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MAXIMUM INDEPENDENT SETS IN DIRECT PRODUCTS OF CYCLES OR TREES WITH ARBITRARY GRAPHS

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Abstract

The direct product of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph, denoted as $G \times H$, with vertex set $V(G \times H) = V(G) \times V(H)$, where vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \times H$ if $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. Let n be odd and m even. We prove that every maximum independent set in $P_n \times G$, respectively $C_m \times G$, is of the form $(A \times C) \cup (B \times D)$, where C and D are nonadjacent in G, and $A \cup B$ is the bipartition of P_n respectively C_m. We also give a characterization of maximum independent subsets of $P_n \times G$ for every even n and discuss the structure of maximum independent sets in $T \times G$ where T is a tree.

Keywords: direct product, independent set.

2010 Mathematics Subject Classification: 05C69, 05C38.

1. Introduction

In this paper we study the structure of maximum independent sets in the direct product of an arbitrary graph with a bipartite graph. In particular the structure of $G \times T$, where T is a tree, is addressed.

Several partial results about the size and structure of maximum independent sets in direct products of graphs are known. In [21] the author proved that for any vertex-transitive graphs G and H we have

$$\alpha(G \times H) = \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\} \tag{1}$$

thereby answering a question posed in [19]. If I is an independent set in G, then $I \times V(H)$ is an independent set in $G \times H$ (similarly $V(G) \times J$ is an independent set in $G \times H$ provided that J is independent in H). These independent sets in the product are obtained from independent sets of factors, and we call such independent sets *canonical*. The lower bound in (1) is easy to prove since it is achieved by a canonical independent set. According to [21], the product $G \times H$ is called *MIS-normal* (maximum-independent-set normal) if all maximum independent sets in $G \times H$ are canonical, and in the same article the author characterizes MIS-normal products of vertex transitive graphs.

The structure of maximum independent sets is also studied in [22]. Here the author proves that for a vertex-transitive graph G, such that G^2 is MIS-normal, any power G^n (n-th power of G with respect to the direct product) is also MIS-normal.

In [5] it is proved that any power of a complete graph is MIS-normal. The same result is proved in [2] where the authors also prove a theorem on independent sets whose size is close to maximum. To be precise, they prove that for every $r \geq 3$, there is a constant $M = M(r)$ such that for every $\epsilon > 0$ we have: if J is an independent set in $G = K_r^n$ such that $|J|/|V(G)| = 1/r - \epsilon$, then there is a canonical independent set I in G such that $|J \triangle I|/|V(G)| < M\epsilon$. Roughly speaking this means that an independent set whose size is close to $\alpha(K_r^n)$ is close to a canonical independent set.

In [14] the authors study the relation between projectivity and the structure of maximum independent sets in powers of vertex transitive graphs, and they give examples of powers of such graphs that are projective yet they have non-canonical maximum independent sets.

Many other results on independence number are given in [9, 10] where the authors determine the independence number when both factors are either a path or a cycle, and in [6, 20] where products of some special families of vertex transitive graphs are considered and their independence and chromatic numbers are determined. The independence number in direct products of arbitrary graphs is studied in [18] (see also [8]) where the author proves that for any graphs G and H we have

$$\alpha(G \times H) \leq \alpha(G)|V(H)| + \alpha(H)|V(G)| - \alpha(G)\alpha(H)\,,$$

and where also a generalization of the independence number and its relation to the Hedetniemi's conjecture is considered. We also mention an incorrect result given in [16, 17], where the authors erroneously claim that $\alpha(G \times P_n) = \max\left\{n\alpha(G), \left\lceil \frac{n}{2} \right\rceil |V(G)|\right\}$.

We also mention the abundance of results on the independence number of Cartesian products of graphs [1, 4, 7, 11, 15], as well as the results on the strong product of graphs that are given in [3, 12, 13].

In this paper we characterize maximum independent sets in $P_n \times G$ for all n, and in $C_n \times G$ for even n. This is a generalization of results given in [9, 10] where the authors establish the independence number when both factors are either a path or a cycle. We prove that every product of a path or an even cycle with an arbitrary graph has a maximum independent set which is a union of two rectangles, more precisely, a set of the form $I = (A \times C) \cup (B \times D)$. It is also shown that $P_n \times G$ admits other maximum independent sets when n is even. Examples of such sets are given and precise characterization of maximum independent sets in $P_n \times G$ for even n is obtained (see Theorem 5). We also give a sufficient condition for a tree T, so that $T \times G$ has a maximum independent set of the form $I = (A \times C) \cup (B \times D)$ (see Theorem 4).

