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FAULT TOLERANT DETECTORS FOR DISTINGUISHING SETS IN GRAPHS

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Abstract

For various domination-related parameters involving locating devices (distinguishing sets) that function as places from which detectors can determine information about the location of an "intruder", several types of possible detector faults are identified. Two of these fault tolerant detector types for distinguishing sets are considered here, namely redundant distinguishing and detection distinguishing. Illustrating these concepts, we focus primarily on open-locating-dominating sets.

Keywords: distinguishing sets, fault tolerant detectors, redundant distinguishing open-locating-dominating set, detection distinguishing open-locating-dominating set.

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1. Introduction

Assume we have a graph $G = (V, E)$ where one of the vertices in $V(G)$ is an "intruder" vertex. A graph G might represent a facility with the intruder being a (vertex) location for a fire, thief or saboteur, or G might be a multiprocessor network with one malfunctioning processor (also to be called the intruder). The concern is to use (the minimum possible number of) detectors to be able to precisely determine the location of the intruder. Various types of detectors can be used. Some, like sonar devices, can be assumed to be able to detect the intruder anywhere in the system. Much work has been done on the study of locating sets/metric bases introduced by Slater [39] and Harary and Melter [11]. Assume that a detector at vertex x can determine the distance $d(x, w)$ to an intruder at w. With an (ordered) set $X = \{x_1, x_2, \ldots, x_k\}$ of vertices, for each $v \in V(G)$ we have a k-tuple $(d(x_1, v), d(x_2, v), \ldots, d(x_k, v))$. Then, X is a *locating-set* or *metric basis* if all of these k-tuples are distinct. We can say that a vertex x resolves two vertices u and v if $d(x, u) \neq d(x, v)$. Then X is locating if for every two vertices u and v in $V(G)$ at least one $x \in X$ resolves u and v. The location number $LOC(G)$ (also called the metric basis number, denoted $MB(G)$) is the minimum cardinality of a locating set $X \subseteq V(G)$. Because $d(x_i, x_i) = 0$, clearly each x_i in X resolves itself with any other vertex. For the recently introduced centroidal bases described in Foucaud, Klasing and Slater [9] the set of detectors in $X = \{x_1, x_2, \ldots, x_k\}$ provide just an ordering of the relative distances to an intruder vertex, not the precise distances.

Others, like heat sensors or motion detectors, might have a limited range. We consider this latter case for which a detector at vertex v can determine the presence of an intruder at w precisely when $vw \in E(G)$ (or possibly when $v = w$). To be able to detect any vertex intruder one needs a dominating or open dominating set. The *open neighborhood* of vertex v is $N(v) = \{x \in V(G) : vx \in E(G)\}$, the set of vertices adjacent to v, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Then $D \subset V(G)$ is a *dominating set* if $\bigcup_{x \in D} N[x] = V(G)$ and an *open dominating set* (also called a *total dominating set*) if $\bigcup_{x \in D} N(x) = V(G)$. The *domination number* is $\gamma(G) = \min\{|D| : D \text{ is a dominating set of } G\}$, and the *open domination number* is denoted by $\gamma_t(G)$ or $\gamma^{op}(G) = \min\{|D| : D \text{ is an open dominating set of } G\}$. For the path $P_{12} = \{v_1, v_2, \ldots, v_{12}\}$, $D = \{v_2, v_5, v_8, v_{11}\}$ is a $\gamma(P_{12})$-set. For this case, note that a heat sensing detector at v_5 can not determine if a fire (intruder) is at location v_4 because it might also be at v_6, although it might be able to distinguish between locations v_4 and v_5.

For the case in which a detector at v can determine if the intruder is at v or if the intruder is in $N(v)$ (but which element in $N(v)$ can not be determined), as introduced in Slater [40, 41, 42], a *locating-dominating set* $L \subseteq V(G)$ is a dominating set for which, given any two vertices u and v in $V(G) - L$, one has

$N(u) \cap L \neq N(v) \cap L$. Every graph G has a locating-dominating set, namely $V(G)$, and the *locating-dominating number* $LD(G)$ is the minimum cardinality of such a set. See, for example, [3, 8, 14].

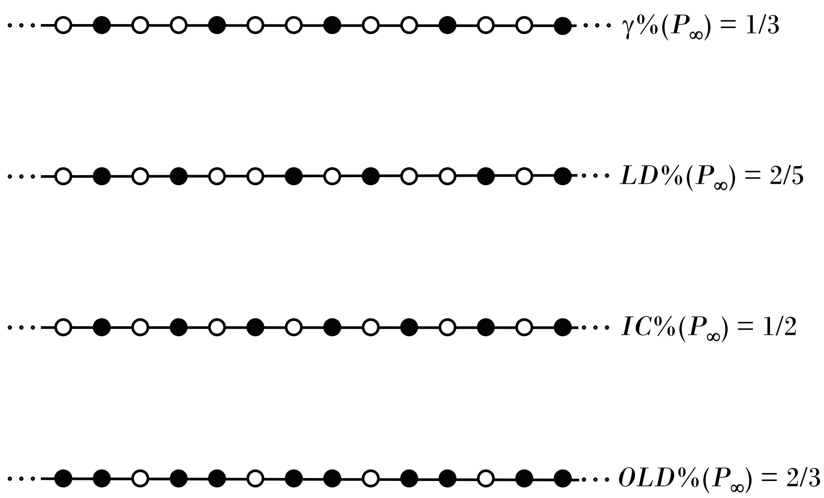
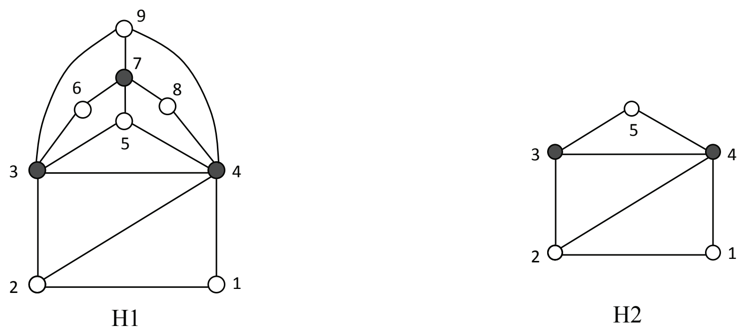


Figure 1. Distinguishing sets for the infinite path.

As introduced by Karpovsky, Charkrabarty and Levitin [21], an *identifying code* $C \subseteq V(G)$ is a dominating set for which given any two vertices u and v in $V(G)$ one has $N[u] \cap C \neq N[v] \cap C$. See, for example, [2, 4, 21, 24]. A graph G has an identifying code when for every pair of vertices, u and v we have $N[u] \neq N[v]$, and the *identifying code number* $IC(G)$ is the minimum cardinality of such a set. When a detection device at vertex v can determine if an intruder is in $N(v)$ but will not/can not report if the intruder is at v itself, then we are interested in open-locating-dominating sets as introduced for the k-cubes Q_k by Honkala, Laihonen and Ranto [20] and for all graphs by Seo and Slater [30, 31]. An open dominating set $S \subseteq V(G)$ is an *open-locating-dominating set* if for all u and v in $V(G)$ one has $N(u) \cap S \neq N(v) \cap S$. A graph G has an open-locating-dominating set when no two vertices have the same open neighborhood, and $OLD(G)$ is the minimum cardinality of such a set. See, for example, [5, 13, 20, 30, 31, 32, 33, 34, 35, 36]. Lobstein [23] maintains a bibliography, currently with more than 280 entries, for work on these topics.

In general, a collection $\mathscr{C} = \{S_1, S_2, \ldots, S_p\}$ of subsets of $V(G)$ is a *distinguishing set* for a graph G if $\bigcup_{1 \leq i \leq p} S_i = V(G)$ and for every pair of distinct vertices u and v in $V(G)$ some S_i contains exactly one of them. So, $L = \{w_1, w_2, \ldots, w_j\}$ is a locating-dominating set if $\mathscr{C}_1 = \{\{w_1\}, N(w_1), \{w_2\},$



(a) (b)

Figure 2. Graphs H1 and H2.

$N(w_2), \ldots, \{w_j\}, N(w_j)\}$ is distinguishing with $p = 2j; S = \{w_1, w_2, \ldots, w_p\}$ is an identifying code if $\mathscr{C}_2 = \{N[w_1], N[w_2], \ldots, N[w_p]\}$ is distinguishing; and $S = \{w_1, w_2, \ldots, w_p\}$ is an open-locating-dominating set if $\mathscr{C}_3 = \{N(w_1), N(w_2), \ldots, N(w_p)\}$ is distinguishing.

