Discussiones Mathematicae
Graph Theory 36 (2016) 59–69
doi:10.7151/dmgt.1842

M_2-EDGE COLORINGS OF CACTI AND GRAPH JOINS

Július Czap and Peter Šugerek

Department of Applied Mathematics and Business Informatics
Faculty of Economics, Technical University of Košice
Němcovej 32, 040 01 Košice, Slovakia

e-mail: julius.czap@tuke.sk
peter.sugerek@tuke.sk

and

Jaroslav Ivančo[1]

Institute of Mathematics, P. J. Šafárik University
Jesenná 5, 040 01 Košice, Slovakia

e-mail: jaroslav.ivanco@upjs.sk

Abstract

An edge coloring φ of a graph G is called an M_2-edge coloring if $|\varphi(v)| \leq 2$ for every vertex v of G, where $\varphi(v)$ is the set of colors of edges incident with v. Let $\mathcal{K}_2(G)$ denote the maximum number of colors used in an M_2-edge coloring of G. In this paper we determine $\mathcal{K}_2(G)$ for trees, cacti, complete multipartite graphs and graph joins.

Keywords: cactus, edge coloring, graph join.

2010 Mathematics Subject Classification: 05C15.

1. Introduction and Notations

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An *edge coloring* of G is an assignment of colors to the edges of G, one color to each edge. If adjacent edges are assigned distinct colors, then the edge coloring is a *proper* edge coloring.

Many generalizations of proper edge colorings have been introduced and studied in research of graph properties. These generalizations are usually obtained by relaxation of the condition that no two edges of the same color are adjacent. For

[1] Supported in part by the Slovak VEGA Grant 1/0652/12.

instance, in an f-*coloring* it is allowed that at each vertex v the same color may occur at most $f(v)$ times, where f is a function which assigns a positive integer $f(v)$ to each vertex v (see [7, 9, 12, 13, 14]). This type of coloring has many important applications in scheduling problems, e.g. the file transfer problem in a computer network (see [2, 3, 8]). If instead we require that the edges incident with each vertex v have at most $f(v)$ different colors, then we obtain the definition of a generalized M_i-edge coloring, which was introduced in [4]. In this paper we deal with M_2-edge colorings.

An edge coloring φ of a graph G is an M_2-*edge coloring* if at most two colors appear at any vertex of G, i.e., $|\varphi(v)| \leq 2$ for every vertex v of G, where $\varphi(v)$ is the set of colors of edges incident with v. The problem is to determine the maximum number $\mathcal{K}_2(G)$ of colors used in an M_2-edge coloring of G. The exact values of this parameter are known only for subcubic graphs [4, 6] and complete graphs [5]. In this paper we determine $\mathcal{K}_2(G)$ for cacti and graph joins, moreover, we answer a question of Soták [11] on the monotonicity of $\mathcal{K}_2$. As corollaries of the main results we obtain the values of $\mathcal{K}_2(G)$ for trees and complete multipartite graphs.

An M_2-edge coloring of G is called *optimal* if it uses $\mathcal{K}_2(G)$ colors. If S is a subset of the vertex set of G, then the subgraph of G induced by S is denoted by $G[S]$.

2. Results

2.1. Cacti

Recall that a connected graph in which every edge belongs to at most one cycle is called a *cactus*.

Lemma 1. *Let $e = uv$ be an edge of a graph G such that* $\max\{\deg(u), \deg(v)\} \leq 2$. *Then*

$$\mathcal{K}_2(G) = 1 + \mathcal{K}_2(G - e).$$

Proof. Since any M_2-edge coloring of $G - e$ using k colors can be extended to an M_2-edge coloring of G using $k + 1$ colors, we obtain $\mathcal{K}_2(G) \geq 1 + \mathcal{K}_2(G - e)$.

On the other hand, any M_2-edge coloring of G using k colors induces an M_2-edge coloring of $G - e$ using at least $k - 1$ colors. Thus, $\mathcal{K}_2(G - e) \geq \mathcal{K}_2(G) - 1$, i.e., $\mathcal{K}_2(G) \leq 1 + \mathcal{K}_2(G - e)$. ■

Lemma 2. *Let v be a pendant vertex of a connected graph G which is adjacent to the vertex u. If the graph $G - u$ contains at least three components, then*

$$\mathcal{K}_2(G) = \mathcal{K}_2(G - v).$$

Proof. Since any M$_2$-edge coloring of $G - v$ using k colors can be extended to an M$_2$-edge coloring of G using k colors, we obtain $\mathcal{K}_2(G) \geq \mathcal{K}_2(G-v)$. So let ψ be an M$_2$-edge coloring of G which uses k colors; it suffices to show that $\mathcal{K}_2(G-v) \geq k$.

