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A FAN-TYPE HEAVY PAIR OF SUBGRAPHS FOR PANCYCLICITY OF 2-CONNECTED GRAPHS

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Abstract

Let G be a graph on n vertices and let H be a given graph. We say that G is pancyclic, if it contains cycles of all lengths from 3 up to n, and that it is H-f_1-heavy, if for every induced subgraph K of G isomorphic to H and every two vertices u, $v \in V(K)$, $d_K(u,v) = 2$ implies $\min\{d_G(u), d_G(v)\} \geq \frac{n+1}{2}$. In this paper we prove that every 2-connected $\{K_{1,3}, P_5\}$-f_1-heavy graph is pancyclic. This result completes the answer to the problem of finding f_1-heavy pairs of subgraphs implying pancyclicity of 2-connected graphs.

Keywords: cycle, Fan-type heavy subgraph, Hamilton cycle, pancyclicity.

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1. Introduction

In the paper we consider only finite, simple and undirected graphs. For terminology and notation not defined here see Bondy and Murty [5].

Let G be a graph on n vertices. G is said to be Hamiltonian, if it contains a cycle C_n, and it is called pancyclic, if it contains cycles of all lengths k for $3 \leq k \leq n$. If G does not contain an induced copy of a given graph H, we say that G is H-free. G is called H-f_i-heavy, if for every induced subgraph S of G isomorphic to H and for every two vertices x, $y \in V(S)$ satisfying $d_S(x,y) = 2$, the following inequality holds: $\max\{d_G(x), d_G(y)\} \geq \frac{n+i}{2}$. For the sake of simplicity, we write f-heavy instead of f_0-heavy. For a family of graphs $\mathcal{H}$ we say that G is $\mathcal{H}$-free ($\mathcal{H}$-f_i-heavy), if G is H-free (H-f_i-heavy) for every graph $H \in \mathcal{H}$.

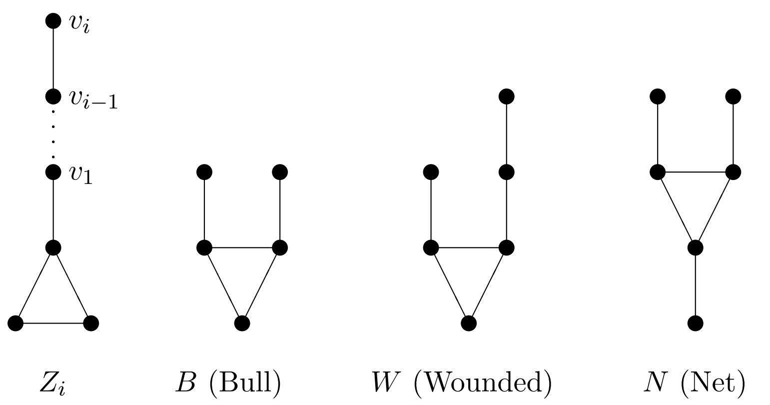

Figure 1. Graphs Z_i, B, W and N.

The complete bipartite graph $K_{1,3}$ is called a claw. Vertex of degree three in the claw is called its center vertex, and other vertices are its end vertices.

In [1] Bedrossian characterised all pairs of forbidden subgraphs implying Hamiltonicity and pancyclicity of 2-connected graphs (graphs Z_i, B, W and N are represented on Figure 1).

Theorem 1 (Bedrossian). *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

Theorem 2 (Bedrossian). *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

Geng-Hua Fan in 1984 proved the following theorem, here stated in the form that uses a notion of f_i-heavy graphs.

Theorem 3 (Fan, [6]). *Every 2-connected P_3-f-heavy graph is Hamiltonian.*

Note that every H-free graph for a given graph H is H-f_i-heavy for every integer i. Having that in mind, one could try to improve Theorem 1, considering f-heavy pairs of graphs instead of forbidden pairs. The following result was obtained by Ning and Zhang.

Theorem 4 (Ning and Zhang, [9]). *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-f-heavy implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, P_6, Z_1, Z_2, B, N or W.*

In 1987 Wojda and Benhocine showed that Fan's condition for Hamiltonicity implies in fact pancyclicity, besides few special cases (where F_{4r} stands for a clique on $2r$ vertices that is connected via perfect matching with r disjoint copies of a path P_2).

Theorem 5 (Benhocine and Wojda, [3]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is P_3-f-heavy, then G is pancyclic unless $G = F_{4r}$ or $G = K_{n/2,n/2}$ or else $n \geq 6$ is even and $G = K_{n/2,n/2} - e$.*

Since none of the special graphs mentioned in Theorem 5 is P_3-f_1-heavy, it is easy to see that every P_3-f_1-heavy graph is pancyclic. P_3 is the only graph having this property. One could now consider a problem of finding all pairs of connected graphs R and S other than P_3 such that every $\{R, S\}$-f_1-heavy graph is pancyclic. By Theorem 2 one of them must be a claw, and the second one must be one of the graphs P_4, P_5, Z_1 or Z_2. Partial answers to this problem were obtained by Bedrossian, Chen, Schelp and Ning.

