Discussiones Mathematicae
Graph Theory 36 (2016) 23–30
doi:10.7151/dmgt.1836

SPLIT EULER TOURS IN 4-REGULAR PLANAR GRAPHS

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Abstract

The construction of a homing tour is known to be NP-complete. On the other hand, the Euler formula puts sufficient restrictions on plane graphs that one should be able to assert the existence of such tours in some cases; in particular we focus on split Euler tours (SETs) in 3-connected, 4-regular, planar graphs (tfps). An Euler tour S in a graph G is a SET if there is a vertex v (called a *half vertex* of S) such that the longest portion of the tour between successive visits to v is exactly half the number of edges of G. Among other results, we establish that every tfp G having a SET S in which every vertex of G is a half vertex of S can be transformed to another tfp G' having a SET S' in which every vertex of G' is a half vertex of S' and G' has at most one point having a face configuration of a particular class. The various results rely heavily on the structure of such graphs as determined by the Euler formula and on the construction of tfps from the octahedron. We also construct a 2-connected 4-regular planar graph that does not have a SET.

Keywords: 4-regular, 3-connected, planar, split Euler tour, NP-complete.

2010 Mathematics Subject Classification: 05C30, 05C75, 05C10.

1. Introduction

Froemke *et al.* [6] have shown that the construction of a homing tour is NP-complete. Herein, we study the special case of split Euler tours (SETs) in 4-regular planar graphs. Lehel [11], Manca [12], and Broersma *et al.* [2], respectively, have studied closely the construction of 4-regular planar 3-connected graphs from the octahedron. By utilizing these results, one may construct infinite classes of such graphs admitting SETs. Moreover, we show that every tfp G having a SET S in which every vertex of G is a half vertex of S can be transformed to another tfp G' having a SET S' in which every vertex of G' is a half vertex of S' and G' has at most one point having a face configuration of a particular class. We also construct a 2-connected 4-regular planar graph that does not have a SET.

2. Preliminaries

We use the terminology of Bondy and Murty [1]. All graphs are finite, simple, and undirected.

This work is devoted to the study of the existence of a *split Euler tour* in 4-regular planar Eulerian graphs. For brevity we will often abbreviate split Euler tour as *SET*. An Euler tour S in a graph G is a SET if there is a vertex v (called a *half vertex* of S) such that the longest portion of the tour between successive visits to v is exactly half the number of edges of G.

If T is a SET of G with half vertex v_0 and n vertices then T may be written as a listing of its vertices $v_0v_1v_2\cdots v_{n-2}v_{n-1}v_0v_1'v_2'\cdots v_{n-1}'v_0$ and we will refer to each of the subtours $v_0v_1v_2\cdots v_{n-2}v_{n-1}v_0$ and $v_0v_1'v_2'\cdots v_{n-1}'v_0$ as *halves* of T. We will often denote such halves by drawing the graph and illustrating one half of a SET by dotted lines and the other half by solid lines.

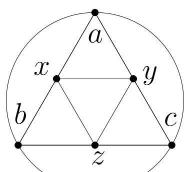

Figure 1

Following the notation of [2], we will say that a graph is a *tfp* if and only if it is 3-connected, 4-regular, and planar. The octahedron $\mathcal{O}$ (see Figure 1) is the basis for the construction of all 4-regular planar 3-connected graphs. We adopt additional notation of [2] in defining three operations, their inverse operations, and associated graphs. By Theorem 1 of [2], there is for each tfp G a finite sequence of tfps $G_0, G_1, \ldots, G_n$ such that $G_0 = \mathcal{O}$, $G_n = G$, G_i is a tfp for each $i = 0, 1, \ldots, n$, and, for each $i = 1, \ldots, n$, G_i is constructible from G_{i-1} from one of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are as below. The first main result of this work is the construction of an infinite family of tfps having a SET S in which each vertex of G is a half vertex of S using combinations of $\mathcal{B}$ and $\mathcal{C}$.