If the color of the edge uv occurs on at least two edges of G, then the coloring ψ of G induces a required coloring of $G - v$.

So we can assume that the color of the edge uv occurs only once in G. In this case all edges incident with u and different from uv have the same color c, $c \neq \psi(uv)$. Let G_1 be a component of $G - u$ which does not contain the vertex v. If we recolor all edges colored by c in the subgraph of G induced by $V(G_1) \cup \{u\}$ with color $\psi(uv)$, then we obtain an M$_2$-edge coloring of G using k colors such that the color of the edge uv occurs on at least two edges of G. This coloring induces an M$_2$-edge coloring of $G - v$ with k colors. Consequently, $\mathcal{K}_2(G) \leq \mathcal{K}_2(G-v)$. ■

Let $V_j(G)$ denote the set of vertices of degree j in G. M$_2$-edge colorings of subcubic graphs were studied in [4].

Proposition 3 [4]**.** *Let G be a graph with maximum degree at most three. Let t denote the maximum number of disjoint cycles of G that contain no vertices of degree two. Then*

$$\mathcal{K}_2(G) = \frac{1}{2}\left(|V(G)| + |V_2(G)|\right) + t.$$

Lemma 4. *Let uv be a pendant edge of a cactus G such that* $\deg(v) = 1$ *and* $\deg(u) = 3$*. If there is a cycle C in G containing the vertex u, then*

$$\mathcal{K}_2(G-v) = \begin{cases} \mathcal{K}_2(G) - 1 & \textit{if } \deg(w) \geq 3 \textit{ for every } w \in V(C), \\ \mathcal{K}_2(G) & \textit{otherwise.} \end{cases}$$

Proof. First assume that $\deg(w) \geq 3$ for every vertex $w \in V(C)$. Since any M$_2$-edge coloring of G using k colors induces an M$_2$-edge coloring of $G - v$ using at least $k-1$ colors, we obtain $\mathcal{K}_2(G-v) \geq \mathcal{K}_2(G) - 1$. So let ψ be an M$_2$-edge coloring of $G - v$ which uses k colors; it suffices to show that $\mathcal{K}_2(G) \geq k+1$.

If $|\psi(u)| = 1$, then ψ can be extended to a required coloring of G. If $|\psi(u)| = 2$, then we modify ψ into an M$_2$-edge coloring ψ' which uses at least k colors, moreover $|\psi'(u)| = 1$. In the following first we show that $|\bigcup_{w \in V(C)} \psi(w)| \leq |V(C)|$, and then describe the modification.

For every vertex $w \in V(C) - \{u\}$ we choose an edge e_w incident with w such that $e_w \notin E(C)$ and $\psi(w) = \{\psi(e_w), \psi(e_1), \psi(e_2)\}$, where e_1 and e_2 are edges of C incident with w. The subgraph H of $G - v$ induced by $\bigcup_{w \in V(C-u)} e_w \cup E(C)$ contains the cycle C, and every vertex of $C - u$ is adjacent to a pendant vertex of H. The maximum degree of H is three, therefore Proposition 3 gives $\mathcal{K}_2(H) = |V(C)|$. Since ψ induces an M_2-edge coloring of H, $|\bigcup_{w\in V(C)}\psi(w)| \leq |V(C)|$.

For every vertex $w \in V(C)$ let c_w be a new color, i.e., c_w does not belong to $\psi(E(G-v))$, and let G_w be the component of $(G-v)-E(C)$ containing the vertex w. The edge coloring ψ' of $G-v$ is defined in the following way

$$\psi'(e) = \begin{cases} c_u & \text{if } e \in E(C),\\ c_w & \text{if } e \in E(G_w) \text{ and } \psi(e) = \psi(e_w),\\ c_u & \text{if } e \in E(G_w) \text{ and } \psi(e) \in \psi(w) - \psi(e_w),\\ \psi(e) & \text{otherwise.} \end{cases}$$

It is easy to see that ψ' is an M_2-edge coloring of $G-v$. The coloring ψ' uses at least $k - |\bigcup_{w\in V(C)}\psi(w)| + |V(C)| \geq k$ colors. Moreover, $\psi'(u) = \{c_u\}$, and so ψ' can be extended to a required coloring of G.