Theorem 6 (Bedrossian, Chen and Schelp, [2]). *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, Z_1\}$-f_1-heavy, then G is pancyclic.*

Theorem 7 (Ning, [8]). *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, Z_2\}$-f_1-heavy or $\{K_{1,3}, P_4\}$-f_1-heavy, then G is pancyclic.*

The last pair from Theorem 2 that could imply pancyclicity is $\{K_{1,3}, P_5\}$. In this paper we prove the following theorem.

Theorem 8. *Let G be a 2-connected graph which is not a cycle. Then G being $\{K_{1,3}, P_5\}$-f_1-heavy implies G is pancyclic.*

Theorems 6, 7 and 8 can be rewritten together in a following form, that extends Theorem 2 and fully answers problem of finding f_1-heavy pairs of graphs implying pancyclicity of 2-connected graphs.

Theorem 9. *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-f_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

In Section 2 we introduce notation used further in the paper and present some of the previous results that will be of use in the proof of Theorem 8. The proof itself is postponed to Section 3.

2. Preliminaries

We first give some additional terminology and notation, and present previous results that will be of use in the proof of Theorem 8.

The subgraph of G induced by the set of vertices $A \subset V(G)$ is denoted $G[A]$. By $G - A$ we denote the subgraph $G[V(G) \setminus A]$. If $A = \{v\}$, we write $G - v$ instead of $G - \{v\}$. Let $A = \{v_1; v_2, v_3, v_4, v_5\}$. If $G[A]$ is isomorphic to P_5, we say that A induces a P_5, where $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$ are the edges of this path. If $A = \{v_1, v_2, v_3, v_4\}$ and $G[A]$ is isomorphic to $K_{1,3}$, we say that $\{v_1, v_2, v_3, v_4\}$ induces $K_{1,3}$ (or induces a claw), where v_1 is a center vertex and v_2, v_3 and v_4 are end vertices of a claw.

For a cycle C we distinguish one of two possible orientations of C. We write xC^+y for the path from $x \in V(C)$ to $y \in V(C)$ following the orientation of C, and xC^-y denotes the path from x to y opposite to the direction of C. For two positive integers k and m, where $k \le m$, we say that G contains $[k, m]$-cycles if there are cycles $C_k, C_{k+1}, \ldots, C_m$ in G. Let $\{v_1, \ldots, v_p\}$ be the set of vertices of a cycle C. For two positive integers k and m, satisfying $k \le m \le p$, by $C[v_k, v_m]$ we denote the set $\{v_k, v_{k+1}, \ldots, v_m\}$.

Let G be a graph on n vertices. Vertex $v \in V(G)$ is called heavy, if $d_G(v) \ge \frac{n}{2}$ and super-heavy, if $d_G(v) \ge \frac{n+1}{2}$. We say that two vertices u and v form a heavy-pair (super-heavy pair), if both u and v are heavy (super-heavy).

Lemma 10 (Benhocine and Wojda, [3])**.** *Let G be a graph on $n \ge 4$ vertices and let C be a cycle of length $n-1$ in G. If $d_G(v) \ge \frac{n}{2}$ for $v \in V(G) \setminus V(C)$, then G is pancyclic.*

Lemma 11 (Bondy, [4])**.** *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d_G(x) + d_G(y) \ge n + 1$, then G is pancyclic.*

Lemma 12 (Hakimi and Schmeichel, [10])**.** *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d_G(x) + d_G(y) \ge n$, then G is pancyclic unless G is bipartite or else G is missing only $(n-1)$-cycles.*

Lemma 13 (Ferrara, Jacobson and Harris, [7])**.** *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 2$ and $d_G(x) + d_G(y) \ge n + 1$, then G is pancyclic.*

3. Proof of Theorem 8

***Proof of Theorem* 8.** The Theorem 8 will be proved by contradiction. Suppose graph G on n vertices satisfies assumptions of the theorem but is not pancyclic. Since the result is easy to verify for $n \le 6$, assume $n \ge 7$. Note that G is Hamiltonian by Theorem 4. If G is $\{K_{1,3}, P_5\}$-free, it is pancyclic by Theorem 2, a contradiction. Hence, there exists in G an induced claw or path P_5 and a super-heavy vertex u. Consider $G' = G - u$. G' is $\{K_{1,3}, P_5\}$-f-heavy. If G' is 2-connected, it is Hamiltonian by Theorem 4 and hence G is pancyclic by Lemma 10, a contradiction. Now assume G' is not 2-connected. Then there exists a vertex $v \in V(G)$ such that $G - \{u, v\}$ is not connected. $G - \{u, v\}$ consists of two components. Let $H_1 = \{x_1, \ldots, x_{h_1}\}$ denote the set of verties of the first component and $H_2 = \{y_1, \ldots, y_{h_2}\}$ be the vertices of the second component. Assume, without loss of generality, that $h_1 \leq h_2$. Let $C = uy_1y_2 \cdots y_{h_2}vx_{h_1} \cdots x_1u$ be a Hamilton cycle in G with the given orientation. Ning in [8] proves the following general observations:

Claim 14 [8]. *There are no super-heavy vertices in* H_1.