2. The Structure of Maximum Independent Sets

Let G and H be graphs and $G \times H$ their direct product. For any $x \in V(G)$ or $y \in V(H)$ we define the *x-layer* H_x and the *y-layer* G_y as follows

$$H_x = \{(x, h) \mid h \in V(H)\} \quad \text{and} \quad G_y = \{(g, y) \mid g \in V(G)\}.$$

We denote by

$$p_G : V(G) \times V(H) \to V(G)$$

the *projection* of $V(G \times H)$ to $V(G)$. The projection is given by $p_G(x, y) = x$. For a set $I \subseteq V(G \times H)$ and vertices $x \in V(G), y \in V(H)$ we use

$$I_x = p_H(I \cap H_x) \ \text{ and } \ I_y = p_G(I \cap G_y).$$

For $X, Y \subseteq V(G)$ we denote by $[X, Y]$ the set of edges with one endvertex in X and the other in Y. We write $X \lll Y$ if there is a matching in $[X, Y]$ that saturates X, and $X \equiv Y$ if there is a perfect matching in $[X, Y]$ (equivalently, a matching that saturates X and Y). For any $x \in V(G)$ we define the set $N(x) = \{u \in V(G) \mid xu \in E(G)\}$ and for $X \subseteq V(G)$ we use $N(X) = \bigcup_{x \in X} N(x)$. For a set $U \subseteq V(G)$, we say that U is *expansive* if for every $X \subseteq U$ we have $|N(X) \cap U| \geq |X|$. Finally we say that X is *nonadjacent* to Y if $N(X) \cap Y = \emptyset$. Note that nonadjacent subsets can have nonempty intersection.

The following observation follows directly from the definition of direct products of graphs.

Observation 1. *Let G and H be graphs and I an independent set in $G \times H$. If $xy \in E(H)$, then I_x is nonadjacent to I_y.*

Theorem 2. *Let n be an even number and let A, B be the bipartition of $V(C_n)$. Every maximum independent set in $C_n \times G$ is equal to*

$$I = (A \times C) \cup (B \times D)$$

for some nonadjacent $C, D \subseteq V(G)$.

Proof. Let $V(C_n) = \{1, 2, \ldots, n\}$ and set $G_i = \{i\} \times V(G)$. Let $C, D \subseteq V(G)$ be nonadjacent sets such that $|C|+|D|$ is maximum. Since C and D are nonadjacent we find that $(A \times C) \cup (B \times D)$ is an independent set in $C_n \times G$.

For an independent set J in $C_n \times G$ denote

$$J_i = p_G(J \cap G_i)\,.$$

Clearly, any two consecutive J_i and J_{i+1} are nonadjacent in G and therefore $|J_i| + |J_{i+1}| \leq |C| + |D|$. Since

$$|J| = (|J_1| + |J_2|) + (|J_3| + |J_4|) + \cdots + (|J_{n-1}| + |J_n|)$$

it follows that

$$|J| \leq |(A \times C) \cup (B \times D)|\,.$$

This proves that $(A \times C) \cup (B \times D)$ is a maximum independent set in $C_n \times G$. Additionally, if J is a maximum independent set in $C_n \times G$, we have $|J_i|+|J_{i+1}| = |C| + |D|$ for $i = 1, 2, \ldots, n$. To prove the theorem assume that J is a maximum independent set. We claim that $J_1 = J_3$. Since $|J_1| + |J_2| = |C| + |D|$ we find that $J_3 \subseteq J_1$. Indeed, if $J_3 \setminus J_1 \neq \emptyset$, then $J_1 \cup J_3$ and J_2 are nonadjacent and $|J_1 \cup J_3| + |J_2| > |C| + |D|$, a contradiction. Additionally, it follows from $|J_2| + |J_3| = |C| + |D|$ that $J_1 \subseteq J_3$, and hence $J_1 = J_3$. It follows inductively that $J_i = J_k$ for $j \equiv k \,(\text{mod } 2)$. ■

Theorem 3. *Let P_n be an odd path and A, B the bipartition of $V(P_n)$. Every maximum independent set in $P_n \times G$ is equal to*

$$I = (A \times C) \cup (B \times D)$$

for some nonadjacent $C, D \subseteq V(G)$.