So we may assume that there is a vertex of degree two in $V(C)$. Since any M_2-edge coloring of $G-v$ using k colors can be extended to an M_2-edge coloring of G using k colors, we obtain $\mathcal{K}_2(G) \geq \mathcal{K}_2(G-v)$. So let φ be an M_2-edge coloring of G which uses k colors; it suffices to show that $\mathcal{K}_2(G-v) \geq k$.

If the color of the edge uv occurs on at least two edges of G, then the coloring φ of G induces a required coloring of $G-v$. Otherwise, we modify φ into an M_2-edge coloring φ' which uses at least k colors, moreover the color $\varphi'(uv)$ occurs on at least two edges of G. Thus, assume that the color of the edge uv occurs only once in G. Similarly as in the previous case we can show that $|\bigcup_{w\in V(C)}\varphi(w)| \leq |V(C)|$.

Set $p = |V(C)|$ and denote the vertices of C by u_1, u_2, $\dots$, u_p in such a way that its edges are u_iu_{i+1} for $i = 1, 2, \dots, p$, the subscripts being taken modulo p. For $i = 1, 2, \dots, p$, let G_i be the component of $G - E(C)$ containing the vertex u_i, let c_i be a color belonging to $\varphi(u_i)$, and let c_i^* be a new color not belonging to $\varphi(E(G))$ such that $|\bigcup_{i=1}^{p}\{c_i\}| \leq |\bigcup_{i=1}^{p}\{c_i^*\}| = p$. The edge coloring φ' of G is given by

$$\varphi'(e) = \begin{cases} c_i^* & \text{if } e = u_iu_{i+1},\\ c_i^* & \text{if } e \in E(G_i) \text{ and } \varphi(e) = c_i,\\ c_{i-1}^* & \text{if } e \in E(G_i) \text{ and } \varphi(e) \in \varphi(u_i) - \{c_i\},\\ \varphi(e) & \text{otherwise.} \end{cases}$$

It is easy to see that φ' is an M_2-edge coloring of G. The coloring φ' uses at least $k - |\bigcup_{i=1}^{p}\varphi(u_i)| + p \geq k$ colors. Moreover, the edges of C incident with u have different colors, and so the color $\varphi'(uv)$ occurs on at least two edges of G. Therefore, the coloring φ' of G induces a required coloring of $G-v$. ∎

Theorem 5. *Let G be a cactus on at least two vertices. Let r denote the number of cycles of G containing a vertex of degree two. Then*

$$\mathcal{K}_2(G) = 1 + |V(G)| - |V_1(G)| - r.$$

Proof. We use induction on the number of edges. If $|E(G)| \leq 2$, then G is a path and $\mathcal{K}_2(G) = |E(G)| = 1 + |V(G)| - |V_1(G)|$ as required. Assume that the claim holds for every cactus with ℓ edges. Let G be a cactus with $\ell + 1$ edges.

First suppose that G contains no pendant vertex. Then G contains an edge $e = uv$ belonging to a cycle such that $\deg(u) = \deg(v) = 2$. Then $G - e$ is a cactus with ℓ edges, and so $\mathcal{K}_2(G-e) = 1+|V(G-e)|-|V_1(G-e)|-r'$, where r' is the number of cycles of $G - e$ containing a vertex of degree two. Clearly, $|V(G - e)| = |V(G)|$, $|V_1(G - e)| = |V_1(G)| + 2$, $r' = r - 1$ and Lemma 1 gives $\mathcal{K}_2(G) = 1 + \mathcal{K}_2(G - e) = 1 + |V(G)| - |V_1(G)| - r$.

So we may assume that G contains a pendant vertex v, with neighbor u, say. Then $G - v$ is a cactus with ℓ edges, hence

$$\mathcal{K}_2(G - v) = 1 + |V(G - v)| - |V_1(G - v)| - r^*, \tag{1}$$

where r^* is the number of cycles of $G - v$ containing a vertex of degree two.

