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MAXIMUM EDGE-COLORINGS OF GRAPHS

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Abstract

An *r-maximum k-edge-coloring* of G is a k-edge-coloring of G having a property that for every vertex v of degree $d_G(v) = d$, $d \geq r$, the maximum color, that is present at vertex v, occurs at v exactly r times. The r-maximum index $\chi'_r(G)$ is defined to be the minimum number k of colors needed for an r-maximum k-edge-coloring of graph G. In this paper we show that $\chi'_r(G) \leq 3$ for any nontrivial connected graph G and $r = 1$ or 2. The bound 3 is tight. All graphs G with $\chi'_1(G) = i, i = 1, 2, 3$ are characterized. The precise value of the r-maximum index, $r \geq 1$, is determined for trees and complete graphs.

Keywords: edge-coloring, r-maximum k-edge-coloring, unique-maximum edge-coloring, weak-odd edge-coloring, weak-even edge-coloring.

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1. Introduction

Throughout this paper we follow the terminology and notation used in [1] and [7]. A graph G is always regarded as being connected, finite with loops and multiple edges allowed. A loopless graph without multiple edges is referred to as a *simple graph*. The parameters $n(G) = |V(G)|$ and $m(G) = |E(G)|$ are called *order* and *size* of G, respectively. Whenever $n(G) = 1$ we say G is *trivial*. For $X \subseteq V(G) \cup E(G)$, the subgraph of G obtained by removing the vertices and edges of X is denoted by $G - X$. If $X = \{x\}$ is a singleton we write $G - x$ rather than $G - \{x\}$. For $R \subseteq E(G)$ we denote by $V(R)$ the set of vertices incident with edges of R.

We refer to each vertex v of even (respectively odd) degree $d_G(v)$ as an *even* (respectively *odd*) vertex of G. In particular, a vertex of degree equal to 0 (respectively 1) is an *isolated* (respectively *pendant*) vertex. A graph is called *even* (respectively *odd*) whenever all its vertices are even (respectively odd). Given a (not necessarily proper) edge-coloring φ of graph G and a vertex $v \in V(G)$, we say that a color c *appears* t *times at* v *under* φ if exactly t edges incident with v receive the color c. By definition, each loop at a vertex v colored with c contributes 2 to the count of appearances of c at v.

The edge-coloring φ is *weak-odd* at v whenever at least one color appears odd number of times at v under φ. If this holds for every nonisolated vertex v of G, then we speak of a *weak-odd edge-coloring* of G. Similarly, we say that an edge-coloring φ is *weak-even* at v, whenever at least one of the colors appears even number of times at v under φ. If this holds for every vertex v of degree at least two in G, then we speak of a *weak-even edge-coloring* of G. A weak-odd (respectively weak-even) edge-coloring of G which uses at most k colors is referred to as a *weak-odd k-edge-coloring* (respectively *weak-even k-edge-coloring*). Whenever G admits a weak-odd (respectively weak-even) edge-coloring, the *weak-odd chromatic index* $\chi'_{wo}(G)$ (respectively *weak-even chromatic index* $\chi'_{we}(G)$) is defined to be the least integer k for which G has a weak-odd (respectively weak-even) k-edge-coloring.

The notion of *odd* edge-coloring of a graph G was introduced by Pyber in [8] as an edge-coloring of G such that every subgraph induced by a color class is odd. He considered simple graphs and proved the following result.

Theorem 1 [8]**.** *Every simple graph admits an odd edge-coloring with at most* 4 *colors. Moreover, the bound* 4 *is tight.*

Lužar, Petruševski, and Škrekovski in [6] considered the same notation for multigraphs and proved an analogous result.