Proof. Let $V(P_n) = \{1, 2, \ldots, n\}$, denote $G_i = \{i\} \times V(G)$, and assume that $|A| > |B|$. Let J be a maximum independent set in $P_n \times G$ and $J_i = p_G(J \cap G_i)$. Let $i_0 \in \{1, 2, \ldots, n-1\}$ be such that

$$|J_{i_0}| + |J_{i_0+1}| \geq |J_i| + |J_{i+1}|$$

for all $i \in \{1, 2, \ldots, n-1\}$ and assume $|J_{i_0}| \geq |J_{i_0+1}|$. We claim that $J_i = J_\ell$ for $i - \ell \equiv 0 \,(\text{mod } 2)$. Let $k \in \{2, 4, \ldots, n-1\}$ be such that $|J_k| \geq |J_i|$ for all $i \in \{2, 4, \ldots, n-1\}$. Since $|J_{i_0}| + |J_{i_0+1}|$ is maximum we find that

$$\begin{aligned} |J_k| + \sum\nolimits_{i=1}^{n} |J_i| &= \sum\nolimits_{i=1}^{k/2} (|J_{2i-1}| + |J_{2i}|) + \sum\nolimits_{i=k/2}^{(n-1)/2} (|J_{2i}| + |J_{2i+1}|) \\ &\leq \frac{n+1}{2} |J_{i_0}| + \frac{n-1}{2} |J_{i_0+1}| + |J_{i_0+1}|\,. \end{aligned} \tag{2}$$

On the other hand $(A \times J_{i_0}) \cup (B \times J_{i_0+1})$ is an independent set of size

$$\frac{n+1}{2}|J_{i_0}| + \frac{n-1}{2}|J_{i_0+1}|$$

and since J is a maximum independent set we find that

$$\sum\nolimits_{i=1}^{n} |J_i| \geq \frac{n+1}{2}|J_{i_0}| + \frac{n-1}{2}|J_{i_0+1}|\,. \tag{3}$$

Since $|J_k|$ is maximum for even k, we see that $|J_k| \geq |J_{i_0+1}|$ (for otherwise we would have $|J_i| < |J_{i_0+1}| \leq |J_{i_0}|$ for all even indices i, a contradiction), and therefore it follows from (2) and (3) that

$$|J_k| + \sum\nolimits_{i=1}^{n} |J_i| = \frac{n+1}{2}|J_{i_0}| + \frac{n-1}{2}|J_{i_0+1}| + |J_{i_0+1}|\,.$$

It follows that each term in both sums,

$$\sum\nolimits_{i=1}^{k/2}(|J_{2i-1}| + |J_{2i}|) \ \text{ and } \ \sum\nolimits_{i=k/2}^{(n-1)/2}(|J_{2i}| + |J_{2i+1}|), \tag{4}$$

is equal to $|J_{i_0}| + |J_{i_0+1}|$. We claim that $J_{k-1} = J_{k+1}$. If not, then $J_{k-1} \nsubseteq J_{k+1}$ or $J_{k+1} \nsubseteq J_{k-1}$. If $J_{k-1} \nsubseteq J_{k+1}$ we can construct an independent set

$$I = \bigcup\nolimits_{i=1}^{k}(\{i\} \times J_i) \cup \left[((A \times (J_{k-1} \cup J_{k+1})) \cup (B \times J_k)) \cap \bigcup\nolimits_{i=k+1}^{n} G_i\right].$$

Since $|J_{k-1} \cup J_{k+1}| + |J_k| > |J_{k+1}| + |J_k| = |J_{i_0}| + |J_{i_0+1}|$ we see that $|I| > |J|$, a contradiction. Hence $J_{k-1} \subseteq J_{k+1}$, and analogously $J_{k+1} \subseteq J_{k-1}$. Thus we have $J_{k-1} = J_{k+1}$. Now assume there is an odd $\ell > k$ such that $J_\ell \neq J_{\ell+2}$, and assume ℓ is the smallest such index. If $J_\ell \nsubseteq J_{\ell+2}$ we can construct an independent set analogous to the one constructed in the case $J_{k-1} \nsubseteq J_{k+1}$ as follows:

$$I = \bigcup\nolimits_{i=1}^{\ell}(\{i\} \times J_i) \cup \left[((A \times (J_\ell \cup J_{\ell+2})) \cup (B \times J_{\ell+1})) \cap \bigcup\nolimits_{i=\ell+1}^{n} G_i\right].$$

So assume that $J_\ell \subseteq J_{\ell+2}$ and $J_{\ell+2} \nsubseteq J_\ell$. In this case we construct an independent set

$$I = \bigcup\nolimits_{i=\ell+1}^{n}(\{i\} \times J_i) \cup \left[((A \times J_{\ell+2}) \cup (B \times J_{\ell+1})) \cap \bigcup\nolimits_{i=1}^{\ell} G_i\right].$$