We shall say that a planar graph G has been *enlarged* by the operations $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ when the configurations A^-, B^-, or C^- have been substituted with A^+, B^+, or C^+, respectively. If G' is a graph that is obtained from enlarging graph G as above, we also say that G' is *constructible* from G. Similarly, we shall say that a planar graph G has been *reduced* by the operations $\mathcal{A}^{-1}$, $\mathcal{B}^{-1}$, or $\mathcal{C}^{-1}$ when the configurations A^+, B^+, or C^+ have been substituted with A^-, B^-, or C^-, respectively.

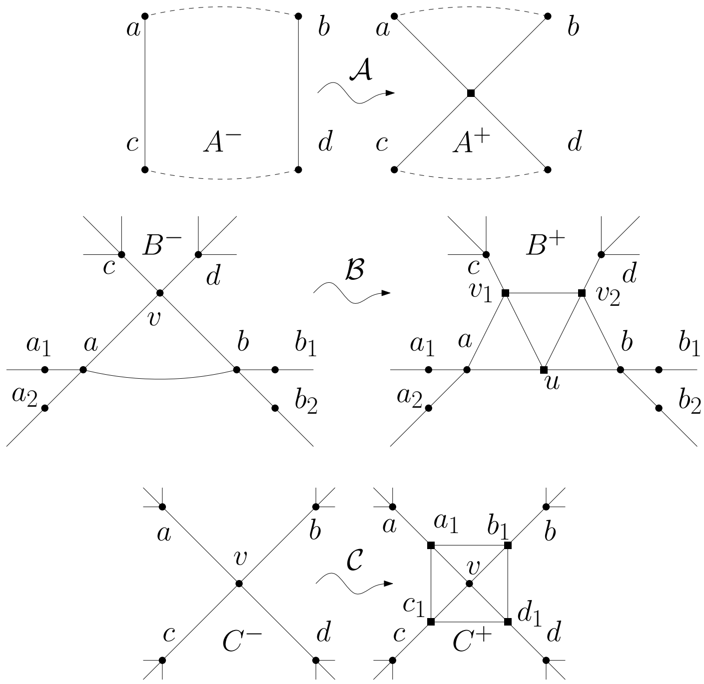

Figure 2

Note that in B^+, there is no edge ab. The generalized versions of the following theorems appear in [12], but we restrict to the case where G is a tfp.

Theorem 1. *If G is a tfp and one of the operations $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ is performed, then the resulting graph is also a tfp.*

Theorem 2. *If G is a tfp and one of the operations $\mathcal{A}^{-1}$, $\mathcal{B}^{-1}$, or $\mathcal{C}^{-1}$ is performed, then the resulting graph is also a tfp.*

Theorem 3. *If G is a tfp and not isomorphic to the graph of the octahedron, $\mathcal{O}$, then G has at least one of the configurations A^+, B^+, or C^+.*

We now classify *control points* in the tfps. From a classical theorem of Lebesgue (see [3], [9], or [10]), it follows that every tfp graph contains a vertex x incident with either

1. three triangles and one k-gon, $k \geq 3$, or
2. with two triangles, one quadrangle, and one l-gon, $4 \leq l \leq 11$; or
3. with two triangles, one pentagon, and one m-gon, $5 \leq m \leq 7$, or
4. with one triangle, two quadrangles, and one n-gon, $4 \leq n \leq 5$.

If vertex x has property (1), it is called a control point of Type I. If x has one of the properties (2), (3), or (4), it is called a control point of Type II.

3. Main Results

We begin with the following lemma, with its simple proof left to the reader.

Lemma 4. *The octahedron $\mathcal{O}$ has a SET S such that each vertex of $\mathcal{O}$ is a half vertex of S.*

Lemma 5. *If G and G' are tfps, G has a SET S such that each vertex of G is a half vertex of S, and G' can be obtained from G by $\mathcal{B}$, then G' has a SET S' such that each vertex of G' is a half vertex of S'.*

Proof. We will fix the notation of operation B in Figure 2. Assume that S has halves S_1 and S_2. We will construct S' with halves S_1' and S_2'. Without loss of generality, suppose $cv \in S_1$.

Case 1: Suppose that $bv \in S_1$, $ab \in S_1$. Without loss of generality, further suppose that $aa_1 \in S_1$. Then $dv \in S_2$, $va \in S_2$, and $aa_2 \in S_2$ (and $bb_1 \in S_2$ and $bb_2 \in S_2$).