Proof. This is true, since every vertex $x \in H_1$ can be adjacent only to u, v and other vertices from H_1. Since $h_1 \leq \frac{n-2}{2}$, we have $d_G(x) \leq h_1 + 1 \leq \frac{n}{2}$. □

Claim 15 [8]. $N_{H_2}(u) \subset N[y_1]$.

Proof. Suppose this is not true. Then there exists a vertex $y \in N_{H_2}(u) \setminus N(y_1)$. But now $\{u; x_1, y_1, y\}$ induces a claw. Since G is $K_{1,3}$-f_1-heavy and x_1 is not super-heavy, y_1 must be super-heavy. Hence, G is pancyclic by Lemma 11, a contradiction. □

Claim 16 [8]. *There are no super-heavy pairs of vertices with distance one or two along the orientation of a Hamilton cycle in* G.

Proof. Otherwise G is pancyclic by Lemma 11 or Lemma 13, a contradiction. □

Case 1. $h_1 = 1$.

Subcase 1.1. $uv \in E(G)$. If all vertices in G are neighbours of u, then G is pancyclic, a contradiction. Hence, there exists $y_i \in N_{H_2}(u)$ such that $uy_{i+1} \notin E(G)$. Let y_i be the first vertex in $C[y_1, y_{h_2-1}]$ with this property and let y_j be the first vertex in $C[y_i, y_{h_2}]$ such that $uy_{j+1} \in E(G)$, where we assume that $y_{h_2+1} = v$. Clearly, $j \geq i + 1$.

Claim 17 [8]. $i \geq 2$.

Proof. Suppose that the claim is not true, i.e. $uy_1 \in E(G)$ and $uy_2 \notin E(G)$. Since u is super-heavy and $u, y_2 \in N(y_1) \setminus N(u)$, by Claim 15 we get $d_G(y_1) \geq \frac{n+1}{2} - 3 + 2 \geq \frac{n-1}{2}$. This means that $\{u, y_1\}$ is a heavy-pair with a distance equal to one along the Hamilton cycle C. Since uC^+vu is an $(n-1)$-cycle in G, G is neither bipartite nor missing $(n-1)$-cycle. Therefore, by Lemma 12, G is pancyclic, a contradiction. □

Claim 18 [8]. $j \geq i + 2$.

Proof. Suppose $j = i + 1$. By the choice of i and j we have $uy_i, uy_{i+2} \in E(G)$ and $uy_{i+1} \notin E(G)$.

Suppose $y_i y_{i+2} \notin E(G)$. Then $\{u; x_1, y_i, y_{i+2}\}$ induces a claw. Since G is $K_{1,3}$-f_1-heavy and x_1 is not super-heavy by Claim 14, $\{y_i, y_{i+2}\}$ is a super-heavy pair with $d_C(y_i, y_{i+2}) = 2$. This contradicts Claim 16.

Now assume $y_i y_{i+2} \in E(G)$. Suppose $y_1 y_{i+1} \notin E(G)$. Set $G' = G - y_{i+1}$. $C' = uC^+ y_i y_{i+2} C^+ u$ is a Hamilton cycle in G' with $d_{C'}(u, y_1) = 1$. Furthermore, we have $d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) \geq \frac{n+1}{2} + \frac{n+1}{2} - 2 = |G'|$. Lemma 12 implies that G' is either pancyclic, bipartite or missing $(n-2)$-cycle. Since G' is Hamiltonian and $C'' = uC^+ y_i y_{i+2} C^+ vu$ is an $(n-2)$-cycle in G', G' is pancyclic. Together with the cycle C, G is pancyclic, a contradiction. Hence, $y_1 y_{i+1} \in E(G)$. Then, by Claim 15, it must be $d_G(y_1) \geq \frac{n+1}{2} - 1$. It follows that $\{u, y_1\}$ is a heavy-pair in G with the distance between them along the cycle C equal to one, and so G is either pancyclic or bipartite or else missing only $(n-1)$-cycle by Lemma 12. Since G is Hamiltonian and $C' = uC^+ vu$ is an $(n-1)$-cycle in G, G is pancyclic, a contradiction. □

Claim 19. *$vy_1 \notin E(G)$.*

Proof. Suppose $vy_1 \in E(G)$ and consider $G' = G - x_1$. Then $d_{G'}(u) \geq \frac{n+1}{2} - 1$ and $d_{G'}(y_1) \geq \frac{n+1}{2} - 2 + 1$, and so $\{u, y_1\}$ is a heavy-pair in G'. Since $C' = uC^+ vu$ is a Hamilton cycle in G' with $d_{C'}(u, y_1) = 1$ and $vy_1 C^+ v$ is an $(n-2)$-cycle in G', G' is pancyclic by Lemma 12. This implies that G is pancyclic, a contradiction.