$|J_{\ell+1}| + |J_{\ell+2}|$ is a term of the second sum in (4), hence it is equal to $|J_{i_0}| + |J_{i_0+1}|$ and since $|J_{k-1}| = |J_{k+1}| = \cdots = |J_\ell| < |J_{\ell+2}|$, we see that $|I| > |J|$, a contradiction. An analogous proof works if there is an odd $\ell < k$ such that $J_{\ell-2} \neq J_\ell$ and we assume that ℓ is the largest such index. If $J_\ell \nsubseteq J_{\ell-2}$, we can construct an independent set

$$I = \bigcup\nolimits_{i=\ell}^{n}(\{i\} \times J_i) \cup \left[((A \times (J_{\ell-2} \cup J_\ell)) \cup (B \times J_{\ell-1})) \cap \bigcup\nolimits_{i=1}^{\ell-1} G_i\right].$$

Since $|J_\ell \cup J_{\ell-2}| + |J_{\ell-1}| > |J_{\ell-2}| + |J_{\ell-1}| = |J_{i_0}| + |J_{i_0+1}|$, we see that $|I| > |J|$, a contradiction. If $J_\ell \subseteq J_{\ell-2}$ and $J_{\ell-2} \nsubseteq J_\ell$, we construct an independent set

$$I = \bigcup_{i=1}^{\ell-1}(\{i\} \times J_i) \cup \left[((A \times J_{\ell-2}) \cup (B \times J_{\ell-1})) \cap \bigcup_{i=\ell}^{n} G_i\right] .$$

$|J_{\ell-2}| + |J_{\ell-1}|$ is a term of the first sum in (4), hence it is equal to $|J_{i_0}| + |J_{i_0+1}|$ and since $|J_{k-1}| = \cdots = |J_\ell| < |J_{\ell-2}|$, we see that $|I| > |J|$, also a contradiction. Therefore $J_{\ell+2} = J_\ell$ for all odd indices ℓ. If $J_\ell \neq J_{\ell+2}$ for an even ℓ, then

$$I = (A \times J_1) \cup \left(B \times \bigcup_{i=1}^{(n-1)/2} J_{2i}\right)$$

is an independent set strictly larger than J, a contradiction. So we proved that $J_\ell = J_1$ for all odd ℓ, and $J_\ell = J_2$ for all even ℓ, which proves the theorem. ■

Theorem 4. *Let T be a tree with bipartition $A \cup B$, where $|A| \geq |B|$. Suppose that for every vertex $x \in B$ there is an edge cover M_x of T such that x is covered at most twice and all the other vertices once. Then for every graph G there are nonadjacent sets $C, D \subseteq V(G)$, such that*

$$I = (A \times C) \cup (B \times D)$$

is a maximum independent set in $T \times G$.

Proof. Suppose that $2\alpha'(T) = |V(T)|$ (where $\alpha'(T)$ is the matching number of T), and let M be a perfect matching in T. Additionally, let C and D be nonadjacent subsets of G, such that $|C| + |D|$ is maximum. We claim that $I = (A \times C) \cup (B \times D)$ is a maximum independent set in $T \times G$. To prove this let J be any independent set in $T \times G$ and note that

$$|J| = \sum_{x \in V(T)} |J_x| = \sum_{uv \in M}(|J_u| + |J_v|) \leq \sum_{uv \in M} |C| + |D| = |I| .$$

Assume now that $2\alpha'(T) = |V(T)| - 1$ and let $A \cup B$ be the bipartition of T with $|A| > |B|$. By assumption, for every vertex $x \in B$ there is an edge cover M_x of T such that every vertex of T is covered exactly once by M_x, except the vertex x, which is covered twice. Let J be a maximum independent set in $T \times G$ and let $x_0 \in B$ be such that $|J_{x_0}| \geq |J_x|$ for every $x \in B$. Let uv be an edge of T such that $|J_u| + |J_v|$ is maximum. Assume that $|J_u| \geq |J_v|$ and set $C = J_u$ and $D = J_v$. We claim that $|J_{x_0}| \geq |D|$. If not then $|J_x| \leq |J_{x_0}| < |D| \leq |C|$ for every $x \in B$. This is a contradiction since either $u \in B$ or $v \in B$. Then we have

$$\begin{aligned} |J| + |J_{x_0}| &= \sum_{ab \in M_{x_0}} (|J_a| + |J_b|) \\ &\leq \sum_{ab \in M_{x_0}} (|C| + |D|) = \frac{1}{2}(|A| + |B| + 1)(|C| + |D|) , \end{aligned}$$