Theorem 2 [6]**.** *Every multigraph admits an odd edge-coloring with at most* 6 *colors. Moreover, the bound* 6 *is tight.*

The weak-odd (respectively weak-even) edge-coloring has been recently introduced by Lužar, Petruševski, and Škrekovski in the paper [7]. They proved

Theorem 3 [7]**.** *Let G be a connected graph of order at least* 2*. Then $\chi'_{wo}(G) \leq 3$ and $\chi'_{we}(G) \leq 2$. Both bounds are tight.*

Let G be a graph. A *unique-maximum k-edge-coloring* of G is an edge-coloring φ with colors $1, 2, \ldots, k$ such that, for each vertex v of G, the maximum color that occurs at v under φ occurs there exactly on one edge. The minimum k

for which G has a unique-maximum k-edge-coloring is denoted by $\chi'_{um}(G)$. It is easy to see that $\chi'_{wo}(G) \leq \chi'_{um}(G)$. Motivations to investigate a unique-maximum k-edge-coloring came from the recent paper [5] where unique-maximum k-edge-colorings with respect to faces of plane graphs are studied and from papers [4] and [3] where the unique-maximum k-vertex-colorings with respect to paths in graphs and hyperedges in hypergraphs, respectively, are considered. In these papers one can find motivations and connections to other problems and applications.

In this paper we introduce a notion of a maximum edge-coloring which in a unique way strengthens both the weak-odd edge-coloring and the weak-even edge-coloring and generalizes a unique-maximum edge-coloring.

An *r-maximum k-edge-coloring* of G is a k-edge-coloring of G with colors $1, 2, \ldots, k$ having a property that for every vertex v of degree $d_G(v) = d$, $d \geq r$, the maximum color that is present at vertex v occurs at v exactly r times. The minimum k for which G has an r-maximum k-edge-coloring is denoted by $\chi'_r(G)$. Notice that $\chi'_{um}(G) = \chi'_1(G)$ and $\chi'_{we}(G) \leq \chi'_2(G)$.

The rest of the paper is organized as follows: In Section 2 we prove that any nontrivial graph has a unique-maximum k-edge-coloring with $k \leq 3$. In Section 3 we show that $\chi'_2(G) \leq 3$ for any nontrivial connected graph. Sections 4 and 5 are devoted to study of r-maximum edge-colorings for $r \geq 1$. In Section 4 trees and in Section 5 complete graphs are considered. In Section 6 we show that for any integer $r \geq 3$ there is a graph G for which $\chi'_r(G)$ is not defined. In Section 7 we state two open questions.

2. Unique-Maximum Edge-Coloring

Theorem 4. *Let G be a connected graph of order at least two. Then*

1. $\chi'_{um}(G) = 1$ *if and only if* $G = K_2$.
2. $\chi'_{um}(G) = 2$ *if and only if G has a maximum matching M such that for every vertex $v \in V(G) - V(M)$ we have $d_G(u) = 1$.*
3. $\chi'_{um}(G) = 3$ *in all other cases.*

Proof. *Case* 1 is trivial.

Case 2. If $\chi'_{um}(G) = 2$, then $G \neq K_2$ and there is a unique-maximum 2-edge-coloring of G. Let M be the set of edges of G colored with color 2. Clearly, M is a matching in G. All other edges of G are colored with color 1 and therefore every vertex $v \in V(G) - V(M)$ is incident with exactly one edge. Consequently, M is a maximum matching. The proof of the opposite implication is easy.

Case 3. Let G be a connected graph different from those mentioned in Cases 1 and 2. We are going to show that G has a unique-maximum 3-edge-coloring with colors 1, 2, and 3. Let M be a maximum matching in G. We assign the edges of M color 3. It is easy to see that for every vertex $v \in V(G) - V(M)$ the set of neighbors $N(v)$ of v is a subset of $V(M)$. For every such vertex v we choose one edge incident with v and assign it color 2. All the remaining uncolored edges are assigned color 1. Clearly this coloring is a unique-maximum one. ■

3. 2-Maximum Edge-Coloring

Theorem 5. *Let G be a nontrivial connected graph. Then $\chi'_2(G) \leq 3$. Moreover, the bound 3 is tight.*

Proof. We are going to show how to color the edges of G with three colors 1, 2 and 3. Let H be a maximum 2-regular subgraph of G. We assign 3 to all edges of H. The remaining edges of the subgraph $G[V(H)]$ are colored with 1.

Let $X = V(G) - V(H)$ and $F = G[X]$. Observe that F is a forest. Let $T_1, T_2, \dots, T_m$ be the components of F (all are trees). Below, we present a procedure for coloring the trees one by one.