□

Claim 20. *$y_1 y_k \notin E(G)$ for $k \in \{i+1, \ldots, j\}$.*

Proof. Suppose there exists $k \in \{i+1, \ldots, j\}$ such that $y_1 y_k \in E(G)$. Since $uy_k \notin E(G)$, by Claims 15 and 19 we have $d_G(y_1) \geq \frac{n+1}{2} - 3 + 2$. Therefore, $\{u, y_1\}$ is a heavy-pair such that $d_C(u, y_1) = 1$. Since G is neither bipartite nor missing $(n-1)$-cycle, it is pancyclic by Lemma 12, a contradiction. □

Claim 21. *$y_i y_{i+2} \notin E(G)$.*

Proof. Suppose $y_i y_{i+2} \in E(G)$. Set $G' = G - \{x_1, y_{i+1}\}$. Then $d_{G'}(u) \geq \frac{n+1}{2} - 1$ and $d_{G'}(y_1) = d_G(y_1) \geq \frac{n+1}{2} - 3 + 1$. Hence, $\{u, y_1\}$ is a heavy-pair in G'. Since $C' = vuC^+ y_i y_{i+2} C^+ v$ is a Hamilton cycle in G' and $d_{C'}(u, y_1) = 1$, Lemma 12 implies that G' is either pancyclic, bipartite or missing only $(|G'| - 1)$-cycle. But u is adjacent to y_2, by the choice of i and Claim 17, and so $uy_2 C'^+ vu$ is a $(|G'| - 1)$-cycle in G'. Hence, G' is pancyclic, implying that G contains $[3, n-2]$-cycles. Since G is Hamiltonian and contains an $(n-1)$-cycle, it is pancyclic, a contradiction. □

Claim 22. *y_i is super-heavy and $i > 2$.*

Proof. By Claims 18 and 21, $\{x_1, u, y_i, y_{i+1}, y_{i+2}\}$ induces a P_5. Since x_1 is not super-heavy and G is P_5-f_1-heavy, y_i must be super-heavy. u is a super-heavy vertex and so it must be $i > 2$, by Claim 16. □

By Claim 22 we have $y_{i-1} \neq y_1$.

Claim 23. *$y_{i-1}y_{i+1} \notin E(G)$, $y_{i-1}y_{i+2} \notin E(G)$.*

Proof. If $y_{i-1}y_{i+1} \in E(G)$, G is pancyclic by Claim 22 and Lemma 10, a contradiction. Now assume $y_{i-1}y_{i+2} \in E(G)$. Then $\{x_1, u, y_{i-1}, y_{i+2}, y_{i+1}\}$ induces a P_5. By Claims 22 and 16, y_{i-1} is not super-heavy. Since x_1 is also not super-heavy, this contradicts G being P_5-f_1-heavy. □

Now consider y_{i+3} (perhaps $y_{i+3} = y_{j+1}$; note that, by Claims 15, 19 and the choice of j, it must be $y_{j+1} \neq v$).

Claim 24. *$y_{i+1}y_{i+3} \notin E(G)$, $y_iy_{i+3} \notin E(G)$, $y_{i-1}y_{i+3} \in E(G)$.*

Proof. Suppose $y_{i+1}y_{i+3} \in E(G)$. Set $G' = G - \{x_1, y_{i+2}\}$. $C' = u\,C^+\,y_{i+1}\,y_{i+3}\,C^+\,v\,u$ is a Hamilton cycle in G' with $d_{C'}(u, y_1) = 1$. Since removing vertices x_1 and y_{i+2} from G does not change the degree of y_1 and lowers degree of u by one, we have $d_{G'}(u) \geq \frac{n+1}{2} - 1$ and $d_{G'}(y_1) \geq \frac{n+1}{2} - 3 + 1$, and so $\{u, y_1\}$ is a heavy-pair in G'. Furthermore, by Claim 17, $uy_2C'^{+}u$ is a $(|G'| - 1)$-cycle in G'. Hence, G' is pancyclic by Lemma 12. This implies that G is pancyclic, a contradiction.

Now suppose $y_iy_{i+3} \in E(G)$ and consider $G' = G - \{x_1, y_{i+1}, y_{i+2}\}$. G' is, again, a Hamiltonian graph, with a Hamilton cycle $C' = uC^+y_iy_{i+3}C^+vu$ and $d_{C'}(u, y_1) = 1$. We have $d_{G'}(u) = d_G(u) - 1 \geq \frac{n+1}{2} - 1$ and $d_{G'}(y_1) = d_G(y_1)$. Hence, by Claim 15, $\{u, y_1\}$ is a super-heavy pair in G' and G' is pancyclic by Lemma 11. Since there are $[n-2, n]$-cycles in G, G is pancyclic, a contradiction.