If $\deg(u) = 2$, then $|V(G - v)| = |V(G)| - 1$, $|V_1(G - v)| = |V_1(G)|$, $r^* = r$ and Lemma 1 with (1) gives $\mathcal{K}_2(G) = 1 + \mathcal{K}_2(G - v) = 1 + |V(G)| - |V_1(G)| - r$.

So we may assume that $\deg(u) \geq 3$. Then $|V(G - v)| = |V(G)| - 1$, $|V_1(G - v)| = |V_1(G)| - 1$, and either $r^* = r$ or $r^* = r + 1$.

If $G - u$ has at least three components, then $r^* = r$ and Lemma 2 with (1) gives $\mathcal{K}_2(G) = \mathcal{K}_2(G - v) = 1 + |V(G)| - |V_1(G)| - r$.

So we may assume that $G - u$ has only two components. Then u is a vertex of a cycle C and $\deg(u) = 3$. If C contains a vertex of degree two, then $r^* = r$ and Lemma 4 with (1) gives $\mathcal{K}_2(G) = \mathcal{K}_2(G - v) = 1 + |V(G)| - |V_1(G)| - r$. Finally, if C contains no vertex of degree two, then $r^* = r + 1$ and Lemma 4 with (1) gives $\mathcal{K}_2(G) = 1 + \mathcal{K}_2(G - v) = 1 + |V(G)| - |V_1(G)| - r$. ■

As any tree is a cactus without cycles we immediately have the following assertion.

Corollary 6. *If T is a tree on at least two vertices, then*

$$\mathcal{K}_2(T) = 1 + |V(T)| - |V_1(T)|.$$

2.2. Joins of graphs

Recall that the *join* of two graphs G and H is obtained from vertex-disjoint copies of G and H by adding all edges between $V(G)$ and $V(H)$.

A *matching* in a graph is a set of pairwise nonadjacent edges. A matching is *perfect* if every vertex of the graph is incident with exactly one edge of the matching. A *maximum matching* is a matching that contains the largest possible number of edges. The number of edges in a maximum matching of a graph G is denoted by $\alpha(G)$.

Proposition 7 [1]. *Let G be a graph of maximum degree at least two. Then*

$$\mathcal{K}_2(G) \geq 1 + \alpha(G).$$

Let $c(G)$ denote the number of components of G. Clearly, $c(G) \leq \alpha(G)$ if G has no isolated vertices. If ψ is an edge coloring of G and X is a set of colors, we write $G -_{\psi} X$ for the graph obtained from G by deleting all edges with colors in X and all resulting isolated vertices.

Lemma 8. *Let ψ be an optimal* M_2*-edge coloring of a graph G and let X be a subset of the color set used by ψ. If $\psi(v) \cap X \neq \emptyset$ for every vertex v of G, then*

$$\mathcal{K}_2(G) - |X| = c(G -_{\psi} X) \leq \alpha(G -_{\psi} X) \leq \alpha(G).$$

Proof. Every component of $G -_{\psi} X$ is monochromatic, since at every vertex of G there is an edge with a color from X. Different components have different colors, as otherwise ψ is not optimal. Thus, the number of components of $G -_{\psi} X$ is equal to the number of colors on its edges, which is $\mathcal{K}_2(G) - |X|$. It is easy to see that the inequalities $c(G -_{\psi} X) \leq \alpha(G -_{\psi} X) \leq \alpha(G)$ hold. ■

Theorem 9. *Let G_1 and G_2 be disjoint graphs such that $|V(G_1)| \leq |V(G_2)|$, and let G be the join of G_1 and G_2. Then*

$$\mathcal{K}_2(G) = \begin{cases} 1 & \text{if } |V(G_1)| = |V(G_2)| = 1, \\ 2 + \alpha(G) & \text{if (i), (ii) or (iii) holds}, \\ 1 + \alpha(G) & \text{otherwise}, \end{cases}$$

where (i), (ii) *and* (iii) *are as follows:*

(i) $|V(G_1)| = 1$ *and there is a perfect matching M in G_2 such that $G_2 - M$ is disconnected,*

(ii) $|V(G_1)| = 2$, *G_2 has an isolated vertex and* $1 + 2\alpha(G_2) = |V(G_2)|$,

(iii) $\alpha(G) = |V(G_1)| = 2$ *and C_4 is an induced subgraph of G.*

Proof. Clearly, the claim holds for $|V(G_1)| = |V(G_2)| = 1$. So we will suppose that $|V(G_2)| \geq 2$.