These definitions extend naturally to locally finite, countably infinite graphs. Percentage parameters for measuring density for locally-finite, countably infinite graphs were defined in Slater [44]. For example, for the $\gamma(G)$ parameter we have $\gamma\%(G)$ defined as follows as the minimum possible percentage of vertices in a dominating set of G. The *closed k-neighborhood* of vertex v is the set of vertices at distance at most k from v, $N^k[v] = \{w \in V(G) : d(v, w) \leq k\}$. For $S \subseteq V(G)$, the density of S is $dens(S) = \max_{v \in V(G)} \limsup_{k \to \infty}(|S \cap N^k[v]|/|N^k[v]|)$. Then, for example, the domination percentage of G is $\gamma\%(G) = \min\{dens(S) : S \subseteq V(G)$ is *dominating*$\}$. Figure 1 illustrates how to achieve the smallest possible percentage of vertices for these distinguishing sets for the infinite path.

Theorem 1. *For the infinite path P_∞ the smallest possible percentage of vertices is*

(a) $\gamma\%(P_\infty) = 1/3$,

(b) (*Slater* [41, 42]) $LD\%(P_\infty) = 2/5$,

(c) (*Karpovsky, Charkrabarty and Levitin* [21]) $IC\%(P_\infty) = 1/2$, *and*

(d) (*Seo and Slater* [30]) $OLD\%(P_\infty) = 2/3$.

Note that for LD-sets, $\mathscr{C}_1$ can be a multiset if, for example, $N(w_1) = N(w_2)$. For a dominating set D the associated collection $\mathscr{C}_4$ has $I(D)$ singleton set entries where $I(D) = \Sigma_{v \in D}(1 + deg\ v)$ is the influence of D as defined in Grinstead and Slater [10]. Each $\{v\}$ appears in $\mathscr{C}_4$, in fact, $|N[v] \cap D|$ times. The redundance is important when we consider fault-tolerance.

Many results about the size of various dominating sets are based on "share" arguments. For a dominating set D and a vertex $v \in D$, the "share" of v in D is a measure of the amount of domination done by v. If $N[u] \cap D = \{v\}$, then v is a *sole dominator* of u and u is said to be a *private neighbor* (PN) of v. In graph $H1$ of Figure 2, vertex 1 is a private neighbor of vertex 4, and vertex 7 is its own PN. Because $N[2] \cap \{3,4,7\} = \{3,4\}$, each of vertex 3 and vertex 4 is considered to have a 1/2-share in dominating vertex 2. If D is a dominating set and $v \in D$, then the *share* of v in D is defined in [44] as $sh(v; D) = \Sigma_{w \in N[v]} 1/|N[w] \cap D|$. For example, in graph $H1$ of Figure 2 we have $N[3] = \{2,3,4,5,6,9\}$ and $sh(3; \{3,4,7\}) = 1/2 + 1/2 + 1/2 + 1/3 + 1/2 + 1/3 = 8/3$. Also, $sh(4; \{3,4,7\}) = 1 + 1/2 + 1/2 + 1/2 + 1/3 + 1/2 + 1/3 = 11/3$, and $sh(7; \{3,4,7\}) = 1/3 + 1/2 + 1 + 1/2 + 1/3 = 8/3$. Note that $\Sigma_{v \in D} sh(v; D) = |V(G)| = n$ for any dominating set D and that $|D| \geq |V(G)|/MAX_{v \in V} sh(v; D)$

Similarly, for an open dominating set D the *open-share* of a vertex v in D is defined as $sh^{op}(v; D) = \Sigma_{w \in N(v)} 1/|N(w) \cap D|$ in Seo and Slater [30]. For example, in graph $H2$ of Figure 2 we have $N(3) = \{2,4,5\}$ and $sh^{op}(3; \{3,4\}) = 1/2 + 1 + 1/2 = 2$. Also, $N(4) = \{1,2,3,5\}$ and $sh^{op}(4; \{3,4\}) = 1 + 1/2 + 1 + 1/2 = 3$. For finite graphs G with an open dominating set D we have $\Sigma_{v \in D} sh^{op}(v; D) = |V(G)|$ and $|D| \geq |V(G)|/MAX_{v \in V} sh^{op}(v; D)$. Vertex set $D \subseteq V(G)$ is *open k-dominating* if every vertex in $V(G)$ is open dominated at least k times and the *open k-share* is defined as $sh^{op}_{\times k}(v; D) = \Sigma_{w \in N(v)} k/|N(w) \cap D|$. If D is open k-dominating, then $\Sigma_{v \in D} sh^{op}(v; D) = k \cdot |V(G)|$ and for any open k-dominating set D we have $|D| \geq k|V(G)|/MAX_{v \in V} sh^{op}_{\times k}(v; D)$.

In this paper, we will focus on open-locating-dominating sets along with open-shares of vertices.

2. Introduction: Fault Tolerant Detection

There are two aspects to using a detector in deciding where an intruder is located. The detector must be able to determine the presence of the intruder and to transmit this information to a command point P where the information can be used. For now we assume that, at any given time, there is at most one faulty detector (and at most one intruder). Several types of faults are described in Slater [46].

Perhaps the simplest fault is that the detector loses the ability to transmit to command point P. This type of fault is discussed in Hernando, Mora, Slater, and Wood for locating sets [15] and in Honkala, Laihonen, and Ranto for identifying codes [20]. Assume we have an OLD-set S for G and that each $v \in S$ is the location for a detection device. At designated points in time, each OLD detector, say at vertex v, transmits a 1 if there is an intruder in $N(v)$ and transmits a 0

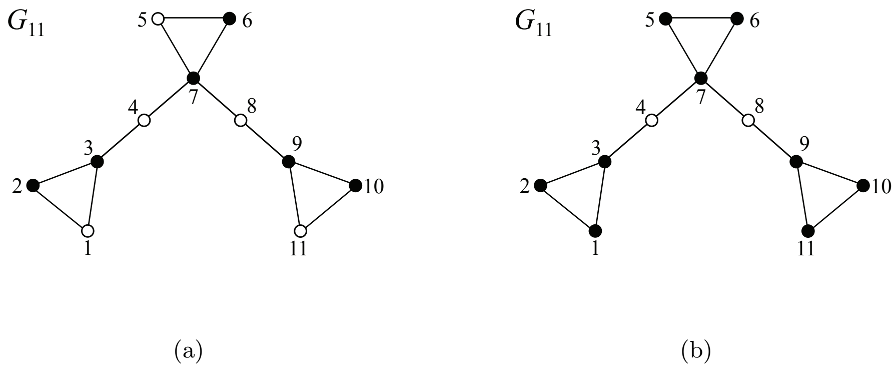


Figure 3. (a) An OLD(G_{11})-set and (b) a RED:OLD(G_{11})-set.

if $N(v)$ does not contain an intruder. In this case, a fault for detector v is clear when no transmission is received. What is required here for $S \subseteq V(G)$ to be a fault tolerant OLD-set is that, for each $v \in S$, the set $S - \{v\}$ must be an OLD-set. We say that an OLD-set S is a *redundant OLD-set* (a RED:OLD-set) if, for each $v \in S$, the set $S - \{v\}$ is an OLD-set, and we let *RED:OLD*(G) be the minimum cardinality of a RED:OLD-set for G. A redundant OLD-set S of G with RED:OLD$(G) = |S|$ is called a RED:OLD(G)-set.

Observation 2. *For the graph G_{11} in Figure* 3*, RED:OLD*$(G_{11}) = 9$.