Finally, if $y_{i-1}y_{i+3} \notin E(G)$, then $\{y_{i-1}, y_i, y_{i+1}, y_{i+2}, y_{i+3}\}$ induces a P_5. Since y_i is super-heavy, neither y_{i-1} nor y_{i+1} can be super-heavy, by Claim 16. This contradicts G being P_5-f_1-heavy. □

Now consider $G' = G - \{x_1, y_i, y_{i+1}, y_{i+2}\}$. By Claim 24, G' is Hamiltonian, with a Hamilton cycle $C' = uC^+y_{i-1}y_{i+3}C^+vu$. We have $d_{G'}(u) = d_G(u) - 2$ and $d_{G'}(y_1) = d_G(y_1) - 1$. Hence, $d_{G'}(u) + d_{G'}(y_1) \geq \frac{n+1}{2} - 2 + \frac{n+1}{2} - 3 = |G'|$. Since, by Claim 17, $uy_2C'^{+}u$ is a $(|G'| - 1)$-cycle in G', G' is pancyclic by Lemma 12, and so G contains $[3, n-4]$-cycles. This implies that G is pancyclic, because it is Hamiltonian, and it contains a cycle uC^+vu of length $n-1$, a cycle uy_2C^+vu of length $n-2$ and a cycle $uC^+y_{i-1}y_{i+3}C^+u$ of length $n-3$. This contradiction completes the proof of this subcase.

Subcase 1.2 [8]. $uv \notin E(G)$. Suppose $uy_2 \notin E(G)$. Since u is super-heavy and $u, y_2 \in N(y_1) \setminus N(u)$, by Claim 15 we have $d_G(y_1) \geq \frac{n+1}{2}$. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$ and G is pancyclic by Lemma 11, a contradiction.

If $uy_2 \in E(G)$, then we have $d_G(y_1) \geq \frac{n-1}{2}$, implying that $\{u, y_1\}$ is a heavy-pair such that $d_C(u, y_1) = 1$. Since G is Hamiltonian and uy_2C^+u is an $(n-1)$-cycle in G, G is neither bipartite nor missing $(n-1)$-cycle, and so G is pancyclic by Lemma 12, a contradiction.

Case 2. $h_1 \geq 2$.

Subcase 2.1. $N_{H_1}(u) = \{x_1\}$. If $uy_2 \notin E(G)$, then $\{x_2, x_1, u, y_1, y_2\}$ induces a P_5. Since x_1 and y_1 are not super-heavy (by Claim 16), this contradicts G being P_5-f_1-heavy. Hence, it must be $uy_2 \in E(G)$.

Suppose that $uv \notin E(G)$. Then, by Claim 15, $d_G(y_1) \geq \frac{n+1}{2} - 1$ and so $d_G(y_1) + d_G(u) \geq n$. Since uy_2C^+u is an $(n-1)$-cycle in G, G is pancyclic by Lemma 12, a contradiction.

Now suppose $uv \in E(G)$ (implying $d_G(y_1) \geq \frac{n+1}{2} - 2$). Assume $h_1 = 2$. Then the graph $G' = G - \{x_1, x_2\}$ is Hamiltonian, with a Hamilton cycle $C' = uC^+vu$ and $d_{C'}(u, y_1) = 1$. Since $d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) - 1 \geq n - 2 = |G'|$, Lemma 12 implies that G' is either pancyclic or missing only $(|G'|-1)$-cycle. But uy_2C^+vu is a cycle of length $|G'| - 1$ in G' and hence G' is pancyclic, implying pancyclicity of G. Now, if $h_1 \geq 3$, it must be $x_1x_3 \in E(G)$ in order to avoid $\{x_3, x_2, x_1, u, y_1\}$ inducing P_5 (x_1 and x_3 are not super-heavy by Claim 14). Then $C' = uC^+x_3x_1u$ is a Hamilton cycle in $G' = G - x_2$ with $d_{C'}(u, y_1) = 1$ and $d_{G'}(u) + d_{G'}(y_1) \geq |G'|$. Since one can easily obtain a cycle of length $|G'| - 1$ in G' by omitting y_1 in the cycle C', G' is pancyclic by Lemma 12 and hence G is pancyclic, a contradiction.

Subcase 2.2. $N_{H_1}(u) \neq \{x_1\}$. Let x_{j_0} denote the last neighbour of u on $C^-[x_1, x_{h_1}]$.