Form SET S' in G' by replacing the path $cvba$ in S_1 with cv_1v_2bua and by replacing the path dva in S_2 with dv_2uv_1a. Note that the case where $va \in S_1$, $ab \in S_1$, and, without loss of generality, $bb_1 \in S_1$ is parallel.

Case 2: Suppose that $va \notin S_1$ and $vb \notin S_1$. Then $vd \in S_1$, $va \in S_2$, and $vb \in S_2$. It must also be the case that $ab \in S_1$ and, without loss of generality, $aa_1 \in S_1$ and $bb_1 \in S_1$.

Form SET S' in G' by replacing the path cvd in S_1 with cvv_1d and the edge ab in S_1 with the path aub, and by replacing the path avb in S_2 with av_1uv_2b.

All other cases are very similar. ■

Lemma 6. *If G and G' are tfps, G has a SET S such that each vertex of G is a half vertex of S, and G' can be obtained from G by $\mathcal{C}$, then G' has a SET S' such that each vertex of G' is a half vertex of S'.*

Proof. We will fix the notation of operation C in Figure 2. Assume that S has halves S_1 and S_2. We will construct S' with halves S_1' and S_2'. Without loss of generality, suppose $av \in S_1$. Now, assume that $vd \in S_1$. Thus $bv \in S_2$ and $cv \in S_2$.

Form SET S' in G' by replacing path avd in S_1 with path $aa_1c_1vb_1d_1d$ and by replacing path bvc in S_2 by path $bb_1a_1vd_1c_1c$. The cases that $vb \in S_1$ or $vc \in S_1$ are similar; all other cases follow by symmetry. ■

Theorem 7. *If G is tfp and there is a finite sequence $G_0, G_1, \ldots, G_n$ such that $G_0 = \mathcal{O}$, $G_n = G$, and, for each $i = 1, \ldots, n$, G_i is constructible from G_{i-1} from one of $\mathcal{B}$ and $\mathcal{C}$, then G has a SET S such that each vertex of G is a half vertex of S.*

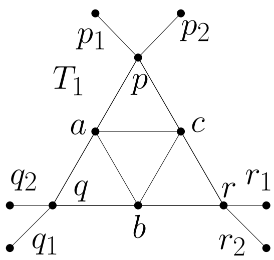

Figure 3

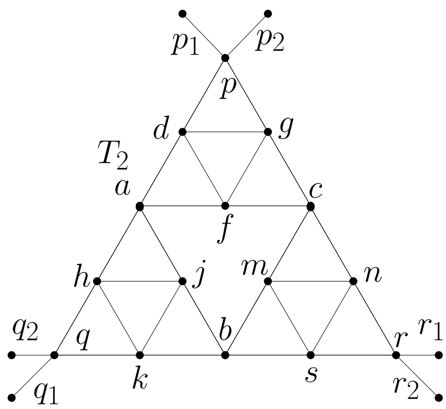

Figure 4

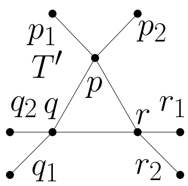

Figure 5

Lemma 8. *Suppose tfp G contains one of the subgraphs, T_1 or T_2, as in Figure 3 and Figure 4, respectively. Further suppose that G has SET S such that each vertex of G is a half vertex of S. Let T' be as in Figure 5 and let G' be the tfp constructed by replacing T_1 or T_2 in G by T'. Then G' is a tfp and has SET S' such that each vertex of G' is in each half of S'.*

Proof. Suppose G contains T_1. S meets T_1 such that each of the paths p_1pp_2 and $q_iqbacrr_j$ with i, j in $\{1, 2\}$ are in one half of S (S_1) and $q_{3-i}qapcbrr_{3-j}$ are in the other (S_2).

We construct G' by replacing T_1 in G by T'. As an example, with $i = 1$ and $j = 1$, we construct SET S' in G' such that S' has halves S'_1 and S'_2 defined so that $q_2qbacrr_2$ in S_1 is replaced by q_2qrr_2 and $q_1qapcbrr_1$ in S_2 is replaced by q_1qprr_1.