The proof is in two parts. In the first part we will prove that $\mathcal{K}_2(G) \leq 2 + \alpha(G)$, with equality only if (i), (ii) or (iii) holds. In the second part we will give constructions to prove that $\mathcal{K}_2(G) \geq 2 + \alpha(G)$ if (i), (ii) or (iii) holds. This will prove the result, since we know from Proposition 7 that $\mathcal{K}_2(G) \geq 1 + \alpha(G)$ in all other cases.

For the first part of the proof, let ψ be an optimal M$_2$-edge coloring of G, and let $\Psi_j = \bigcup_{v \in V(G_j)} \psi(v)$ for $j \in \{1, 2\}$. There are some cases to consider.

Case 1. $|\Psi_j| \leq 2$ for some $j \in \{1, 2\}$. Define $H = G -_\psi \Psi_j$; evidently $H \subseteq G_{3-j}$. Since every edge between G_1 and G_2 has a color from Ψ_j, we can deduce from Lemma 8 that $\mathcal{K}_2(G) = |\Psi_j| + c(H) \leq 2 + \alpha(G)$.

Suppose that $\mathcal{K}_2(G) = 2 + \alpha(G)$. Then $|\Psi_j| = 2$ and $c(H) = \alpha(H) = \alpha(G_{3-j}) = \alpha(G)$. The last equality holds only if $|V(G_j)| = 1$ (which implies $j = 1$) and G_{3-j} (=G_2) has a perfect matching. Now $c(H) = \alpha(G_2)$ implies that $E(H)$ is a perfect matching of G_2, denote it M. Let $V(G_1) = \{v\}$, $\Psi_j = \psi(v) = \{c_1, c_2\}$, and for $i \in \{1, 2\}$ let U_i comprise the vertices of G_2 that are joined to v by an edge of color c_i. Then U_1, U_2 partition $V(G_2)$ into disjoint nonempty sets. If e is an edge of G_2 joining vertices $u_1 \in U_1$ and $u_2 \in U_2$, then its color is neither c_1 nor c_2 (otherwise at u_1 or u_2 appear edges of colors c_1 and c_2 as well as an edge of M), and so e is an edge of M. Thus there is no edge of $G_2 - M$.

Case 2. $|\Psi_j| \geq 3$ for each $j \in \{1, 2\}$. In this case $|V(G_1)| \geq 2$.

Case 2.1. $|\psi(v_1) \cup \psi(v_2)| \leq 2$ for some $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

Let $\Psi_{1,2} = \psi(v_1) \cup \psi(v_2)$ and $D = G -_\psi \Psi_{1,2}$. Since every vertex v of G is adjacent to v_1 or v_2, and so $\psi(v) \cap \Psi_{1,2} \neq \emptyset$, we can deduce from Lemma 8 that $\mathcal{K}_2(G) = |\Psi_{1,2}| + c(D) \leq 2 + \alpha(D)$. Since $D \subseteq G - \{v_1, v_2\}$, we have $\alpha(D) + 1 \leq \alpha(G)$. Thus $\mathcal{K}_2(G) \leq 1 + \alpha(G)$.

Case 2.2. $|\psi(v_1) \cup \psi(v_2)| \geq 3$ for all $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

In this case $|\psi(w)| = 2$ for every vertex $w \in V(G)$, since $|\psi(v_1) \cap \psi(v_2)| \geq 1$.

Case 2.2.1. For each $i \in \{1, 2\}$ there are vertices $u_i, v_i \in V(G_i)$ such that $\psi(u_i) \cap \psi(v_i) = \emptyset$.

Observe that $\psi(w) \subseteq \psi(u_i) \cup \psi(v_i)$ for every vertex $w \in V(G_{3-i})$ because w is adjacent to u_i and v_i. Therefore $\psi(u_1) \cup \psi(v_1) = \psi(u_2) \cup \psi(v_2)$ and ψ uses exactly four colors. This implies that $\mathcal{K}_2(G) \leq 1 + \alpha(G)$ when $\alpha(G) \geq 3$. Moreover, if $\alpha(G) = 2$, then the vertices u_1, u_2, v_1, v_2 induce a 4-cycle and $|V(G_1)| = 2$, therefore (iii) holds.