and since $|A| = |B| + 1$ we find that

$$|J| + |J_{x_0}| \leq |(A \times C) \cup (B \times D)| + |D|\,.$$

Finally, since $|J_{x_0}| \geq D$ it follows that

$$|J| \leq |(A \times C) \cup (B \times D)|\,.$$

Since $(A \times C) \cup (B \times D)$ is an independent set we have also

$$|J| \geq |(A \times C) \cup (B \times D)|\,,$$

and hence the equality holds, which proves the theorem. ■

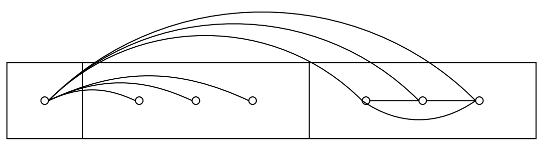

Figure 1. The graph G.

We have shown so far that every product of an odd path (or an even cycle) with an arbitrary graph admits only maximum independent sets of the form $(A \times C) \cup (B \times D)$. Moreover, every product of an even path with an arbitrary graph admits maximum independent sets of this form (see Corollary 6 and Theorem 5), but not all maximum independent sets are neccesarily such. We next give an example of a product of a tree T with a graph G, such that no maximum independent set in $T \times G$ is of the form $(A \times C) \cup (B \times D)$. Consider the graph G shown in Figure 1, and the product $T \times G$ and the set I shown in Figure 2. Clearly, the set I is an independent set in $T \times G$ and its size is 43. On the other hand for any independent set $J = (A \times C) \cup (B \times D)$ we have $|J| < 43$ (in order to maximize J we have $C = Q \cup R$ and $D = Q$, or alternatively $C = V(G)$ and $D = \emptyset$).

To characterize maximum independent sets in products $P_n \times G$ where n is even we denote $V(P_n) = \{1, 2, \ldots, n\}$ and we let $G_k = \{k\} \times V(G)$. For a set $I \subseteq V(P_n \times G)$ and $k \leq n$ we define

$$I_k = p_G(I \cap G_k)\,.$$

Maximum independent sets in $P_n \times G$ can be characterized by subset relations (as given in (i) and (ii) of Theorem 5) and by matchings between certain parts

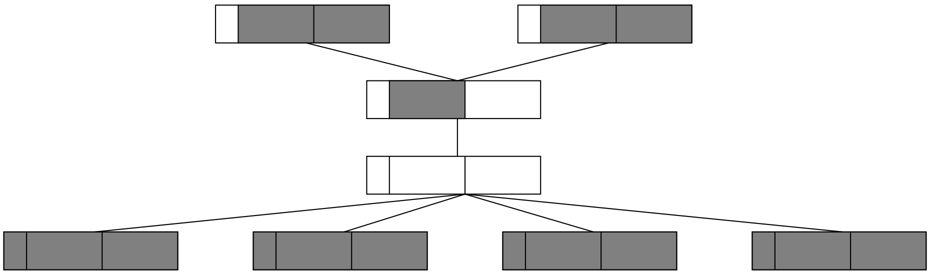

Figure 2. The product $T \times G$. The independent set I is marked with gray color.

of consecutive sets I_k (as given in (iii) and (iv) of Theorem 5). The main idea of the proof is that I is a maximum independent set in $P_n \times G$ if and only if I_k and I_{k+1} are nonadjacent and $|I_k| + |I_{k+1}|$ is maximum for all odd k. This follows from the fact that I is an independent set in $P_n \times G$ if and only if I_k and I_{k+1} are nonadjacent for all k and

$$|I| = \sum_{l=1}^{n/2} (|I_{2l-1}| + |I_{2l}|).$$

It turns out that $|I_k| + |I_{k+1}|$ is maximum for all odd k if and only if (i) through (vi) of Theorem 5 is true. Here conditions (i) through (iv) guarantee that $|I_k| + |I_{k+1}| = |I_{k+2}| + |I_{k+3}|$ for all odd k, and the additional conditions (v) and (vi) guarantee that $|I_1| + |I_2|$ is maximum.

Theorem 5. *Let n be an even number and I an independent set in $P_n \times G$. Then I is a maximum independent set in $P_n \times G$ if and only if the following is true:*

(i) $I_{k+2} \subseteq I_k$ *for every odd* k.

(ii) $I_k \subseteq I_{k+2}$ *for every even* k.

(iii) $I_{k+3} \setminus (I_k \cup I_{k+1}) \equiv (I_k \setminus I_{k+2}) \cap I_{k+1}$ *for every odd* k.