Proof. The set $W = \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$ is a RED:OLD-set for G_{11}. (See Figure 3(b).) For example, if $R = W - \{6\}$, then $N(5) \cap R = \{7\}$, $N(7) \cap R = \{5\}$, $N(6) \cap R = \{5, 7\}$, $N(4) \cap R = \{3, 7\}$, etc. Hence, *RED:OLD*$(G_{11}) \leq 9$. Suppose a vertex set $S \subseteq V(G_{11})$ does not contain all three vertices in one of the triangles, for example, $\{1, 2, 3\} \nsubseteq S$. Let R be obtained from S as follows. If $3 \in S$, let $R = S - \{3\}$. Because $|S \cap \{1, 2, 3\}| \leq 2$ we can assume $2 \notin S$, but then vertex 1 is not open dominated by R and hence R is not an OLD-set. Assume $3 \notin S$. For S to be an OLD-set, $N(2) \cap S = \{1\} \subseteq S$ and $N(1) \cap S = \{2\} \subseteq S$. Let $R = S - \{2\}$. Again, vertex 1 is not open dominated by R. It follows that $\{1, 2, 3, 5, 6, 7, 9, 10, 11\}$ is the unique RED:OLD(G_{11})-set. ■

A second type of fault discussed in Slater [46] for LD-sets involves a device whose detection capability fails but which still transmits. That is, at command point P one must allow for the fact that one of the detectors transmitting a 0 is doing so incorrectly. Suppose, for example, that $S = V(G_{11})$ and vertex 3 transmits a 1 (correctly indicating that $N(3) = \{1, 2, 4\}$ contains an intruder) while every other vertex transmits a 0. The intruder can be at vertex 1 with vertex 2 incorrectly transmitting 0, at vertex 2 with vertex 1 incorrectly transmitting 0, or at vertex 4 with vertex 7 incorrectly transmitting 0. That is, G_{11} can not handle this type of fault. We call OLD-set S a detector OLD-set (a DET:OLD-set) if the location of an intruder can be correctly identified when at most one detector incorrectly transmits a 0. Let $DET{:}OLD(G)$ be the minimum cardinality of a DET:OLD-set for G, and call the DET:OLD-set for G a DET:OLD(G)-set if $|S|= DET{:}OLD(G)$.

Using notation different from that in Slater [46] and here, faults of this second type were considered in Slater [44] for locating-dominating sets. (With the obvious similar notation for LD-sets we have the following. For cycle C_8, $RED{:}LD(C_8) = 6 < DET{:}LD(C_8) = 7$.)

Theorem 3. *For cycle C_n with $n \neq 4$, $RED{:}OLD(C_n) = n$ and $DET{:}OLD(C_n)$ is undefined.*

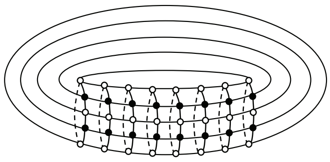
Proof. Clearly $V(C_n)$ is a RED:OLD-set for C_n, so $RED{:}OLD(C_n)$ is defined. Suppose we have cycle $C_n : v_1, v_2, \ldots, v_n$, and we let $S = V(C_n) - \{v_i\}$. Then $S - \{v_{i+2}\}$ does not open dominate v_{i+1}, so S is not a RED:OLD-set. Hence, $RED{:}OLD(C_n) = n$.

If $S = V(C_n)$ and v_i transmits 1 while every other vertex transmits 0, then the intruder can be at v_{i-1} (with v_{i-2} detector faulty) or at v_{i+1} (with v_{i+2} detector faulty). So, $V(C_n)$ is not a DET:OLD-set. ∎

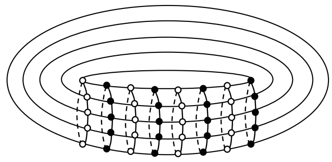
We let $G\square H$ denote the Cartesian product of G and H. Figure 4 illustrates that $OLD(C_5\square C_8) \leq 16$, $RED{:}OLD(C_5\square C_8) \leq 20$, and $DET{:}OLD(C_5\square C_8) \leq 30$. Note that each $v \in V(C_5\square C_8))$ has $deg\ v = 4$. If $v \in S$ where S is an OLD-set for a graph G and $degv = k$, then v can have at most one PN and each other $x \in N(v)$ is dominated at least twice. Hence, $sh^{op}(v; S) \leq 1 + (deg\ v - 1)/2 = (1 + deg\ v)/2$. In particular, for $(C_5\square C_8)$ each $sh^{op}(v; S) \leq 5/2$ and $|S| \geq 40(2/5) = 16$. So, $OLD(C_5\square C_8) = 16$. That $RED{:}OLD(C_5\square C_8) = 20$ and $DET{:}OLD(C_5\square C_8) = 30$ can be shown using arguments similar to those in Theorems 12 and 13.

The collection $\mathscr{C} = \{S_1, S_2, \ldots, S_t\}$ with $S_i \subseteq V(G)$ for $1 \leq i \leq t$ is *distinguishing* for a graph G if $\bigcup_{1\leq i\leq t} S_i = V(G)$ and for each pair u, v of distinct vertices there is some S_i containing exactly one of them. For $v \in V(G)$ let $\mathscr{C}(v) = \{i : v \in S_i\}$ and $A\triangle B$ is the symmetric difference of sets A and B. Then $\mathscr{C}$ is distinguishing if $\mathscr{C}(v) \neq \emptyset$ for all $v \in V(G)$ and $|\mathscr{C}(u)\triangle\mathscr{C}(v)| \geq 1$ for all pairs u, v.

We say that S_i distinguishes u from v if $u \in S_i$ and $v \notin S_i$, and S_i is said to distinguish u and v if either S_i distinguishes u from v or v from u. Collection $\mathscr{C}$ will be said to 2-*distinguish u and v* if $\mathscr{C}$ contains distinct elements S_i and S_j each of which distinguishes u and v. In particular, perhaps $u \in S_i$, $u \notin S_j$ and $v \notin S_i$, $v \in S_j$. Collection $\mathscr{C}$ will be said to $2^{\#}$-*distinguish u and v* if $\mathscr{C}$ contains



(a)



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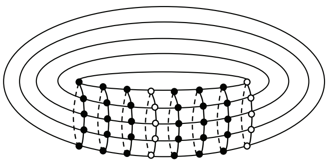


(c)

Figure 4. (a) OLD($C_5\square C_8$)-set (b) RED:OLD($C_5\square C_8$)-set, and (c) DET:OLD($C_5\square C_8$)-set.

distinct elements S_i and S_j such that either S_i and S_j both distinguish u from v or both distinguish v from u.

The next two theorems were presented in Slater [46] without proof. Collection $\mathscr{C}$ is called *redundant-distinguishing* if $\mathscr{C} - S_i$ is distinguishing for $1 \leq i \leq t$.

Theorem 4 (Slater [46]). $\mathscr{C} = \{S_1, S_2, \ldots, S_t\}$ *is redundant-distinguishing if and only if each* $|\mathscr{C}(u)| \geq 2$ *and* $\mathscr{C}$ 2*-distinguishes each pair* u, v *of distinct vertices* (*that is,* $|\mathscr{C}(u) \triangle \mathscr{C}(v)| \geq 2$).

Proof. Assume $\mathscr{C}$ is redundant-distinguishing. If $|\mathscr{C}(u)| < 2$ and $u \in S_j$, then $\bigcup_{1 \leq i \leq t, i \neq j} S_i$ does not contain u (that is, u will not even be detected). Hence, $|\mathscr{C}(u)| \geq 2$ for each $u \in V(G)$. If only S_j distinguishes u and v, then $\mathscr{C} - S_j$ is not distinguishing. (That is, if S_j does not transmit, then each other S_i transmits a 1 if both u and v are in S_i or 0 if both u and v are not in S_i. So $\mathscr{C} - S_j$ will not distinguish there being an intruder at u or at v.) Hence $|\mathscr{C}(u) \triangle \mathscr{C}(v)| \geq 2$.

For the converse, if there is no intruder then every S_i that transmits will transmit a 0. Assume there is an intruder at vertex u. Consider $\mathscr{C} - S_i$ (that is, perhaps S_i does not transmit). Because $|\mathscr{C}(u)| \geq 2$ at least one S_j with $j \neq i$ contains vertex u (so this S_j transmits a 1). Let $v \in V(G) - \{u\}$. If $v \notin S_j$

then S_j transmitting 1 determines that v is not the intruder vertex. Assume $\{u,v\} \subseteq S_j$. There are two elements of $\mathscr{C}$, say S_h and S_k, that distinguish u and v. We can assume $k \neq i$. Then if $u \in S_k$ and $v \notin S_k$, element S_k transmits a 1, and we know that v is not the intruder vertex. If $u \notin S_k$ and $v \in S_k$, element S_k transmits a 0 and we know that v is not the intruder vertex. ∎

Collection $\mathscr{C}$ is *detection-distinguishing* if $\mathscr{C}$ can distinguish under the condition that one S_i can falsely report "no intruder" in S_i. Note that any S_i indicating that the intruder vertex u is in S_i (that is, transmitting a 1) is assumed to be correctly reporting. At any given time, at most one S_i can falsely indicate non-membership.