Case 2.2.2. For some $i \in \{1, 2\}$ there are vertices $u_i, v_i \in V(G_i)$ such that $\psi(u_i) \cap \psi(v_i) = \emptyset$, and $\psi(u) \cap \psi(v) \neq \emptyset$ for all $u, v \in V(G_{3-i})$.

Let $\psi(u_i) = \{a, b\}$ and $\psi(v_i) = \{c, d\}$. As every vertex $w \in V(G_{3-i})$ is adjacent to u_i and v_i, it follows that $\psi(w)$ is equal to $\{a, c\}, \{a, d\}, \{b, c\}$ or $\{b, d\}$. Since $\psi(u) \cap \psi(v) \neq \emptyset$ for all $u, v \in V(G_{3-i})$, we may assume without loss of generality that $\psi(w)$ equals $\{a, c\}$ or $\{b, c\}$ for each $w \in V(G_{3-i})$. Moreover, there are vertices of both types, since $|\Psi_{3-i}| \geq 3$, say $\psi(w_1) = \{a, c\}$ and $\psi(w_2) = \{b, c\}$.

Let $Q = G -_{\psi} \{a, b, c\}$; evidently $Q \subseteq G_i - u_i$. By Lemma 8, $\mathcal{K}_2(G) = 3 + c(Q) \leq 3 + \alpha(G_i - u_i) \leq 2 + \alpha(G)$.

Suppose that $\mathcal{K}_2(G) = 2 + \alpha(G)$. Then $c(Q) = \alpha(G_i - u_i) = \alpha(G) - 1$. The last equality implies that $|V(G_{3-i})| = 2$ (since $2 \leq |V(G_1)| \leq |V(G_2)|$) and $G_i - u_i$ has a perfect matching. Now $c(Q) = \alpha(G_i - u_i)$ implies that $E(Q)$ is a perfect matching of $G_i - u_i$. Since each vertex x of $G_i - u_i$ is incident with an edge of Q, whose color is not in $\{a, b, c\}$, x must be joined to w_1 and w_2 by edges of color c, and so is not adjacent to u_i. Thus u_i is an isolated vertex of G_i. As $2 = |V(G_{3-i})| < |V(G_i)|$, $i = 2$ thus (ii) holds.

Case 2.2.3. $\psi(u) \cap \psi(v) \neq \emptyset$ for all $u, v \in V(G)$. Let F be the graph whose vertices are the colors used by ψ, where vertices c_1, c_2 are adjacent whenever there is a vertex $w \in V(G)$ such that $\psi(w) = \{c_1, c_2\}$. The assumption of this case says that each two edges of F are adjacent, hence F is a 3-cycle or a star. If it is a 3-cycle, then $\mathcal{K}_2(G) = 3 \leq 1 + \alpha(G)$ (since $2 \leq |V(G_1)| \leq |V(G_2)|$). If F is a star, then there is a color c which appears at every vertex of G, and then $\mathcal{K}_2(G) \leq 1 + \alpha(G)$ by Lemma 8 with $X = \{c\}$.

This completes the first part of the proof. In the second part, we show that $\mathcal{K}_2(G) \geq 2 + \alpha(G)$ if (i), (ii) or (iii) holds. First suppose that (i) holds. Let $V(G_1) = \{v\}$ and let $U_1 \cup U_2$ be a partition of $V(G_2)$ such that only edges of M join U_1 to U_2. Color all edges of $G[U_1 \cup \{v\}] - M$ with one color, all edges of $G[U_2 \cup \{v\}] - M$ with a second color, and every edge of M with a different color. It is easy to see that this gives an M_2-edge coloring of G using $2 + |M| = 2 + \alpha(G)$ colors, so $\mathcal{K}_2(G) \geq 2 + \alpha(G)$, as required.

Now suppose that (ii) holds. Let $V(G_1) = \{v_1, v_2\}$, let u be the isolated vertex of G_2, and let M be a perfect matching of $G_2 - u$. Use color 1 on uv_1, color 2 on uv_2, color 3 on every other edge not in M, and a different color for every edge of M. This gives an M_2-edge coloring of G using $3 + |M| = 2 + \alpha(G)$ colors, consequently $\mathcal{K}_2(G) \geq 2 + \alpha(G)$.