Claim 25. *$N_{H_1}[u]$ induces a clique.*

Proof. Suppose the claim is not true. Then there exist two neighbours of u in H_1 that are not adjacent. Together with u and y_1 they induce a claw. By Claims 14 and 16, u is the only super-heavy vertex in this claw. This contradicts G being $K_{1,3}$-f_1-heavy. □

Claim 26. *Let $A = \{x_{a+1}, \ldots, x_{a+p}\}$ be a maximal set of consecutive non-neighbours of u in H_1. Then x_a is adjacent to every vertex from A.*

Proof. Since the statement is trivial for $p = 1$, assume $p \geq 2$. Since A is maximal, x_a must be adjacent to u. Assume that the claim is not true, i.e., there exists x_{a+i} for some $2 \leq i \leq p$ such that $x_ax_{a+i-1} \in E(G)$ and $x_ax_{a+i} \notin E(G)$. Then $\{y_1, u, x_a, x_{a+i-1}, x_{a+i}\}$ induces a P_5 with u being its only super-heavy vertex. This contradicts G being P_5-f_1-heavy. □

Corollary 27. *If* $x_{j_0} \neq x_{h_1}$, *then* x_{j_0} *is adjacent to vertices* $x_{j_0+1}, \ldots, x_{h_1}$.

Claim 28. $N_{H_2}(u) \neq H_2$.

Proof. Suppose $N_{H_2}(u) = H_2$. Then G contains $[3, h_2+1]$ and $[n-h_2+1, n]$-cycles. Since $h_2 \geq \frac{n-2}{2}$, G is pancyclic, a contradiction. □

By Claim 28, we can choose a vertex $y_k \in N_{H_2}(u)$ such that $uy_{k+1} \notin E(G)$, where $y_{k+1} \in H_2$. Let y_k be the first vertex on $C[y_1, y_{h_2-1}]$ with this property. Note that if $y_1 \neq y_k$, then $y_1y_k \in E(G)$, by Claim 15.

Claim 29. *There are* $[n-h_1-k+3, n]$-*cycles in* G.

Proof. By Corollary 27, there exists a cycle $C' = uy_kC^+x_{h_1}x_{j_0}u$ of length $n - h_1 - k + 3$ (if $x_{j_0} \neq x_{h_1}$) or $n - h_1 - k + 2$ (if $x_{j_0} = x_{h_1}$). Since u is adjacent to all of the vertices $y_1, \ldots, y_{k-1}$, C' can be extended to the cycle $uy_{k-1}y_kC^+x_{h_1}x_{j_0}u$. This way we can append all the vertices from $C[y_1, y_{k-1}]$ to C', one-by-one. Hence, G contains $[n-h_1-k+3, n-h_1+2]$-cycles. Now, by Corollary 27, we can append to the just obtained cycle $C'' = uC^+x_{h_1}x_{j_0}u$ (of length $n-h_1+2$) vertices from $C[x_{h_1-1}, x_{j_0+1}]$, which gives us $[n-h_1+2, n-j_0+1]$-cycles. Since neighbours of u in H_1 induce a clique, we can add them one-by-one to the longest of just obtained cycles. Finally, by Claim 26, appending the non-neighbours of u from H_1 to the longest cycle can be performed in a similar way as it has been done with the vertices $\{x_{j_0+1}, \ldots, x_{h_1-1}\}$. This gives us cycles of all lengths from $n - h_1 - k + 3$ up to n. □

Claim 30. $N_{H_1}(u) = H_1$.

Proof. Suppose this is not true, i.e. there exists a vertex $x_a \in N_{H_1}(u)$ such that $x_{a+1} \in H_1$ and $ux_{a+1} \notin E(G)$. Then $\{x_{a+1}, x_a, u, y_k, y_{k+1}\}$ induces a P_5. Since x_a is not super-heavy, y_k must be super-heavy. Since u is super-heavy, it must be $k \geq 3$, by Claim 16.

Suppose $uv \in E(G)$. Set $G' = G - H_1$. Then $C' = y_ky_1C^+y_{k-1}uvC^-y_k$ is a Hamilton cycle in G' with $d_{C'}(y_1, y_k) = 1$. Since $uv \in E(G)$, Claim 15 implies that $d_{G'}(y_1) = d_G(y_1) \geq \frac{n+1}{2} - d_{H_1}(u) - 1$. Since $d_{H_1}(u) \leq h_1 - 1$ and y_k is super-heavy, we get $d_{G'}(y_1) + d_{G'}(y_k) \geq |G'| + 1$. Hence, G' is pancyclic by Lemma 11 and there are $[3, n-h_1]$-cycles in G. Since $k \geq 3$, G is pancyclic by Claim 29, a contradiction.

Now suppose $uv \notin E(G)$. If $x_{j_0} = x_{h_1}$, set $G' = G - \{x_1, \ldots, x_{h_1-1}\}$. Then $|G'| = n-h_1+1$ and $C' = y_ky_1C^+y_{k-1}ux_{h_1}C^-y_k$ is a Hamilton cycle in G'. Since $d_{C'}(y_1, y_k) = 1$ and $d_{G'}(y_1) + d_{G'}(y_k) = d_G(y_1) + d_G(y_k) \geq \frac{n+1}{2} - d_{H_1}(u) + \frac{n+1}{2} \geq |G'| + 1$, G' is pancyclic by Lemma 11 and there are $[3, n-h_1+1]$-cycles in G. This implies, again, by Claim 29, that G is pancyclic, a contradiction.