Theorem 5 (Slater [46]). *$\mathscr{C}$ is detection-distinguishing if and only if each $|\mathscr{C}(u)| \geq 2$ and for each pair u,v we have either $|\mathscr{C}(u) \setminus \mathscr{C}(v)| \geq 2$ or $|\mathscr{C}(v) \setminus \mathscr{C}(u)| \geq 2$ (that is, $\mathscr{C}\,2^{\#}$-distinguishes u and v).*

Proof. Assume $\mathscr{C}$ is detection-distinguishing. Suppose that $|\mathscr{C}(u)| \leq 1$. If any element in $\mathscr{C}$ contains u, let it be S_i. If $u \in S_i$, then S_i might incorrectly report a 0. That is, we can have every S_k transmitting a 0 when u is the intruder location. Hence $|\mathscr{C}(u)| \geq 2$ for all $u \in V(G)$. Suppose that $\mathscr{C}$ does not $2^{\#}$-distinguish u and v. If there is a (unique) element of $\mathscr{C}$ that distinguishes u from v, call it S_i, and if there is a (unique) element of $\mathscr{C}$ that distinguishes v from u, let it be S_j. We have $u \in S_i$, $v \notin S_i$, $u \notin S_j$, $v \in S_j$, and if $h \neq i,j$ then $|S_h \cap \{u,v\}| = 0$ or 2. If every S_h with $h \neq i,j$ has $|S_h \cap \{u,v\}| = 0$, then we can have an undetected intruder at u if S_i transmits incorrectly (or at v if S_j transmits incorrectly). If $\{u,v\} \subseteq S_h$ for some $S_h \in \mathscr{C}$, suppose each S_k with $\{u,v\} \subseteq S_k$ transmits a 1 while all others (including S_i and S_j) transmit a 0. The intruder can be at u or v. Hence, $\mathscr{C}$ must $2^{\#}$-distinguish each pair u and v.

For the converse, because each $|\mathscr{C}(u)| \geq 2$, every S_i transmits 0 if and only if there is no intruder. Assume there is an intruder at vertex u, then $|\mathscr{C}(u)| \geq 2$ implies some S_j containing u transmits a 1. Suppose that $v \notin S_j$, then because S_j transmits a 1 we know that v is not the intruder vertex. Suppose that $\{u,v\} \subseteq S_j$. If both S_h and S_k distinguish u from v, they both transmit a 1, and we know that v is not the intruder vertex. If both S_h and S_k distinguish v from u, they both transmit a 0. At least one is not faulty, and we know that v is not the intruder vertex. ∎

We emphasize that we are considering distinguishing sets S that are robust with respect to possible detector faults (elements of S). Honkala and Laihonen [19], for example, consider robustness under conditions where the graph G is subject to modifications, specifically allowing edge additions or deletions. For the case with one unknown edge addition, their optimum identifying code for the infinite hexagonal mesh is actually $2^{\#}$-distinguishing. Perhaps the *k-distinguishing* and *$k^{\#}$-distinguishing* concepts will be useful in other settings.

In this paper we consider the case where $\mathscr{C} = \{N(v_1), N(v_2), \ldots, N(v_n)\}$ is the collection of open neighborhoods for a graph G.

3. RED:OLD% and DET:OLD% for Infinite Grids

Much work has been done on distinguishing parameters for infinite grid graphs. See, for example, [33, 34]. We denote the infinite 3-regular hexagonal graph as HEX, embedded as in Figure 5 in order to facilitate coordinatizing the vertex set. We denote the 4-regular infinite square grid as shown in Figure 6, obtained as the product $Z\Box Z$, as SQ, and the infinite 6-regular triangular graph as TRI. Graph TRI is embedded as in Figure 7 so that for a vertex $v = (i, j)$ the open neighborhood is $N((i, j)) = \{(i, j+1), (i+1, j), (i+1, j-1), (i, j-1), (i-1, j), (i-1, j+1)\}$.

If $L \subseteq V(HEX)$ is an LD-set, then $sh(v, L) \leq 1 + 1 + 1/2 + 1/2 = 3$ and $LD\%(HEX) \geq 1/3$; if $C \subseteq V(HEX)$ is an IC-set, then $sh(v, C) \leq 1 + 1/2 + 1/2 + 1/2 = 5/2$ and $IC\%(HEX) \geq 2/5$; and if $S \subseteq V(HEX)$ is an OLD-set, then $sh^{op}(v, S) \leq 1 + 1 + 1/2 = 2$ and $OLD\%(HEX) \geq 1/2$.

Theorem 6. *For the infinite hexagonal graph HEX,*
(a) (*Honkala and Laihonen* [18]) $LD\%(HEX) = 1/3$.
(b) (*Cohen et al.* [6] *and Cukierman and Yu* [7]) $5/12 \leq IC\%(HEX) \leq 3/7$, *and*
(c) (*Seo and Slater* [30]) $OLD\%(HEX) = 1/2$.

If $L \subseteq V(SQ)$ is an LD-set, then $sh(v, L) \leq 1 + 1 + 1/2 + 1/2 + 1/2 = 7/2$ and $LD\%(SQ) \geq 2/7$; if $C \subseteq V(SQ)$ is an IC-set, then $sh(v, C) \leq 1 + 4(1/2) = 3$ and $IC\%(SQ) \geq 1/3$; and if $S \subseteq V(SQ)$ is an OLD-set, then $sh^{op}(v, S) \leq 1 + 3(1/2) = 5/2$ and $OLD\%(SQ) \geq 2/5$.

Theorem 7. *For the infinite square graph SQ,*
(a) (*Slater* [43, 44]) $LD\%(SQ) = 3/10$,
(b) (*Ben-Haim and Litsyn* [1]) $IC\%(SQ) = 7/20$, *and*
(c) (*Seo and Slater* [30]) $OLD\%(SQ) = 2/5$.

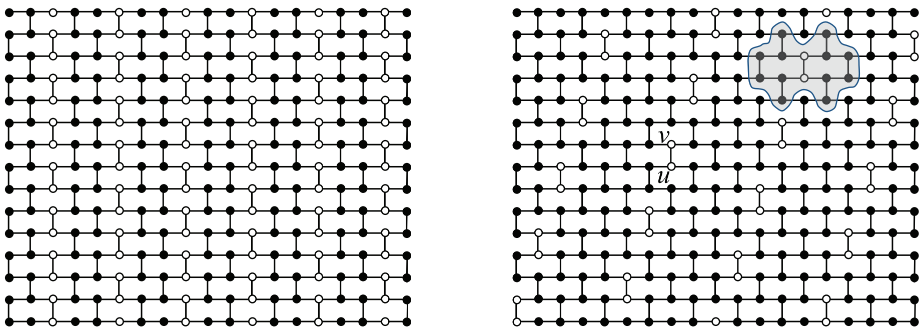
If $L \subseteq V(TRI)$ is an LD-set, then $sh(v, L) \leq 1 + 1 + 5(1/2) = 9/2$ and $LD\%(HEX) \geq 2/9$; if $C \subseteq V(TRI)$ is an IC-set, then $sh(v, C) \leq 1 + 6(1/2) = 4$ and $IC\%(TRI) \geq 1/4$; and if $S \subseteq V(TRI)$ is an OLD-set, then $sh^{op}(v, S) \leq 1 + 5(1/2) = 7/2$ and $OLD\%(TRI) \geq 2/7$. In Seo and Slater [30] we showed that $2/7 \leq OLD\%(TRI) \leq 1/3$. Honkala [17] improved this to show that

$OLD\%(TRI) \leq 6/19$. The precise value is $4/13$, as shown by Kincaid, Oldham, and Yu [22].

Theorem 8. *For the infinite triangular graph TRI,*
(a) (*Honkala* [16]) $LD\%(TRI) = 13/57$,
(b) (*Karpovsky, Charkrabarty and Levitin* [21]) $IC\%(TRI) = 1/4$, *and*
(c) (*Kincaid, Oldham, and Yu* [22]) $OLD\%(TRI) = 4/13$.

As noted, we will consider RED:OLD-sets and DET:OLD-sets for HEX, SQ, and TRI. By Theorems 4 and 5 each RED:OLD-set and DET:OLD-set of a graph G must double open dominate G.