We may iterate the construction above to yield the result for T_2.

As an example, suppose S meets T_2 such that each of the paths p_1pp_2 and $q_2qkbjhadgfcnmsrr_2$ are in half S_1 of S and $q_1qhkjagdpfcnmsrr_1$ is in the other half, S_2.

Construct G' by replacing T_2 in G by T'. We construct S' in G' so that S' has halves S'_1 and S'_2 defined so that $q_2qkbjhadgfcnmsrr_2$ in S_1 is replaced by q_2qrr_2 and $q_1qhkjagdpfcnmsrr_1$ is replaced by q_1qrr_2 in S_2.

The remaining cases are very similar; their constructions are left to the reader. ■

In the following theorem, the term *elementary transformation* will mean the replacement of a triangular face of tfp G with one of the "triangular" subgraphs of T_1 or T_2 of Lemma 8.

Theorem 9. *Every tfp G having a SET S such that each vertex of G is a half vertex of S can be transformed using* (*at most*) *two elementary transformations to a tfp G' having a SET S' in which every vertex of G' is a half vertex of S and G' has at most one control point of Type* II.

Proof. Let G be a tfp having a SET S such that each vertex of G is a half vertex of S. If G has fewer than 2 control points of Type II, then we are done. Hence, we will assume G has at least 2 control points of Type II.

Case 1: There is a Type II control point w with face with two triangles, one quadrangle, and one l-gon, $4 \leq l \leq 11$. We replace one of the triangular faces on which w lies with the appropriate triangular subgraph of T_1 or T_2 such that w is no longer a control point of Type II.

The resulting graph is a tfp with fewer control points of Type II than G and, by Lemma 8, this graph has a SET such that each vertex of the graph is in each half of the SET.

Case 2: Suppose we do not have Case 1 and H has a control point w with face with two triangles, one pentagon, and one m-gon, $5 \leq m \leq 7$. Again, we replace a triangular face with T_1 so that w is no longer a control point of Type II. The resulting graph is a tfp with fewer control points of Type II than G and, by Lemma 8, this graph has a SET such that each vertex of the graph is in each half of the SET.

Case 3: Suppose we have neither Case 1 nor Case 2. Then there exists a control point w with face with one triangle, two quadrangles, and one n-gon, $4 \leq n \leq 5$. We use the same technique as in Case 2.

We iteratively apply the algorithm implied by the 3 cases (in the order presented) until the constructed graph contains fewer than 2 control points of Type II. ■

In a related result, one can show that each tfp H with no Type II control points and with all faces sizes 3 or 4 has an SET S such that each vertex of H is a half vertex of S. One method of proof is to induct on the number of 4-faces of H.

We also offer the conjecture that the question of determining whether each tfp G has a SET S such that each vertex of G is a half vertex of S is NP-complete. In particular, we suspect that it is possible to show directly its equivalence to the NP-complete question of deciding whether a planar, cubic, 3-connected graph with minimum face size 5 admits a Hamiltonian circuit [8].

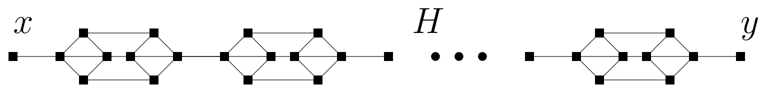

Figure 6

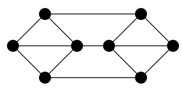

Figure 7

Theorem 10. *There exists a* 2*-connected* 4*-regular planar graph that does not have a SET.*

Proof. Let G be any 2-connected 4-regular planar graph, such as the octahedron, with n vertices. Define H to be the graph in Figure 6, where H contains n copies of the graph in Figure 7.

Let G' be the graph obtained by replacing an edge xy in G by graph H. It is clear that G' is 2-connected, 4-regular, and planar, but cannot contain any SET. ■

Acknowledgements

The authors are grateful to Professor Arthur Hobbs and to Professor J. Grossman for valuable email correspondence regarding the topics in this work. The authors also acknowledge with gratitude the referee for many helpful comments and suggestions.

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Received 19 December 2013
Revised 23 February 2015
Accepted 19 March 2015