Finally, suppose that (iii) holds. Let $C = v_1u_1v_2u_2$ be an induced 4-cycle of G such that $V(G_1) = \{v_1, v_2\}$ and $u_1, u_2 \in V(G_2)$. (Note that $\alpha(G_2) \leq 1$, since $\alpha(G) = 2$. Therefore, at least one vertex of C must be in G_1. Then the opposite vertex must be in G_1 as well, since C_4 is an induced cycle.) Color every edge of C with a different color, color every other edge incident with v_1 or u_1 the same as v_1u_1, and color every remaining edge the same as v_2u_2. This gives an M_2-edge coloring of G using four colors, so that $\mathcal{K}_2(G) \geq 4 = 2 + \alpha(G)$. ■

As a complete $(k+1)$-partite graph is the join of a totally disconnected graph and a complete k-partite graph, it is a straightforward exercise to obtain the following assertion.

Corollary 10. *Let $k \geq 2$ and $1 \leq n_1 \leq \cdots \leq n_k$ be integers. If G is a complete k-partite graph $K_{n_1,\ldots,n_k}$, then*

$$\mathcal{K}_2(G) = \begin{cases} 1 & \textit{if } G = K_{1,1}, \\ 3 & \textit{if } G = K_{1,1,1}, \\ 4 & \textit{if } G = K_{1,2,2} \textit{ or } G = K_{2,n_2}, \\ 1+\alpha(G) & \textit{otherwise.} \end{cases}$$

Note that $\mathcal{K}_2(G)$ for the complete multipartite graph G can be easily computed because $\alpha(K_{n_1,n_2,\ldots,n_k}) = \min\left\{\sum_{i=1}^{k-1} n_i, \left\lfloor \frac{1}{2}\sum_{i=1}^{k} n_i \right\rfloor\right\}$, see [10].

3. Non-Monotonicity of $\mathcal{K}_2$

In [5] the following assertion was proved.

Theorem 11. [5] *For any positive integer ℓ there are graphs G_1, G_2 and G_3 such that $G_1 \subseteq G_2 \subseteq G_3$, $\mathcal{K}_2(G_1) = \mathcal{K}_2(G_3)$ and $\mathcal{K}_2(G_2) \geq \mathcal{K}_2(G_3) + \ell$.*

Soták [11] asked the following.

Problem 12. Let k be a positive integer. Are there graphs G_1, G_2 and G_3 such that $G_1 \subseteq G_2 \subseteq G_3$, $\mathcal{K}_2(G_1) = \mathcal{K}_2(G_3)$ and $\mathcal{K}_2(G_2) \leq \mathcal{K}_2(G_3) - k$?

In the following we show that the answer is positive.

Let k be a positive integer. Denote by T_{2k} the tree with

$$V(T_{2k}) = \{v_{i,j} | i = 1, 2, \ldots, 2k;\ j = 1, 2, 3, 4\},$$

$$E(T_{2k}) = \{v_{i,1}v_{i+1,1} | i = 1, 2, \ldots, 2k-1\} \cup \{v_{i,j}v_{i,j+1} | i = 1, 2, \ldots, 2k;\ j = 1, 2, 3\},$$

see Figure 1 for illustration.

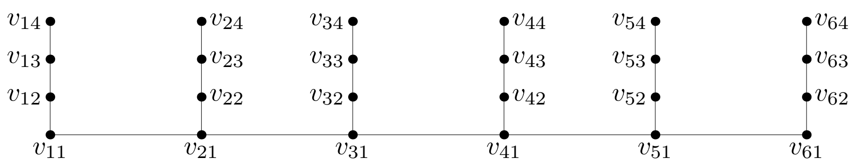

Figure 1. The tree T_6.

By Corollary 6 every optimal coloring of T_{2k} uses $6k+1$ colors.

Let T_{2k}^{+} be a graph obtained from T_{2k} by adding the edges $v_{i,j}v_{i+1,j}$, $i = 1, 3, \ldots, 2k-1$, $j = 2, 3$ (see Figure 2 for illustration).

Using Proposition 3 we obtain $\mathcal{K}_2(T_{2k}^{+}) = \frac{1}{2}(8k+2) + k = 5k+1$.

Let T_{2k}^{++} be a graph obtained from T_{2k}^{+} by adding the edges $v_{i,4}v_{i+1,4}$, $i = 1, 3, \ldots, 2k-1$, see Figure 3 for illustration.