Observation 9. *If a graph G is r-regular of order n, then $DET{:}OLD(G) \geq RED{:}OLD(G) \geq \gamma^{op}_{\times 2}(G) \geq 2n/r$. If a countably infinite graph G is regular of degree r, then $DET{:}OLD\%(G) \geq RED{:}OLD\%(G) \geq \gamma^{op}_{\times 2}\%(G) \geq 2/r$.*



(a) (b)

Figure 5. $RED{:}OLD\%(HEX) \leq 2/3$ and $DET{:}OLD\%(HEX) \leq 6/7$.

Theorem 10. $RED{:}OLD\%(HEX) = \gamma^{op}_{\times 2}\%(HEX) = 2/3$.

Proof. By Observation 9, $RED{:}OLD\%(HEX) \geq \gamma^{op}_{\times 2}\%(HEX) \geq 2/3$. As indicated in Figure 5(a) by the darkened vertices, let $S = \{(i,j) \in V(HEX) : i \equiv 1 \text{ or } 2 \pmod 3)\}$. Each vertex v has $|N(v) \cap S| = 2$. Let $v \in V(HEX)$, and let $N(v) \cap S = \{w, x\}$. Note that no other vertex u is adjacent to both w and x. That is, for any vertices v and u there is a vertex (w or x) that distinguishes v from u. Likewise, there is a vertex that distinguishes u from v. Because S 2-distinguishes any u, v pair, by Theorem 4 we have $RED{:}OLD\%(HEX) \leq 2/3$. ■

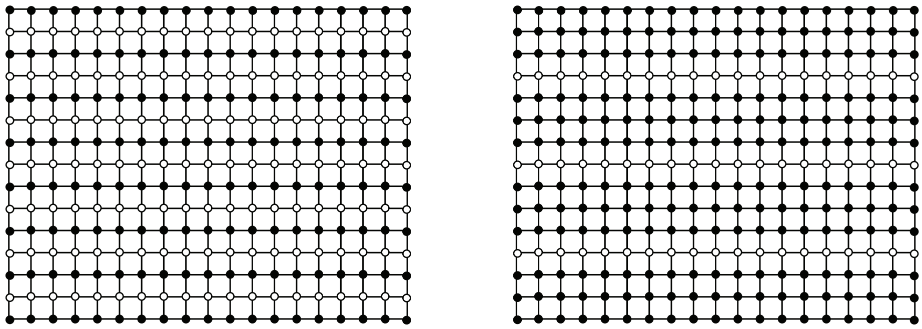
Theorem 11. $DET{:}OLD\%(HEX) = 6/7$.

Proof. We can assume that HEX is labeled as in Figure 5(b) so that S is the set of darkened vertices and adjacent vertices $u = (0,0)$ and $v = (0,1)$ are not in S. First, consider a vertex u not in S. Without loss of generality, say $u = (0,0)$ and $N(u) \cap S = \{(-1,0),(1,0)\}$. If w is any vertex not adjacent to $(-1,0)$ and not adjacent to $(1,0)$, then both $(-1,0)$ and $(1,0)$ distinguish $u = (0,0)$ from w, so u and w are $2^{\#}$-distinguished. Also, $(-3,0)$ and $(-2,1)$ distinguish $(-2,0)$ from u, $(-2,-1)$ and $(0,-1)$ distinguish $(-1,-1)$ from u, $(2,1)$ and $(3,0)$ distinguish $(2,0)$ from u, and $(0,-1)$ and $(2,-1)$ distinguish $(1,-1)$ from u. By symmetry, if u is any vertex not in S and w is any other vertex, then u and w are $2^{\#}$-distinguished. Second, if $u \in S$ and all three neighbors of u are in S, $N[u] \subseteq S$, then the girth $g(HEX) = 6$ implies that no other vertex w is adjacent to two vertices in $N(u)$, and so u is $2^{\#}$-distinguished from any vertex w. Finally, consider a vertex in S adjacent to one not in S, say $u = (1,0)$ with $N(u) \cap S = \{(2,0),(1,-1)\}$. Clearly (2,0) and (1,-1) will $2^{\#}$-distinguish u from any w not adjacent to $(2,0)$ and not adjacent to $(1,-1)$. Any vertex w in $N((2,0)) \cup N((1,-1))$ with $w \neq (1,0)$ has $N[w] \subseteq S$, and $u = (1,0)$ is $2^{\#}$-distinguished from w. Because S is tiled as in the upper-right corner of Figure 5(b) with $12/14 = 6/7$ of the vertices, $DET{:}OLD\%(HEX) \leq 6/7$.

We will show that if S is any DET:OLD(HEX)-set with $u \notin S$ then the six vertices at distance two from u, and all of the vertices at distance four from u, must be in S. Thus, with each $u \notin S$ we can associate the six vertices at distance two from u with the vertex u and conclude that S has at least $6/7$ of the vertices in $V(HEX)$. Suppose $u = (0,0) \notin S$. By symmetry we can assume $N(u) = \{(1,0),(-1,0),(0,1)\}$. These vertices must be double open-dominated by S, so for the set $N^2((0,0))$ of vertices at distance two from (0,0) we have $N^2((0,0)) = \{(2,0),(1,-1),(-2,0),(-1,-1),(-1,1),(1,1)\} \subseteq S$. To see that the set $N^4((0,0))$ of vertices at distance four from $(0,0)$ satisfies $N^4((0,0)) \subseteq S$, consider $v = (0,1)$ and $w = (1,2)$. Vertices v and w must be $2^{\#}$-distinguished. But $N(v) = \{(0,0),(-1,1),(1,1)\}$ with $(0,0) \notin S$ and $(1,1) \in N(w)$. Only $(-1,1)$ distinguishes v from w and so S does not $2^{\#}$-distinguish v from w. Hence S must $2^{\#}$-distinguish w from v. Specifically, $\{(0,2),(2,2)\} \subseteq S$. The other cases are similar, and $N^4((0,0)) \subseteq S$, completing the proof that $DET{:}OLD\%(HEX) \geq 6/7$. ∎

Theorem 12. *$RED{:}OLD\%(SQ) = \gamma^{op}_{\times 2}\%(SQ) = 1/2$.*

Proof. By Observation 9, $RED{:}OLD\%(SQ) \geq \gamma^{op}_{\times 2}\%(SQ) \geq 2/4 = 1/2$. As in Figure 6(a), let $S = \{(i,j) \in V(SQ) : j \text{ is even}\}$. Each vertex v has $|N(v) \cap S| = 2$, say $N(v) \cap S = \{w,x\}$. No other vertex dominates both w and x. Hence, for any two vertices u and v, each of u and v has a vertex that distinguishes it from the other. Thus, u and v are 2-distinguished, and we have $RED{:}OLD\%(SQ) \leq 1/2$. ∎



(a) (b)

Figure 6. $RED{:}OLD\%(SQ) \leq 1/2$ and $DET{:}OLD\%(SQ) \leq 3/4$.

Theorem 13. $DET{:}OLD\%(SQ) = 3/4$.

Proof. As in Figure 6(b), let $S = \{(i,j) \in V(SQ) : j \equiv 1, 2 \text{ or } 3 \pmod 4\}$. If $j \equiv 2 \pmod 4$, then $N((i,j)) \subseteq S$. Because no other vertex w is adjacent to more than two vertices in $N((i,j))$, there are at least two vertices in $N((i,j))$ that distinguish (i,j) from w, and so (i,j) and w are $2^{\#}$-distinguished. If $j \equiv 1$ or 3 (mod 4)), then $|N((i,j)) \cap S| = 3$. Clearly, if a vertex w is adjacent to at most one vertex in $N((i,j)) \cap S$, then there are at least two vertices in $N((i,j)) \cap S$ which will distinguish (i,j) from w and (i,j) and w are $2^{\#}$-distinguished. The only vertices $w = (h,k)$ with w adjacent to two vertices in $N((i,j)) \cap S$ have $k \equiv 2$ (mod 4). As noted, such a vertex w and (i,j) are $2^{\#}$-distinguished. Finally, if (i,j) and (h,k) satisfy $j \equiv 0 \pmod 4$ and $k \equiv 0 \pmod 4$, then $(i, j+1)$ and $(i, j-1)$ will $2^{\#}$-distinguish (i,j) from (h,k). Hence, $DET{:}OLD\%(SQ) \leq 3/4$.