Hence, $x_{j_0} \neq x_{h_1}$. Now we can consider $G' = G - (H_1 \setminus \{x_{j_0}, x_{h_1}\})$. By Corollary 27, $x_{j_0}x_{h_1} \in E(G)$ and so $C' = y_k y_1 C^+ y_{k-1} u x_{j_0} x_{h_1} C^- y_k$ is a Hamilton cycle in G'. Again, $d_{C'}(y_1, y_k) = 1$ and $d_{G'}(y_1) + d_{G'}(y_k) \geq n+2-h_1 = |G'|$. This implies, by Lemma 12, that G' is either pancyclic or missing only $(n-h_1+1)$-cycle. In either case G is pancyclic by Claim 29, a contradiction. □

Corollary 31. *By Claims* 25 *and* 30*, we have* $G[H_1 \cup \{u\}] \simeq K_{h_1+1}$*, which implies that there are both* $[3, h_1+1]$*- and* $[n-h_1+1, n]$*-cycles in* G*. Since* $n = h_1 + h_2 + 2$*, we can rewrite the latter interval in the form* $[h_2+3, n]$*, that will be handy in the following.*

Claim 32. *Let* $y_i \in H_2$ *be a super-heavy vertex for some* $i \in \{3, \ldots, h_2\}$*. Then* $d_G(u, y_i) = 2$*.*

Proof. Since both u and y_i are super-heavy, it obviously must be $d_G(u, y_i) \leq 2$. Suppose that the claim is not true, i.e. that there exists a super-heavy vertex $y_i \in H_2$ adjacent to u. Claim 16 gives us following observations: it must be $i \geq 3$ and neither y_{i-1} nor y_{i+1} is super-heavy (we assume that $y_{h_2+1} = v$). Furthermore, u must be adjacent to either y_{i-1} or y_{i+1}, since otherwise $\{y_i; u, y_{i-1}, y_{i+1}\}$ would induce a claw, by Lemma 10 contradicting G being $K_{1,3}$-f_1-heavy. Note that, by Claim 15, $y_1 y_i \in E(G)$.

Suppose $uy_{i+1} \in E(G)$. If $i \neq h_2$, then $y_1 y_{i+1} \in E(G)$, by Claim 15, and now $C' = u y_i C^- y_1 y_{i+1} C^+ u$ is a Hamilton cycle in G with $d_{C'}(u, y_i) = 1$. Since $\{u, y_i\}$ is a super-heavy pair, G is pancyclic by Lemma 11, a contradiction. We will obtain a contradiction in the same way if $i = h_2$ and $y_1 y_{i+1} \in E(G)$. Assume now that $i = h_2$ and $y_1 y_{i+1} \notin E(G)$. Consider $G' = G - (H_1 \cup \{v\})$. Then $|G'| = n - h_1 - 1$, $d_{G'}(u) = d_G(u) - h_1 - 1$ and $d_{G'}(y_{h_2}) = d_G(y_{h_2}) - 1$. Since both u and y_{h_2} are super-heavy, we have $d_{G'}(u) + d_{G'}(y_{h_2}) \geq |G'|$. Since $uC^+ y_{h_2} u$ is a Hamilton cycle in G' and $y_1 C^+ y_{h_2} y_1$ is an $(n-1)$-cycle in G', G' is pancyclic by Lemma 12. This implies that G is pancyclic, a contradiction.

It must be then $uy_{i+1}, y_1 y_{i+1} \notin E(G)$ and $uy_{i-1}, y_1 y_{i-1} \in E(G)$. Assume $uv \notin E(G)$. Then, by Claims 15 and 30, we have $d_G(y_1) \geq \frac{n+1}{2} - h_1$. Now, consider $G' = G - \{x_1, x_2, \ldots, x_{h_1-1}\}$ with a Hamilton cycle $C' = u y_{i-1} C^- y_1 y_i C^+ x_{h_1} u$. The distance between y_1 and y_i along C' is equal to one and the sum of their degrees in G' is $d_{G'}(y_1) + d_{G'}(y_i) = d_G(y_1) + d_G(y_i) \geq |G'|$. Hence, by Lemma 12, G' is either pancyclic or missing only $(n-h_1)$-cycle. Together with Corollary 31 this implies that G itself is either pancyclic or missing only $(n-h_1)$-cycle. Now, if $uy_2 \in E(G)$ then $uy_2 C^+ x_{h_1} u$ is a cycle of length $n - h_1$ in G and G is pancyclic. Hence, $uy_2 \notin E(G)$, implying that $i - 1 > 2$. If $i - 1 = 3$, then $y_1 y_3 \in E(G)$ and a cycle $u y_1 y_3 C^+ x_{h_1} u$ has lenght $n - h_1$, a contradiction. Hence, $i - 1 > 3$, but then it must be $y_1 y_3 \in E(G)$ (in order to avoid $\{x_1, u, y_1, y_2, y_3\}$ inducing P_5)

and, again, a cycle $uy_1y_3C^+x_{h_1}u$ has length $n-h_1$. This implies G is pancyclic, a contradiction.

