D_u) \cup \{u_1\}$ is a semi-TD-set of T', and so $\gamma_{t2}(T') \le |D| - |D_u| + 1 \le \gamma_{t2}(T) - |X_H|$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T') + |X_H| = \gamma_{t2}(T') + 2k_0$.

Suppose $v \notin \mathcal{A}_{t2}(T')$ and let S' be a $\gamma_{t2}(T')$-set that does not contain the vertex v. Then, the set $S' \cup X_H$ is a $\gamma_{t2}(T)$-set not containing v, implying that $v \notin \mathcal{A}_{t2}(T)$. Therefore, by contraposition, if $v \in \mathcal{A}_{t2}(T)$, then $v \in \mathcal{A}_{t2}(T')$.

Conversely, suppose $v \in \mathcal{A}_{t2}(T')$. Suppose to the contrary that $v \notin \mathcal{A}_{t2}(T)$. Let D be a $\gamma_{t2}(T)$-set that does not contain v. Analogous to our earlier arguments, we can choose such a set D so that $D \cap D(u) = X_H \cup \{u_1\}$. Thus, $D \setminus X_H$ is a semi-TD-set of T' of cardinality $|D| - |X_H| = \gamma_{t2}(T) - |X_H| = \gamma_{t2}(T')$. The set $D \setminus X_H$ is therefore a $\gamma_{t2}(T')$-set that does not contain v, a contradiction. Hence, if $v \in \mathcal{A}_{t2}(T')$, then $v \in \mathcal{A}_{t2}(T)$. Thus, property $P_{\mathcal{A}}$ holds. Analogous arguments show that property $P_{\mathcal{N}}$ holds. □

Claim L.2. *If $k_2 \ge 2$, then $\gamma_{t2}(T') = \gamma_{t2}(T) - 2k_0 - k_2 + 2$ and properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold.*

Proof. Suppose $k_2 \ge 2$. Thus, T' is obtained from T by operation $\mathcal{O}_6$. Let $P : u_1u_2u_3u_4$ be the path P_4 added to $T - D(u)$ when constructing T', where u is adjacent to u_1. Let $H = T[D(u)]$ and let $X_H = (S(T) \cup \mathrm{Gr}(u)) \cap V(H)$. We note that $|X_H| = 2k_0 + k_2$. By Observation 5, there exists a $\gamma_{t2}(T')$-set S that contains the vertex u_3. If $u_2 \in S$, then we may replace u_2 in S with u_1. Hence we may choose S so that $S \cap \{u_1, u_2, u_3, u_4\} = \{u_1, u_3\}$. The set $S \setminus \{u_1, u_3\}$ can therefore be extended to a semi-TD-set of T by adding to it the set X_H, implying that $\gamma_{t2}(T) \le |S \setminus \{u_1, u_3\}| + |X_H| = \gamma_{t2}(T') + |X_H| - 2$.

Conversely, let D be a $\gamma_{t2}(T)$-set and let $D_u = D \cap D(u)$. The set D_u contains at least two vertices from each $L^0(u)$-path and one vertex from each $L^2(u)$-path, implying that $|D_u| \ge 2k_0 + k_2 = |X_H|$. The set $(D \setminus D_u) \cup \{u_1, u_3\}$ is a semi-TD-set of T', and so $\gamma_{t2}(T') \le |D| - |D_u| + 2 \le \gamma_{t2}(T) - |X_H| + 2$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T') + |X_H| - 2 = \gamma_{t2}(T') + 2k_0 + k_2 - 2$.

Suppose $v \notin \mathcal{A}_{t2}(T')$ and let S' be a $\gamma_{t2}(T')$-set that does not contain the vertex v. Analogous to our earlier arguments, we can choose such a set S' so that $S' \cap \{u_1, u_2, u_3, u_4\} = \{u_1, u_3\}$. The set $(S' \setminus \{u_1, u_3\}) \cup X_H$ is a semi-TD-set of cardinality $|S'| + |X_H| - 2 = \gamma_{t2}(T') + |X_H| - 2 = \gamma_{t2}(T)$ and is thus a $\gamma_{t2}(T)$-set not containing v, implying that $v \notin \mathcal{A}_{t2}(T)$. Therefore, by contraposition, if $v \in \mathcal{A}_{t2}(T)$, then $v \in \mathcal{A}_{t2}(T')$.