It remains to show that $DET{:}OLD\%(SQ) \geq 3/4$. Note that SQ is bipartite with bipartition $V_1 = \{(i,j) : i + j \equiv 0 \pmod 2\}$ and $V_2 = \{(i,j) : i + j \equiv 1 \pmod 2\}$. Let S be a DET:OLD(SQ)-set with $S_i = S \cap V_i$ for $i = 1, 2$. We show that each S_i contains at least 3/4 of the vertices in V_i. Here we use a discharging argument starting with a weight of one only on each $v \in S$. Weights will be discharged onto the vertices in $V(SQ) - S$ so that each $x \in V(SQ) - S$ has a resulting weight of at least three, showing that S has at least 3/4 of the vertices in $V(SQ)$. If $v \in S_i$, let $j = 3 - i$. We discharge the weights in S_i onto vertices in $V_j - S_j$. Let $C(v)$ be the set of vertices that are in $V_j - S_j$ and are as close to v as possible, that is $C(v) = \{x \in V_j - S_j : dist(x,v) \leq dist(y,v)$ for every $y \in V_j - S_j\}$. For each $v \in S_i$ we discharge a weight of $1/|C(v)|$ to each $x \in C(v)$.

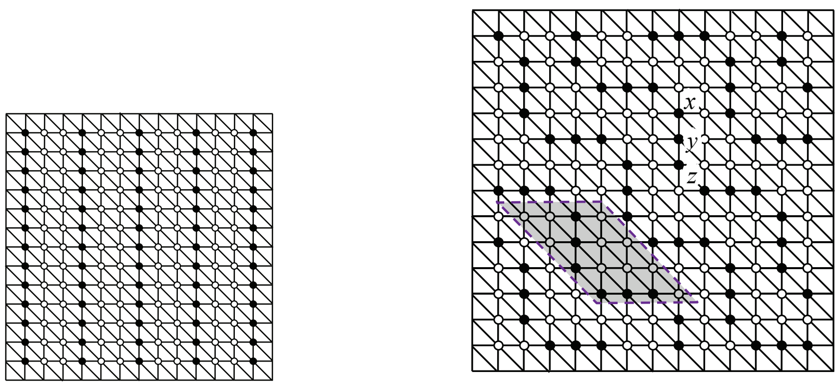
Consider a vertex $x \in V(G) - S$. Without loss of generality, assume $x = (0,0)$. For $u = (1,0)$ and $v = (2,1)$ the only vertex in $N(u)$ that can distinguish u from v is $(1,-1)$. Hence, we must $2^{\#}$-distinguish v from u with $(2,2)$ and $(3,1)$, and so $\{(2,2),(3,1)\} \subseteq S$.

Symmetric cases show that we have $A = \{(2,2),(3,1),(1,3),(-1,3),(-2,2),(-3,1),(-3,-1),(-2,-2),(-1,-3),(1,-3),(2,-2),(3,-1)\} \subseteq S$. Let $R = \{(0,2),(1,1),(2,0),(1,-1),(0,-2),(-1,-1),(-2,0),(-1,1)\}$. If $R \subseteq S$, then each of $(1,1),(1,-1),(-1,-1)$ and $(-1,1)$ discharges a weight of one to x, and each of $(0,2),(2,0),(0,-2)$ and $(-2,0)$ discharges at least 1/2 to x, so x receives a weight of at least six.

Suppose one of $(0,2),(2,0),(0,-2)$ and $(-2,0)$ is not in S.

Without loss of generality, assume $(0,-2) \notin S$. Similar to showing that $A \subseteq S$, if $(0,-2) \notin S$ we have $\{(1,1),(2,0),(3,-1),(3,-3),(2,-4),(1,-5),(-1,-5),(-2,-4),(-3,-3),(-3,-1),(-2,0),(-1,1)\} \subseteq S$. Now S must 2-dominate $(0,-1)$ so $\{(-1,-1),(1,-1)\} \subseteq S$. Each of $(1,1),(2,0),(1,-1),(-1,-1),(-2,0)$ and $(-1,1)$ discharges at least 1/2 to $x = (0,0)$, so x receives a weight of at least three. Suppose $\{(0,2)(2,0),(0,-2),(-2,0)\} \subseteq S$ and one of $(1,1),(1,-1),(-1,-1)$ and $(-1,1)$ is not in S. Without loss of generality, assume $(1,-1) \notin S$. As for $(0,0)$ and A, we now also have $(-1,1),(4,0),(4,-2),(3,-3),(2,-4)$ and $(0,-4)$ in S. If either $(1,1)$ or $(-1,-1) \notin S$, then either $(1,0)$ or $(0,-1)$ would not be double dominated, a contradiction.

So we can assume that $\{(1,1),(-1,-1)\} \subseteq S$. Now $x = (0,0)$ receives a weight of exactly 1 from $(-1,1)$, exactly 1/2 from each of $(1,1),(2,0),(0,-2)$ and $(-1,-1)$, and at least 1/2 from $(0,2)$ and $(-2,0)$, so at least four. In all cases, $(0,0)$ has a resulting weight of at least three. ■



(a) (b)

Figure 7. $\gamma^{op}_{x(2)}\%(TRI) \leq 1/3$ and $RED{:}OLD\%(TRI) \leq 3/8$.

We have $RED{:}OLD\%(HEX) = \gamma^{op}_{\times 2}\%(HEX) = 2/3$ and $RED{:}OLD\%(SQ) = \gamma^{op}_{\times 2}\%(SQ) = 1/2$. We next show that $RED{:}OLD\%(TRI) > \gamma^{op}_{\times 2}\%(TRI)$. In

Theorem 14 we consider how much of the open share of a vertex v comes from a specific vertex x in an open dominating set S. Let $sh^{op}(v, S(x)) = 1/|N(x) \cap S|$.

Theorem 14. *For the infinite triangular grid TRI which is regular of degree six, we have $\gamma^{op}_{x(2)}\%(TRI) = 1/3$ and $RED{:}OLD\%(TRI) = 3/8$.*

Proof. The darkened vertices of TRI in Figure 7(a) consists of all of the vertices in every third column of TRI. That is, we have the set $S = \{(i,j) \in V(TRI) : i \equiv 0 \pmod 3)\}$, and S double open dominates TRI, so $\gamma^{op}_{\times 2}\%(TRI) \leq 1/3$. For any double open dominating set D of a 6-regular graph, each vertex $v \in D$ open dominates its six neighbors. To double open dominate n vertices it thus requires at least $2n/6$ vertices of D, and it follows that $\gamma^{op}_{\times 2}\%(TRI) \geq 1/3$. Hence, $\gamma^{op}_{\times 2}\%(TRI) = 1/3$.

The darkened vertices of TRI in Figure 7(b) form a RED:OLD(TRI)-set S, so $RED{:}OLD\%(TRI) \leq 3/8$. Next, we will show that if S is a RED:OLD(TRI)-set, then the average share of $v \in S$ is at most $8/3$ and hence $RED{:}OLD\%(TRI) \geq 3/8$. (Note that in a RED:OLD(TRI)-set S individual vertices can have a share larger than $8/3$. For example, if $\{x = (0,1), y = (0,0), z = (0,-1)\} \subseteq S$ as identified in Figure 7(b), then $sh^{op}((0,0), S) = 1/2+1/2+1/4+1/2+1/2+1/4 = 5/2$ and $sh^{op}((0,1), S) = sh^{op}((0,-1), S) = 1/2+1/2+1/2+1/2+1/4+1/2 = 11/4 > 8/3$.)

Assume S is a RED:OLD(TRI)-set and $v \in S$. Each x in $N(v)$ is double open dominated, so $sh^{op}(v, S(x)) \leq 1/2$, and so $sh^{op}(v, S) \leq 6(1/2) = 3$. We will show that $sh^{op}(v, S) = 6(1/2)$ is not possible, nor can we have $sh^{op}(v, S) = 5(1/2)+(1/3) = 23/6$. Thus, if $sh^{op}(v, S) > 8/3$, then $sh^{op}(v, S) = 5(1/2)+1/4 = 11/4 = 8/3+1/12$ or $sh^{op}(v, S) = 5(1/2)+1/5 = 27/10 = 8/3+1/30$. We can "discharge" any excess over $8/3$ (namely, $1/12$ or $1/30$) onto nearby vertices in S so that no vertex has a resulting value greater than $8/3$.