(iv) $I_k \setminus (I_{k+2} \cup I_{k+3}) \equiv (I_{k+3} \setminus I_{k+1}) \cap I_{k+2}$ *for every odd* k.

(v) $(V(G) \setminus (I_1 \cup I_4)) \cup (I_4 \setminus (I_1 \cup I_2)) \lll (I_2 \cap I_3) \cup ((I_1 \setminus I_3) \cap I_2)$.

(vi) *The sets* $I_2 \setminus I_1$ *and* $I_3 \setminus I_4$*, and* $(I_1 \setminus I_3) \cap (I_4 \setminus I_2)$ *are expansive.*

Proof. For every odd k we denote

$$\begin{aligned}
A_k &= (I_k \setminus I_{k+2}) \cap (I_{k+3} \setminus I_{k+1}),\\
B_k &= I_k \setminus (I_{k+2} \cup I_{k+3}),\\
C_k &= I_{k+2} \setminus I_{k+3},\\
D_k &= (I_{k+3} \setminus I_{k+1}) \cap I_{k+2},\\
E_k &= I_{k+1} \cap I_{k+2},\\
F_k &= (I_k \setminus I_{k+2}) \cap I_{k+1},
\end{aligned}$$

$G_k = I_{k+1} \setminus I_k$,
$H_k = I_{k+3} \setminus (I_k \cup I_{k+1})$,
$R_k = V(G) \setminus (I_k \cup I_{k+3})$.

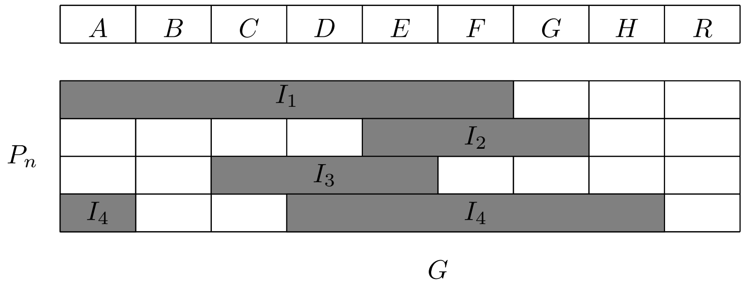

Figure 3. The sets $I_1, \ldots, I_4$.

Assume that I is a maximum independent set in $G \times P_n$. We have to prove (i)–(vi).

Proof of (i) and (ii). Since I is a maximum independent set, $|I_k| + |I_{k+1}|$ is maximum for every odd k. If $I_{k+2} \nsubseteq I_k$ for an odd k, then subsets $I_k \cup I_{k+2}$ and I_{k+1} are nonadjacent and $|I_k \cup I_{k+2}| + |I_{k+1}| > |I_k| + |I_{k+1}|$, a contradiction. Similarly, if there is an even k, such that $I_k \nsubseteq I_{k+2}$, then I_{k+1} and $I_k \cup I_{k+2}$ are nonadjacent and $|I_{k+1}| + |I_k \cup I_{k+2}| > |I_{k+1}| + |I_{k+2}|$, a contradiction.

Proof of (iii) and (iv). We claim that $H_k \equiv F_k$ for every odd k. Since I_k and I_{k+1} are nonadjacent for all k, we see that $N(F_k) \subseteq H_k \cup R_k$.

Let $X \subseteq F_k$ and note that $F_k \cap I_{k+2} = \emptyset$ and hence $X \cap I_{k+2} = \emptyset$. If $|N(X) \cap H_k| < |X|$, then $|N(X) \cap I_{k+3}| < |X|$, because $N(X) \cap I_{k+3} \subseteq H_k$. Therefore $|I_{k+2} \cup X| + |I_{k+3} \setminus N(X)| > |I_{k+2}| + |I_{k+3}|$, which is a contradiction.

Let $X \subseteq H_k$. If $|N(X) \cap F_k| < |X|$, then $|I_k \cup G_k \cup X| + |I_{k+1} \setminus (G_k \cup N(X))| > |I_k| + |I_{k+1}|$, which is also a contradiction, since these two sets are nonadjacent. This proves that there is a perfect matching between H_k and F_k. The proof of (iv) is similar.

Proof of (v). Denote $R = V(G) \setminus (I_1 \cup I_4)$.

We have to prove that for all $X \subseteq R \cup H_1$, $|N(X) \cap (E_1 \cup F_1)| \geq |X|$. But if for some $X \subseteq R \cup H_1$, $|N(X) \cap (E_1 \cup F_1)| < |X|$, then $I_1 \cup X \cup G_1$ and $I_2 \setminus (G_1 \cup N(X))$ are nonadjacent sets and $|I_1 \cup X \cup G_1| + |I_2 \setminus (G_1 \cup N(X))| > |I_1| + |I_2|$, a contradiction.