Let $T = T_i$, $i = 1, 2, \dots, m$. Choose a vertex $v_0 \in V(T)$ as a root of T. Let $e_1, e_2 \dots, e_s$ be the edges of T incident with v_0, and let $h_1, h_2 \dots, h_t$ be the edges of G having v_0 as one endvertex and the other endvertex in H.

If $s \geq 2$ (or $s = 1$ and $t \geq 1$, or $s = 0$ and $t \geq 2$) then the edges e_1 and e_2 (e_1 and h_1, or h_1 and h_2, respectively) are colored with 2 and the remaining uncolored edges incident with v_0 are colored with 1.

If $s = 1$ and $t = 0$ or $s = 0$ and $t = 1$, then the edge e_1 or h_1, respectively, is colored with 2.

In all the above cases all edges incident with v_0 are colored and we say that the vertex v_0 is *cultivated*. Put $u = v_0$.

Next we choose a vertex v of T which is not yet cultivated but is adjacent with a cultivated vertex u along the colored edge $uv = e_0$. Let, as above, $e_1, e_2, \dots, e_s$ (and $h_1, h_2, \dots, h_t$) be uncolored edges having v as one endvertex and the other endvertex in T (and in H, respectively).

If $s = t = 0$ we consider the vertex v to be cultivated.

If $s + t \geq 1$ we distinguish two cases according to the color of e_0.

Case 1. The edge $e_0 = uv$ is assigned 2. Then $s \geq 1$ or $s = 0$ and $t \geq 1$. In the former case the edge e_1, and, in the latter case, the edge h_1 is assigned 2 and all other yet uncolored edges incident with v are colored with 1. After this the vertex v is cultivated.

Case 2. The edge $e_0 = uv$ is assigned 1. If $s \geq 2$ (or $s = 1$ and $t \geq 1$, or $s = 0$, $t \geq 2$) the edges e_1 and e_2 (or e_1 and h_1, or h_1 and h_2, respectively) are colored with 2 and the remaining uncolored edges incident with v are colored

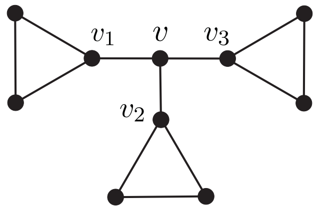

Figure 1. Graph H_3 which needs 3 colors for a 2-maximum edge-coloring.

with 1. Now the vertex v is cultivated. If $s = 1$ and $t = 0$ or $s = 0$ and $t = 1$, then the edge e_1 or h_1, respectively, is colored with 1. Notice that in these cases there is $d_G(v) = 2$. Also in these cases the vertex v is cultivated.

Next we are looking for a not yet cultivated vertex v of T adjacent to a cultivated vertex u of T along colored edge $e_0 = uv$. If we find such a pair of vertices u and v we repeat the above described (general) procedure. If there is no such a pair u and v of vertices in T we say that the component (tree) T is cultivated.

We repeat the procedure described above until all the trees T_i are cultivated, obtaining a 2-maximum 3-edge-coloring of G.

To finish the proof of our theorem it remains to show that there are graphs requiring at least three colors in any 2-maximum edge-coloring. Let $K_{1,n}, n \geq 3$, be a star with a central vertex v and leaves $v_1, v_2, \ldots, v_n$. Attach to each leaf a 3-cycle. The result is a graph H_n. See Figure 1 for H_3. Suppose that there is a 2-maximum 2-edge-coloring of H_n. The edges of every 3-cycle must be assigned color 2. But then all edges incident with v must have color 1, a contradiction. ■

Notice that if the graph G considered in the above proof is a tree, then there is no 2-regular subgraph of G and color 3 is not used. This immediately provides:

Corollary 6. *Let T be a nontrivial tree. Then $\chi'_2(T) = 1$ if $\Delta(T) \leq 2$ and $\chi'_2(T) = 2$ if $\Delta(T) \geq 3$.*

It is easy to see that $\chi'_2(G) = 1$ if and only if $\Delta(G) \leq 2$. It is an open problem to characterize all graphs G with $\chi'_2(G) = k$ for any $k \in \{2, 3\}$. At this moment we are able to prove the following:

Theorem 7. *Let G be a graph with $\delta(G) \geq 3$. Then*

$$\chi'_2(G) = \begin{cases} 2 & \text{if } G \text{ has a 2-factor,} \\ 3 & \text{otherwise.} \end{cases}$$

Proof. The graph G has $\delta(G) \geq 3$, therefore at least 2 colors have to be used. If G has a 2-factor, it suffices to assign color 2 to the edges of the 2-factor and color 1 to the remaining edges.