Without loss of generality, assume $v = (0,0) \in S$ and $N(v) = \{v_1 = (0,1), v_2 = (1,0), v_3 = (1,-1), v_4 = (0,-1), v_5 = (-1,0), v_6 = (-1,1)\}$. If two adjacent vertices in $N(v)$ are in S, we can assume $v_1 = (0,1) \in S$ and $v_2 = (1,0) \in S$. Then v_2 and v_6 are 2-distinguished, so $sh^{op}(v, S(v_2)) + sh^{op}(v, S(v_6)) \leq \max\{1/3+1/3, 1/4+1/2\} = 3/4$. Likewise, $\{v, v_2\} \subseteq N(v_3) \cap N(v_1)$ and $sh^{op}(v, S(v_1)) + sh^{op}(v, S(v_3)) \leq \max\{1/3+1/3, 1/4+1/2\} = 3/4$. Thus, $sh^{op}(v, S) \leq 3/4+3/4+1/2+1/2 = 5/2 < 8/3$. If two opposite vertices in $N(v)$ are in S, we can assume $\{v, v_1, v_4\} \subseteq S$. Then $\{v, v_1\} \subseteq N(v_2) \cap N(v_6)$ and S 2-distinguishes $\{v_2, v_6\}$, so $sh^{op}(v, S(v_2)) + sh^{op}(v, S(v_6)) \leq \max\{1/3+1/3, 1/4+1/2\} = 3/4$, and likewise, $sh^{op}(v, S(v_3)) + sh^{op}(v, S(v_5)) \leq \max\{1/3+1/3, 1/4+1/2\} = 3/4$. Again $sh^{op}(v, S) \leq 5/2 < 8/3$.

If neither of these two cases apply and $|N(v) \cap S| > 2$, we can assume $N(v) \cap S = \{v_1, v_3, v_5\}$. Then, $sh^{op}(v, S) \leq 3(1/2) + 3(1/3) = 5/2 < 8/3$.

If $sh^{op}(v, S) > 8/3$, we can assume that $N(v) \cap S = \{v_1, v_3\}$. Because $\{v_2, v_6\}$

and $\{v_2, v_4\}$ are 2-distinguished, either both $|N(v_4) \cap S| \geq 3$ and $|N(v_6) \cap S| \geq 3$ (and then $sh^{op}(v, S) \leq 8/3$), or else $|N(v_2) \cap S| \geq 4$.

In summary, if $sh^{op}(v, S) > 8/3$, we can assume $N(v) \cap S = \{v_1, v_3\}$, $|N(v_i) \cap S| = 2$ if $i \in \{1, 3, 4, 5, 6\}$ and $|N(v_2) \cap S| = 4$ or 5. We consider two cases for the discharging.

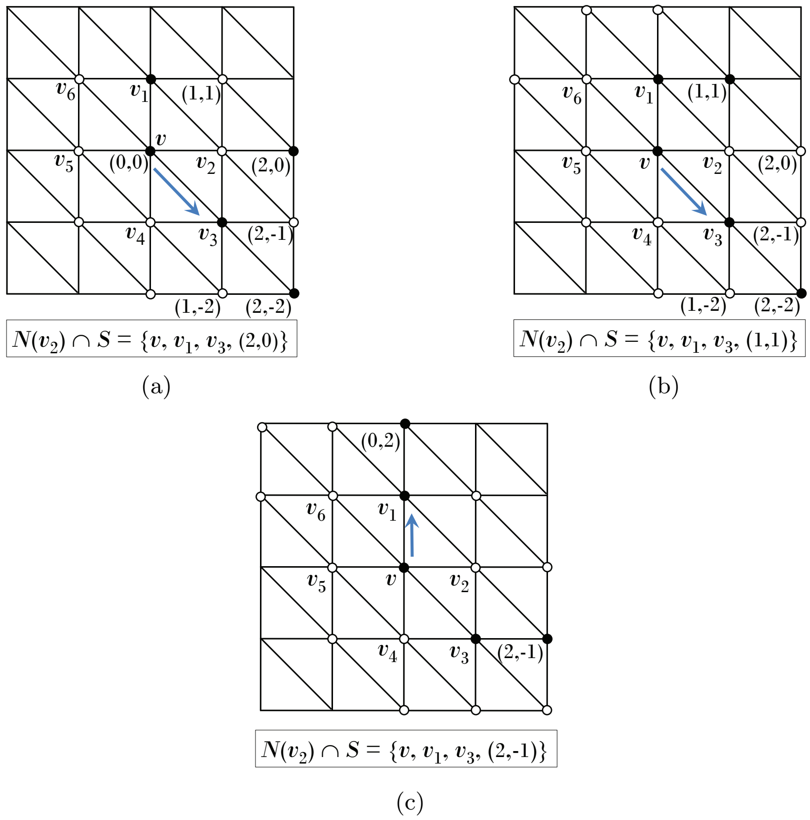


Figure 8. $|N(v_2) \cap S| = 4$ and $sh^{op}(v, S) = 11/4$.

Case 1. Assume $|N(v_2) \cap S| = 4$ and $sh^{op}(v, S) = 11/4$. First, assume $N(v_2) \cap S = \{v, v_1, v_3, (2, 0)\}$. In particular, $(2, -1) \notin S$ and $(1, 1) \notin S$ as in Figure 8(a). Because $sh^{op}(v, S(v_4)) = 1/2, N(v_4) \cap S = \{v, v_3\}$ and $(1, -2) \notin S$ and then $N(v_3) \cap S = \{v, (2, -2)\}$. Because $\{v, v_3, (2, -2)\} \subseteq S$, from before we have $sh^{op}(v_3, S) \leq 5/2$. We discharge a value of $1/12$ from v onto v_3, so v has a remaining value of $11/4$ -$1/12 = 8/3$.

Second, assume $N(v_2) \cap S = \{v, v_1, v_3, (1, 1)\}$ as in Figure 8(b). In particular, $(2, 0) \notin S$ and $(2, -1) \notin S$. Again, $N(v_4) \cap S = \{v, v_3\}$ and so $(1, -2) \notin S$ and $(2, -2) \in S$. As above, we can discharge a value of $1/12$ from v onto v_3. Symmetrically, if $N(v_2) \cap S = \{v, v_1, v_3, (2, -1)\}$ as in Figure 8(c), then $N(v_6) \cap$

$S = \{v, v_1\}$ and $(0,2) \in S$, $sh^{op}(v_1, S) \leq 5/2$, we discharge $1/12$ from v onto v_1.

Case 2. Assume $|N(v_2) \cap S| = 5$ and $sh^{op}(v, S) = 27/10$. First, assume $N(v_2) \cap S = \{v, v_1, v_3, (2,0), (2,-1)\}$ as in Figure 9(a). Then $v_2 \notin S$, $(1,1) \notin S$, $v_6 \notin S$, and $sh^{op}(v, S(v_6)) = 1/2$ implies that $(-1,2) \notin S$ and so $N(v_1) \cap S = \{v, (0,2)\}$.

Because $sh^{op}(v_1, S(v_2)) = 1/5$ and $sh^{op}(v_1, S((1,1))) + sh^{op}(v_1, S((-1,2))) \leq 3/4$, we have $sh^{op}(v_1, S) \leq 1/5 + 3/4 + 3(1/2) = 49/20 < 5/2$. We discharge a value of $1/30$ from v to v_1.

Second, assume $N(v_2) \cap S = \{v, v_1, v_3, (1,1), (2,0)\}$ as in Figure 9(b). Then $(2,-1) \notin S$, $v_2 \notin S$, $v_4 \notin S$, and $sh^{op}(v, S(v_4)) = 1/2$ implies that $(1,-2) \notin S$ and so $N(v_3) \cap S = \{v, (2,-2)\}$. Because $sh^{op}(v_3, S(v_2)) = 1/5$ and $sh^{op}(v_3, S((2,-1)))+ sh^{op}(v_3, S((1,-2))) \leq 3/4$, we have $sh^{op}(v_3, S) \leq 49/20 < 5/2$. We discharge a value of $1/30$ from v to v_3.