Conversely, suppose $v \in \mathcal{A}_{t2}(T')$. Suppose to the contrary that $v \notin \mathcal{A}_{t2}(T)$. Let D be a $\gamma_{t2}(T)$-set that does not contain v. Analogous to our earlier arguments, we can choose a set D so that $D \cap D(u) = X_H$. Therefore, $(D \setminus X_H) \cup \{u_1, u_3\}$ is a semi-TD-set of cardinality $|D| - |X_H| + 2 = \gamma_{t2}(T) - |X_H| + 2 = \gamma_{t2}(T')$ and is thus a $\gamma_{t2}(T')$-set that does not contain v, a contradiction. Hence, if $v \in \mathcal{A}_{t2}(T')$, then $v \in \mathcal{A}_{t2}(T)$. Thus, property $P_{\mathcal{A}}$ holds. Analogous arguments show that property $P_{\mathcal{N}}$ holds. $\square$

Claim L follows from Claim L.1 and Claim L.2. This completes the proof of Claim L. $\square$

By Claim L, we may assume that $k_2 = 0$, for otherwise the desired result follows. By our earlier assumptions, $k_1 = k_3 = k_4 = 0$. Thus, $L(u) = L^0(u)$. Since u is a branch vertex, $k_0 \geq 2$.

Claim M. *Suppose $k_0 \geq 2$. Then, T' is obtained from T by operation $\mathcal{O}_7$ and $\gamma_{t2}(T') = \gamma_{t2}(T) - 2k_0 + 2$ and properties $P_{\mathcal{A}}$ and $P_{\mathcal{N}}$ hold.*

Proof. Let $\{u_5, v_5\} \subseteq L^0(u)$ and let $uu_1u_2u_3u_4u_5$ and $uv_1v_2v_3v_4v_5$ be the respective (u, u_5)-path and (u, v_5)-path in T. Thus, T' is obtained from T by operation $\mathcal{O}_7$. Renaming vertices if necessary, we may assume that $T' = T - (D(u) \setminus \{u_1, u_2, u_3, u_4, u_5\})$. Let $H = T[D(u) \setminus \{u_1, u_2, u_3, u_4, u_5\}]$ and let $X_H = (S(T) \cup \mathrm{Gr}(u)) \cap V(H)$. We note that $|X_H| = 2k_0 - 2$. Every $\gamma_{t2}(T')$-set can be extended to a semi-TD-set of T by adding to it the set X_H, implying that $\gamma_{t2}(T) \leq \gamma_{t2}(T') + |X_H|$.

Conversely, let D be a $\gamma_{t2}(T)$-set and let $D_u = D \cap D(u)$. The set D_u contains at least two vertices from each $L^0(u)$-path, implying that $|D_u| \geq 2k_0 = |X_H| + 2$. The set $(D \setminus D_u) \cup \{u_2, u_4\}$ is a semi-TD-set of T', and so $\gamma_{t2}(T') \leq |D| - |D_u| + 2 \leq \gamma_{t2}(T) - |X_H|$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T') + |X_H| = \gamma_{t2}(T') + 2k_0 - 2$.

Suppose $v \notin \mathcal{A}_{t2}(T')$ and let S' be a $\gamma_{t2}(T')$-set that does not contain the vertex v. Then, the set $S' \cup X_H$ is a $\gamma_{t2}(T)$-set not containing v, implying that $v \notin \mathcal{A}_{t2}(T)$. Therefore, by contraposition, if $v \in \mathcal{A}_{t2}(T)$, then $v \in \mathcal{A}_{t2}(T')$.

Conversely, suppose $v \in \mathcal{A}_{t2}(T')$. Suppose to the contrary that $v \notin \mathcal{A}_{t2}(T)$. Let D be a $\gamma_{t2}(T)$-set that does not contain v and chosen so that $|D \cap D(u)|$ is a minimum. Let $D_u = D \cap D(u)$. If $|D_u| \geq |X_H| + 3$, then the set $(D \setminus D_u) \cup (X_H \cup \{u, u_2, u_4\})$ is a semi-TD-set of T of cardinality $|D| - |D_u| + |X_H| + 3 \leq |D| = \gamma_{t2}(T)$ and is therefore a $\gamma_{t2}(T)$-set containing fewer vertices of $D(u)$ than does D, a contradiction. Hence, $|D_u| \leq |X_H| + 2$. Analogous to our earlier arguments, $|D_u| \geq |X_H| + 2$. Consequently, $|D_u| = |X_H| + 2$ and $(D \setminus D_u) \cup \{u_2, u_4\}$ is a semi-TD-set of T' of cardinality $|D| - |D_u| + 2 = \gamma_{t2}(T) - |X_H| = \gamma_{t2}(T')$. Thus, $(D \setminus D_u) \cup \{u_2, u_4\}$ is a $\gamma_{t2}(T')$-set that does not contain v, a contradiction. Hence, if $v \in \mathcal{A}_{t2}(T')$, then $v \in \mathcal{A}_{t2}(T)$. Thus, property $P_{\mathcal{A}}$ holds. Analogous arguments show that property $P_{\mathcal{N}}$ holds. $\square$

Theorem 8 follows from Claims I, J, K, L and M. $\blacksquare$

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Appendix

We now present an example to illustrate Theorem 2. Applying our pruning process discussed in Section 2 to the rooted tree T with root v illustrated in Figure 1(a), we proceed as follows.