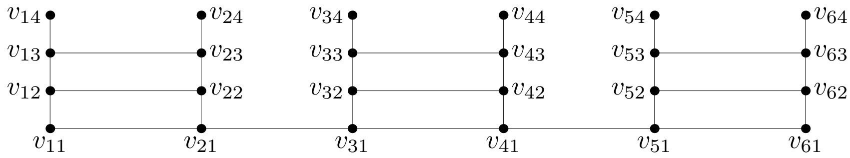

Figure 2. The graph T_6^+.

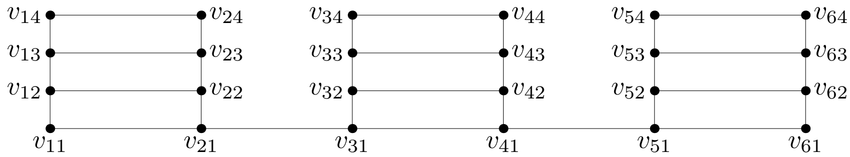

Figure 3. The graph T_6^{++}.

Again, by Proposition 3 we obtain $\mathcal{K}_2(T_{2k}^{++}) = \frac{1}{2}(8k + 2k + 2) + k = 6k + 1$.

Clearly, $T_{2k} \subseteq T_{2k}^+ \subseteq T_{2k}^{++}$ and $\mathcal{K}_2(T_{2k}) = \mathcal{K}_2(T_{2k}^{++}) = 6k + 1$. Moreover, $\mathcal{K}_2(T_{2k}^+) = \mathcal{K}_2(T_{2k}^{++}) - k$.

Acknowledgement

We thank two anonymous reviewers for the comments and suggestions that have led to a substantial improvement of this paper.

References

[1] K. Budajová and J. Czap, M_2*-edge coloring and maximum matching of graphs*, Int. J. Pure Appl. Math. **88** (2013) 161–167.
doi:10.12732/ijpam.v88i2.1

[2] H. Choi and S.L. Hakimi, *Scheduling file transfers for trees and odd cycles*, SIAM J. Comput. **16** (1987) 162–168.
doi:10.1137/0216013

[3] E.G. Coffman Jr., M.R. Garey, D.S. Johnson and A.S. LaPaugh, *Scheduling file transfers*, SIAM J. Comput. **14** (1985) 744–780.
doi:10.1137/0214054

[4] J. Czap, M_i*-edge colorings of graphs*, Appl. Math. Sciences **5** (2011) 2437–2442.
doi:10.12988/ams

[5] J. Czap, *A note on* M_2*-edge colorings of graphs*, Opuscula Math. **35** (2015) 287–291.
doi:10.7494/OpMath.2015.35.3.287

[6] M. Gionfriddo, L. Milazzo and V. Voloshin, *On the upper chromatic index of a multigraph*, Comput. Sci. J. Moldova **10** (2002) 81–91.

[7] S.L. Hakimi and O. Kariv, *A generalization of edge-coloring in graphs*, J. Graph Theory **10** (1986) 139–154.
doi:10.1002/jgt.3190100202

[8] H. Krawczyk and M. Kubale, *An approximation algorithm for diagnostic test scheduling in multicomputer systems*, IEEE Trans. Comput. **C-34** (1985) 869–872.
doi:10.1109/TC.1985.1676647

[9] S.I. Nakano, T. Nishizeki and N. Saito, *On the f-coloring of multigraphs*, IEEE Trans. Circuits Syst. **35** (1988) 345–353.
doi:10.1109/31.1747

[10] D. Sitton, *Maximum matching in complete multipartite graphs*, Furman Univ. Electronic J. Undergraduate Math. **2** (1996) 6–16.

[11] R. Soták, Personal communication.

[12] X. Zhang and G. Liu, *f-colorings of some graphs of f-class* 1, Acta Math. Sin. (Engl. Ser.) **24** (2008) 743–748.
doi:10.1007/s10114-007-6194-9

[13] X. Zhang and G. Liu, *Some sufficient conditions for a graph to be of C_f* 1, Appl. Math. Lett. **19** (2006) 38–44.
doi:10.1016/j.aml.2005.03.006

[14] X. Zhang and G. Liu, *Some graphs of class* 1 *for f-colorings*, Appl. Math. Lett. **21** (2008) 23–29.
doi:10.1016/j.aml.2007.02.009

Received 12 December 2014
Revised 13 April 2015
Accepted 13 April 2015