Third, assume $N(v_2) \cap S = \{v, v_1, v_3, (1,1), (2,-1)\}$ as in Figures 9(c) and 9(d). Then $(2,0) \notin S$, $v_2 \notin S$, and $sh^{op}(v, S(v_1)) = 1/2$ implies that $(0,2) \notin S$. Assume $(1,2) \in S$ (Figure 9(c)). Now $sh^{op}(v_1, S) \leq 1/3 + 1/5 + 4(1/2) = 38/15$. We discharge a value of $1/30$ from v to v_1. Assuming $(1,2) \notin S$ (Figure 9(d)), we have $N((1,1)) \cap S = \{v_1, (2,1)\}$. Note that $sh^{op}((1,1), S(v_2)) = 1/5$ and $sh^{op}((1,1), S((2,0)))+ sh^{op}((1,1), S((1,2))) \leq 3/4$, so $sh^{op}((1,1), S) \leq 49/20$. We discharge a value of $1/30$ from v to (1,1). Note that this is the only time we discharge from v to a vertex not in $\{v_1, v_3\}$.

DISCHARGING SUMMARY. We conclude by showing that any vertex w for which some vertex v discharges a value onto w will have a resulting value of at most $8/3$. First, suppose w is a vertex receiving a discharge value from a nonadjacent vertex v. We can assume $w = (1,1)$ and $v = (0,0)$ as above and we discharge $1/30$ from v to (1,1). Then $sh^{op}(v_1, S) \leq 5(1/2) + 1/5 = 27/10 = 8/3+1/30$, and (1,1) receives a value of at most $1/30$ from $v_1 = (0,1)$ for a total of at most $1/30 + 1/30 = 1/15$ from v and v_1. Then $N((1,1)) \cap S = \{v_1, (2,1)\}$. The maximum remaining value (1,1) can receive is $\max\{1/12, 1/30 + 1/30\} = 1/12$. Hence the maximum resulting value on (1,1) is $49/20 + 1/15 + 1/12 = 156/60 < 8/3$. Second, assume w receives any discharge only from an adjacent vertex. If $sh^{op}(w, S) \leq 5/2$, then the resulting value at w is at most $5/2+1/12+1/12 = 8/3$. The final case is from the third part of Case 2 (with $(1,2) \in S$ and $sh^{op}(v, S) \leq 38/15$) and with $w = v_1$. In this case we are discharging $1/30$ from v to v_1 (and are assuming the nonadjacent vertex v_3 is not discharging to v_1). If v_1 receives any discharge from (1,1), we have $sh^{op}((1,1), S) \leq 5(1/2) + 1/5 = 8/3 + 1/30$. Hence the resulting total at v_1 is at most $38/15 + 1/30 + 1/30 = 39/15 < 8/3$. ■

To date our best construction shows that $DET{:}OLD\%(TRI) \leq 5/9$.

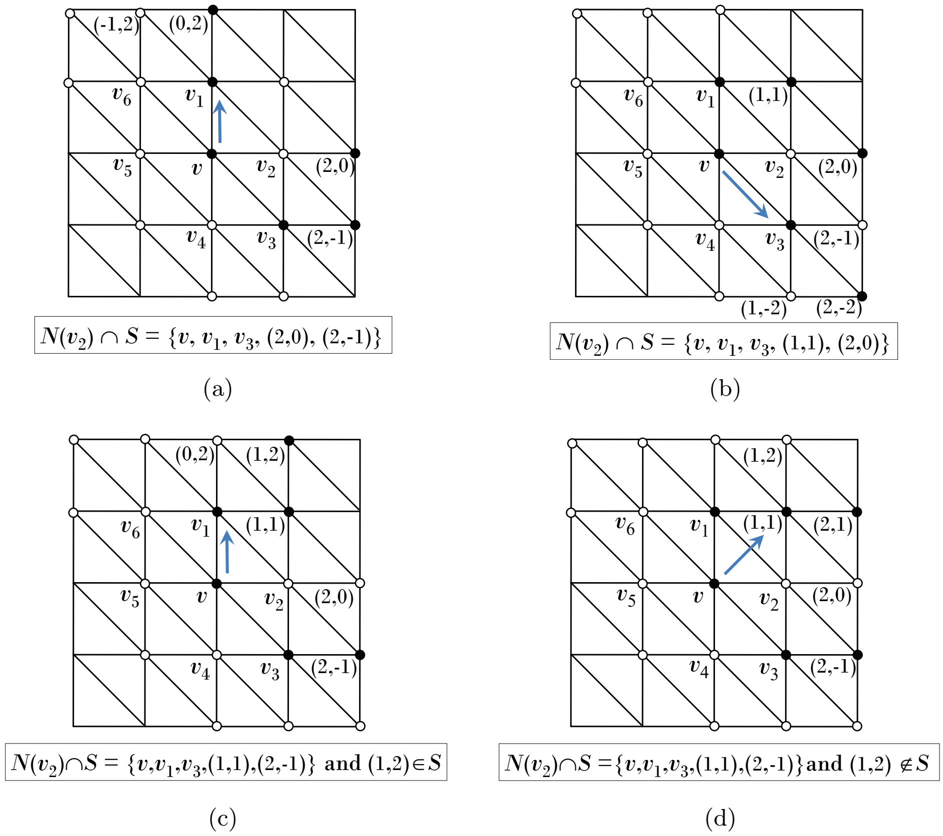


Figure 9. $|N(v_2) \cap S| = 5$ and $sh^{op}(v, S) = 27/10$.

4. Summary

For various domination-related parameters involving locating devices that function as places from which detectors can determine information about the location of an "intruder", several types of possible detector faults are identified in Slater [46]. Two of these are considered here, namely RED and DET. Another involves the notion of a "liar". The idea of a liar's fault coincides, for example, with DET for parameters OLD and IC. That is, $DET{:}OLD(G) \equiv LR{:}OLD(G)$ and $DET{:}IC(G) \equiv LR{:}IC(G)$. However, for the basic domination parameter γ they are different.

For liar's domination, as introduced in Slater [45] and Roden and Slater [27, 28], a detector at vertex v can uniquely identify any intruder vertex in $N[v]$, but with a liar's fault, given that there is an intruder at $x \in N[v]$, the detector at v can report any $y \in N[v]$ as the intruder location. See [12, 25, 26, 28, 29]. Interestingly, as it relates to Theorem 4 and Theorem 5, we have the following characterization of liar's dominating sets.

Theorem A (Slater [45])**.** *Vertex set $L \subseteq V(G)$ is a liar's dominating set if and only if*

(1) *L double dominates every $v \in V(G)$ and*

(2) *for every pair u,v of distinct vertices we have $|(N[u] \cup N[v]) \cap L| \geq 3$.*

Theorem B (Slater [45])**.** *For every connected graph G of order $n \geq 3$ we have $\gamma_{\times 2}(G) \leq LR{:}\gamma(G)$, and if G has minimum degree $\delta(G) \geq 2$, then $\gamma_{\times 2}(G) \leq LR{:}\gamma(G) \leq \gamma_{\times 3}(G)$.*

From Theorem A we see that a liar's dominating set must double dominate and, for each pair u and v, there must be three detectors that distinguish u and v. In particular, from Theorem 5 we see that $DET{:}\gamma(G) \leq LR{:}\gamma(G)$.

A fourth type of fault is the basic error correcting code problem allowing arbitrary transmission errors. A detector might be able to detect the intruder location, but there can be an error in any one detector's transmission to command point P, including the possibility of a false alarm. ER will be used to indicate this type of possible fault. Again using similar notation, we have theorems like the following, as in Slater [46].

Theorem C. $\gamma(G) \leq RED{:}\gamma(G) = DET{:}\gamma(G) \leq LR{:}\gamma(G) \leq ER{:}\gamma(G)$.

Theorem D. $LOC(G) \leq RED{:}LOC(G) = DET{:}LOC(G) \leq LR{:}LOC(G) = ER{:}LOC(G)$.

Theorem E. $OLD(G) \leq RED{:}OLD(G) \leq DET{:}OLD(G) = LR{:}OLD(G) \leq ER{:}OLD(G)$.

Various related problems are under study. In particular, Sewell and Slater [38] have $RED{:}IC(G)$ and $DET{:}IC(G)$ and also $RED{:}LD(G)$ and $DET{:}LD(G)$ under study. (See Sewell [37].)

The specific $LR{:}LD$ parameter appears to be interesting. In this case a detector at vertex v observing that there is an intruder x in $N[v]$ might report any of the three possibilities that no intruder is in $N[v]$, that $x = v$, or that $x \in N(v)$.

There are other graphs that can be considered. In [34], we have considered domination related parameters and distinguishing sets for different classes of the infinite and finite tumbling block graphs. Investigating the redundant distinguishing and detection distinguishing open-locating-dominating sets on these graphs will also be interesting.

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