• The branch vertices b_3 and b_4 are both at maximum distance 3 from v in T. We select b_3, where $|L^3(b_3)| = 1$. Thus, b_3 is a type-(T.1) branch vertex and we delete $D(b_3)$ and attach a path of length 3 to b_3.

• The branch vertex at maximum distance from v in the resulting tree (illustrated in Figure 1(b)) is the vertex b_4. Since $|L^1(b_4)| > 2$ and every leaf-descendant of b_4 belongs to $L^1(b_4)$, the vertex b_4 is therefore a type-(T.3) branch vertex and we delete $D(b_4)$ and attach a path of length 1 to b_4.

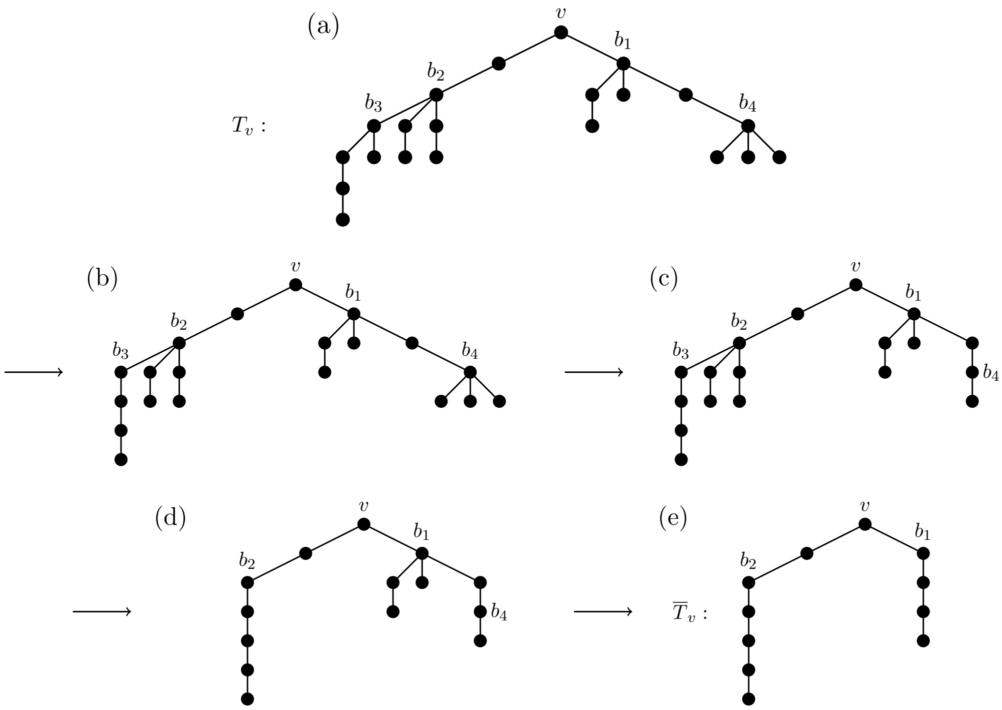

Figure 1.The pruning of a tree rooted at v.

• The branch vertex at maximum distance from v in the resulting tree (illustrated in Figure 1(c)) is the vertex b_2. Since $|L^4(b_2)| = 1$ and $L^1(b_2) = L^3(b_2) = \emptyset$, the vertex b_2 is a type-(T.4) branch vertex and we delete $D(b_2)$ and attach a path of length 4 to b_2.

• The branch vertex at maximum distance from v in the resulting tree (illustrated in Figure 1(d)) is the vertex b_1. Since $|L^3(b_1)| = 1$, the vertex b_1 is a type-(T.1) branch vertex and we delete $D(b_1)$ and attach a path of length 3 to b_1. The resulting pruned tree $\overline{T}_v$ is illustrated in Figure 1(e).

• Since $|\overline{L}^1(v)| = 1$ and $|\overline{L}^4(v)| = 1$, by Theorem 2, we deduce that $v \notin \mathcal{A}_{t2}(T) \cup \mathcal{N}_{t2}(T)$.

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