Now suppose $uv \in E(G)$. By Claim 15, $d_G(y_1) \geq \frac{n+1}{2} - h_1 - 1$. Now, set $G' = G - H_1$. G' is Hamiltonian with a Hamilton cycle $C' = uy_{i-1}C^-y_1y_iC^+vu$. Since $d_{C'}(y_1, y_i) = 1$ and $d_{G'}(y_1) + d_{G'}(y_i) = d_G(y_1) + d_G(y_i) \geq n - h_1 = |G'|$, Lemma 12 implies that G' is either pancyclic or missing only a cycle of length $n-h_1-1$. The same is true about G itself, since it contains $[n-h_1+1, n]$-cycles. Note that uC^+vu is an $(n-h_1)$-cycle in G. From this cycle we can obtain an $(n-h_1-1)$-cycle by omitting y_1 or y_2, depending on the existence of the edges uy_2 and y_1y_3, just like in the previous case. Hence, G is pancyclic, a contradiction.

□

Corollary 33. $G[N_{H_2}[u]] \simeq K_{d_{H_2}(u)+1}$.

Proof. Otherwise there would exist two non-adjacent neighbours of u in H_2, say y_a and y_b. Then $\{u; x_1, y_a, y_b\}$ induces a claw. By Claims 14 and 32, u is the only super-heavy vertex of this claw. This contradicts G being $K_{1,3}$-f_1-heavy. □

Let y_m denote the last neighbour of u in $C[y_1, y_{h_2}]$.

Claim 34. *Let $A = \{y_{a+1}, \dots, y_{a+p}\}$ be a maximal set of consecutive non-neighbours of u in H_2. Then y_a is adjacent to every vertex from A.*

Proof. Since the statement is trivial for $p = 1$, assume $p \geq 2$. Since A is maximal, y_a must be adjacent to u. By Claim 32, y_a is not super-heavy. Assume that the claim is not true, i.e., there exists y_{a+i} for some $2 \leq i \leq p$ such that $y_ay_{a+i-1} \in E(G)$ and $y_ay_{a+i} \notin E(G)$. Then $\{x_1, u, y_a, y_{a+i-1}, y_{a+i}\}$ induces a P_5 with x_1 and y_a being not super-heavy. This contradicts G being P_5-f_1-heavy. □

Corollary 35. *If $y_m \neq y_{h_2}$, then y_m is adjacent to vertices $y_{m+1}, \dots, y_{h_2}$.*

Now, by Claim 30 and Corollary 35, $C' = uy_my_{h_2}vx_{h_1}u$ is a cycle of length four or five, depending on whether $m = h_2$ or not. By appending consecutively vertices $y_{h_2-1}, \dots, y_{m+1}$ to C', we obtain $[5, h_2 - m + 4]$-cycles, by Corollary 35. To the longest of just created cycles we can append neighbours of u from H_2, one-by-one, since they induce a clique. The same can be done, by Claim 34, with the vertices from H_2 that are not adjacent to u. This procedure gives us $[5, h_2 + 3]$-cycles. But Corollary 31 says, that there are $[h_2 + 3, n]$-cycles in G, so altogether we have $[5, n]$-cycles in G. Note that ux_1x_2u is a triangle. Now, since there is either an induced claw or an induced path P_5 in G, there exists a super-heavy vertex other than u, say w. By Claim 14, w either belongs to the set H_2 or $w = v$. Suppose $w \in H_2$. Then by Claim 32 $uw \notin E(G)$. Since both u and w are super-heavy, they must have at least two common neighbours, and together with them they form a cycle of length four. Hence, G is pancyclic, a contradiction. This implies that the only super-heavy vertices in G are u and v.

Again, if $uv \notin E(G)$, u and v have at least two common neighbours and we have a cycle C_4 in G. So it must be $uv \in E(G)$. Now if $vx_1 \notin E(G)$ and $vy_1 \notin E(G)$, $\{u; x_1, y_1, v\}$ induces a $K_{1,3}$. Since x_1 and y_1 are not super-heavy, this contradicts G being $K_{1,3}$-f_1-heavy. Hence v must be adjacent to either x_1 or y_1. In either case we get a cycle C_4: $x_1x_{h_1}vux_1$ in the previous, and $x_{h_1}vy_1ux_{h_1}$ in the latter. This implies that G is pancyclic, a contradiction. The proof is complete. ■

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