Suppose that G does not have a 2-factor and $\chi'_2(G) = 2$. Let H be a subgraph of graph G induced on edges assigned color 2. Since every vertex of H is incident with exactly two edges assigned color 2, H is a 2-regular subgraph and since every vertex of G has degree at least 3, there is no vertex which is not incident with edges assigned color 2, so H is also spanning subgraph. Therefore, H is a 2-factor of G, which is a contradiction. According to Theorem 5, $\chi'_2(G) \leq 3$ for every graph G, hence $\chi'_2(G) = 3$ in this case. ■

4. Trees

For trees we are able to prove more general results as mentioned in Corollary 6. Namely the following holds:

Theorem 8. *Let T be a nontrivial tree with maximum degree $\Delta(T)$ and r be a positive integer. Then $\chi'_r(T) = 1$ if $\Delta(T) \leq r$ and $\chi'_r(G) = 2$ if $\Delta(T) > r$.*

Proof. Proof is by induction on the number of inner vertices of T. The only tree with exactly one inner vertex is the star $S_n = K_{1,n}, n \geq 2$, with a central vertex u and leaves $v_1, v_2, \ldots, v_n$. Let $e_i = uv_i, i = 1, 2, \ldots, n$, be the pendant edges of S_n. Now define an r-maximum 2-edge-coloring φ of S_n as follows:

If $n \leq r$ we put $\varphi(e_i) = 1$ for all $i = 1, 2, \ldots, n$.

If $n > r$ we let $\varphi(e_i) = 2$ for all $i = 1, 2, \ldots, r$, and $\varphi(e_i) = 1$ for $i = r+1, \ldots, n$.

Consider now a tree T having at least two inner vertices. Let v be an inner vertex of T adjacent to only one other inner vertex w_1 of T. The other vertices adjacent to v are leaves $w_2, w_3, \ldots, w_{d_T(v)}$. Let $e_i = vw_i, i = 1, \ldots, d_T(v)$, be edges incident with v.

Let $T' = T - \{w_2, w_3 \ldots, w_{d_T(v)}\}$. By induction hypothesis there exists an r-maximum 2-edge-coloring φ of T'. Next we show how to extend this coloring to a required coloring φ of T. We consider two cases according to the color of the edge e_1.

Case 1. Let $\varphi(e_1) = 1$. If $d_T(v) \leq r$, then we put $\varphi(e_i) = 1$ for all $i = 2, 3, \ldots, d_T(v)$. If $d_T(v) > r$, then we put $\varphi(e_i) = 2$ for $i = 2, 3, \ldots, r+1$, and $\varphi(e_i) = 1$ for all $i = r+2, \ldots, d_T(v)$.

Case 2. Let $\varphi(e_1) = 2$. If $d_T(v) < r$, then we put $\varphi(e_i) = 1$ for all $i = 1, 2, \ldots, d_T(v)$. If $d_T(v) \geq r$, then we let $\varphi(e_i) = 2$ for all $i = 2, 3, \ldots, r$, and $\varphi(e_i) = 1$ for $i = r+1, r+2, \ldots, d_T(v)$.

It is easy to see that so obtained 2-edge-coloring has the required property.

■

5. COMPLETE GRAPHS

Trivially $\chi'_r(K_n) = 1$ if and only if $r \geq n-1$.

Theorem 9. *Let r, n be positive integers, $1 \leq r \leq n-2$. Then*

1. $\chi'_r(K_n) = 2$ *for even n or n odd and r even.*
2. $\chi'_r(K_n) = 3$ *for n odd and r odd.*

Proof. We distinguish two cases according to the parity of n.