Proof of (vi). Now we will prove that G_1 is expansive. If not, $|N(X) \cap G_1| < |X|$ for some $X \subset G_1$. But then $I_1 \cup X$ and $I_2 \setminus N(X)$ are nonadjacent and $|I_1 \cup X| + |I_2 \setminus N(X)| > |I_1| + |I_2|$, a contradiction.

Similarly, if C_1 is not expansive, then $|N(X) \cap C_1| < |X|$ for some $X \subset C_1$, and hence we have $|I_3 \setminus N(X)| + |I_4 \cup X| > |I_3| + |I_4|$ for nonadjacent sets

$I_3 \setminus N(X)$ and $I_4 \cup X$.

And at last A_1 is expansive. Note that A_1 is adjacent to H_1 and $|H_1| = |F_1|$. If A_1 is not expansive, then $|N(X) \cap A_1| < |X|$ for some $X \subset A_1$ and we also have a contradiction, because $|I_3 \cup X \cup F_1| + |I_4 \setminus (N(X) \cup H_1)| > |I_3| + |I_4|$ for nonadjacent sets $I_3 \cup X \cup F_1$ and $I_4 \setminus (N(X) \cup H_1)$.

Assume now that (i) through (vi) is true. First we claim that

$$A_k \cup B_k \cup F_k = I_k \setminus I_{k+2}.$$

To see this note that $I_k \setminus (I_{k+2} \cup I_{k+3}) = (I_k \setminus I_{k+2}) \cap (I_k \setminus I_{k+3})$ and therefore the left side of the above equality is

$$(I_k \setminus I_{k+2}) \cap [(I_{k+3} \setminus I_{k+1}) \cup (I_k \setminus I_{k+3}) \cup I_{k+1}] = (I_k \setminus I_{k+2}) \cap (I_k \cup I_{k+1} \cup I_{k+3}) = I_k \setminus I_{k+2}$$

which proves the claim. Similarly we see that

$$A_k \cup H_k \cup D_k = I_{k+3} \setminus I_{k+1}.$$

It follows from (iii) and (iv) that

$$|I_{k+3} \setminus (I_k \cup I_{k+1})| = |(I_k \setminus I_{k+2}) \cap I_{k+1}|$$

and

$$|I_k \setminus (I_{k+2} \cup I_{k+3})| = |(I_{k+3} \setminus I_{k+1}) \cap I_{k+2}|$$

and hence

$$|I_k \setminus I_{k+2}| = |I_{k+3} \setminus I_{k+1}|\,.$$

By (i) and (ii) we have

$$|I_k| - |I_{k+2}| = |I_k \setminus I_{k+2}| = |I_{k+3} \setminus I_{k+1}| = |I_{k+3}| - |I_{k+1}|$$

and therefore $|I_k| + |I_{k+1}| = |I_{k+2}| + |I_{k+3}|$ for every odd k. To prove that I is a maximum independent set in $G \times P_n$ we have to prove that $|I_1| + |I_2|$ is maximum. That is, for any pair of nonadjacent subsets J_1 and J_2 in G we have

$$|I_1| + |I_2| \geq |J_1| + |J_2|.$$

In Figure 3 the sets I_1, I_2, I_3 and I_4 are shown. Note that the picture is as general as possible because $I_3 \subseteq I_1$ and $I_2 \subseteq I_4$.

Let $R = V(G) \setminus (I_1 \cup I_4)$. It follows from (iii) and (iv) that $H_k \equiv F_k$ and $B_k \equiv D_k$ for every odd k. From (vi) we have G_1, C_1, A_1 are expansive, and (v) implies that $R \cup H_1 \lll E_1 \cup F_1$. Assume that J_1 and J_2 are nonadjacent. We claim that

$$|J_2 \setminus I_2| \leq |I_1 \setminus J_1| \text{ and } |J_1 \setminus I_1| \leq |I_2 \setminus J_2|\,.$$