Case 1. Let $n = 2k$. It is well known that the graph K_{2k} can be factorable into $k-1$ Hamiltonian cycles $H_1, H_2, \ldots, H_{k-1}$ and a 1-factor P (see [2] p.426).

Subcase 1.1. If $r = 2l$, then $1 \leq l \leq k-1$. Denote by F_{2l} a $2l$-regular factor of G formed by the union of the Hamiltonian cycles $H_1, H_2, \ldots, H_l$. If we assign 2 to the edges of F_{2l} and 1 to the remaining edges of K_n we obtain a $2l$-maximum 2-edge-coloring of K_{2k}.

Subcase 1.2. If $r = 2l+1$, then $1 \leq l \leq k-2$. Denote by F_{2l+1} a $(2l+1)$-regular factor of G created by the union of the 1-factor P and Hamiltonian cycles $H_1, H_2, \ldots, H_l$. The rest of the proof is as above.

Case 2. Let $n = 2k+1$. The graph K_{2k+1} can be factorable into k Hamiltonian cycles $H_1, H_2, \ldots, H_k$ (see [2] p. 424).

Subcase 2.1. If $r = 2l$, then $1 \leq l \leq k-1$. In this case we proceed in the same way as in Subcase 1.1.

Subcase 2.2. If $r = 2l+1$, then $1 \leq l \leq k-1$. First we use the above constructed $(2l+1)$-maximum 2-edge-coloring of the subgraph $K_{2k} = K_{2k+1} - v$ for some vertex v replacing color 2 by 3. Next we assign 2 to exactly $2l+1$ edges incident with v and 1 to the remaining $2k-2l-1$ edges at v. Hence, we have $\chi'_{2l+1}(K_{2n+1}) \leq 3$.

To finish the proof of our theorem it remains to show that any $(2l+1)$-maximum m-edge-coloring of K_{2k+1} requires $m \geq 3$ colors. Suppose that there is a $(2l+1)$-maximum 2-edge-coloring of K_{2k+1}. Consider a subgraph H of K_{2k+1} induced by edges assigned 2. Obviously, H is a $(2l+1)$-factor of K_{2k+1}; i.e. an odd-factor on odd number of vertices, a contradiction. ■

6. $r \geq 3$ AND GENERAL GRAPHS

Trivially $\chi'_r(G) = 1$ if and only if $\Delta(G) \leq r$. The following easy observation can be useful.

Theorem 10. *Let G be a nontrivial graph and r be a positive integer. If G has an r-regular subgraph H such that for every vertex $v \in V(G) - V(H)$, $d_G(v) \leq r$ then $\chi'_r(G) \leq 2$.*

On the other hand we have:

Theorem 11. *Let $r \geq 3$ be an integer. Then there is a graph G for which no r-maximum edge-coloring is defined.*

Proof. For $i \in \{1, 2\}$ let $H_i = K_{r+1}^{(i)} - e^{(i)}$ be a complete graph on $r + 1$ vertices with an edge $e^{(i)} = u^{(i)}v^{(i)}$ deleted. Identify the vertex $u^{(1)}$ (respectively $v^{(1)}$) of H_1 with the vertex $u^{(2)}$ (respectively $v^{(2)}$) of H_2. The resulting graph G has $2r$ vertices all of which, except of u^* and v^*, are of degree r, where $u^*(v^*)$ is the result of the indentification of $u^{(1)}$ and $u^{(2)}$ (respectively $v^{(1)}$ and $v^{(2)}$). The degree of u^* and v^* is $2r - 2$.

Let x be a vertex of G, $u^* \neq x \neq v^*$, incident with an edge colored with color k. Then all edges incident with x must be colored with k. This coloring enforces all other edges of G to be colored with k. This yields a contradiction because $d_G(u^*) = 2r - 2 > r$. ∎

7. An Open Problem

We expect an affirmative answer to the following question:

Problem 12. Is $\chi'_r(G) \leq 3$ for any integer $r \geq 3$ and any nontrivial graph G admitting an r-maximum edge-coloring?

We state also one more open problem:

Problem 13. For a given integer $r \geq 3$ characterize all graphs admitting an r-maximum edge-coloring.

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