Since A_1 is expansive we have $|(J_2 \setminus I_2) \cap A_1| = |J_2 \cap A_1| \leq |N(J_2 \cap A_1) \cap A_1|$. Since J_1 and J_2 are nonadjacent, we find that $(N(J_2 \cap A_1) \cap A_1) \cap J_1 = \emptyset$ and so $N(J_2 \cap A_1) \cap A_1 \subseteq (I_1 \setminus J_1) \cap A_1$. So we have $|(J_2 \setminus I_2) \cap A_1| \leq |(I_1 \setminus J_1) \cap A_1|$. Similarly we find that $|(J_2 \setminus I_2) \cap C_1| \leq |(I_1 \setminus J_1) \cap C_1|$. Since $R \cup H_1 \lll E_1 \cup F_1$ we have $|(J_2 \setminus I_2) \cap (R \cup H_1)| \leq |(I_1 \setminus J_1) \cap (E_1 \cup F_1)|$. From $B_1 \equiv D_1$ follows $|(J_2 \setminus I_2) \cap B_1| \leq |(I_1 \setminus J_1) \cap D_1|$ and $|(J_2 \setminus I_2) \cap D_1| \leq |(I_1 \setminus J_1) \cap B_1|$. Combining all inequalities we get $|J_2 \setminus I_2| \leq |I_1 \setminus J_1|$. The proof of $|J_1 \setminus I_1| \leq |I_2 \setminus J_2|$ is analogous. To prove that $|I_1| + |I_2| \geq |J_1| + |J_2|$ we write $|J_k \setminus I_k| = |J_k| - |J_k \cap I_k|$ and $|I_k \setminus J_k| = |I_k| - |J_k \cap I_k|$ for $k = 1, 2$. A straightforward calculation gives the desired inequality. Since for any independent set J, J_k and J_{k+1} are nonadjacent we have $|J_k| + |J_{k+1}| \leq |I_1| + |I_2| = |I_k| + |I_{k+1}|$ for every odd k. This proves $|J| \leq |I|$. ■

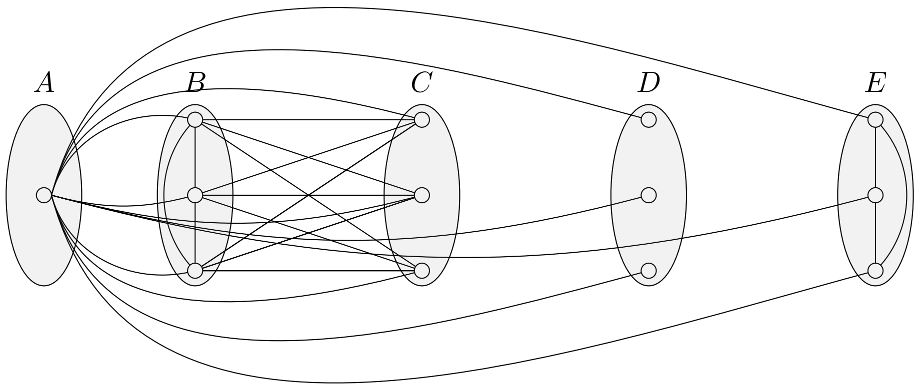

Figure 4. A graph G such that $P_8 \times G$ has a maximum independent set which is not a union of two rectangles.

We give an example of a graph G, such that $P_8 \times G$ has maximum independent set which is not a union of two rectangles. The graph G is shown in Figure 4 and the maximum independent set in $P_8 \times G$ is shown in Figure 5. It is straightforward to check that (i)–(vi) of Theorem 5 holds.

Corollary 6. *For every even n and every graph G there is a maximum independent set I in $P_n \times G$ such that*

$$I = (A \times C) \cup (B \times D)$$

for some nonadjacent $C, D \subseteq V(G)$.

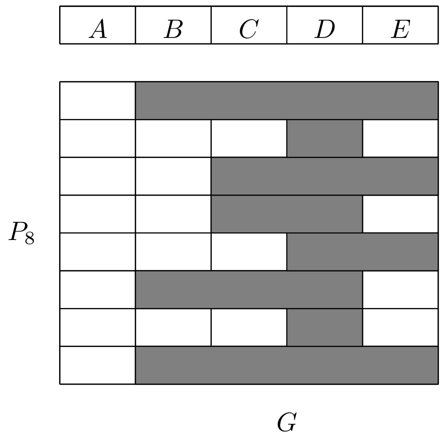

Figure 5. A maximum independent set in $P_8 \times G$.

Proof. The argument is similar to the argument in the above proof. If I is a maximum independent set in $P_n \times G$, then I_k and I_{k+1} are nonadjacent and $|I_k| + |I_{k+1}|$ is maximum for every odd k. The converse is also true. Therefore if we choose $I_\ell = C$ for every odd ℓ and $I_\ell = D$ for every even ℓ, and C and D are maximum nonadjacent, we find that I has the desired structure as claimed. ∎

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