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RAINBOW TETRAHEDRA IN CAYLEY GRAPHS

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Abstract

Let Γ_n be the complete undirected Cayley graph of the odd cyclic group $\mathbf{Z}_n$. Connected graphs whose vertices are rainbow tetrahedra in Γ_n are studied, with any two such vertices adjacent if and only if they share (as tetrahedra) precisely two distinct triangles. This yields graphs G of largest degree 6, asymptotic diameter $|V(G)|^{1/3}$ and almost all vertices with degree: (a) 6 in G; (b) 4 in exactly six connected subgraphs of the $(3,6,3,6)$-semiregular tessellation; and (c) 3 in exactly four connected subgraphs of the $\{6,3\}$-regular hexagonal tessellation. These vertices have as closed neighborhoods the union (in a fixed way) of closed neighborhoods in the ten respective resulting tessellations.

Keywords: rainbow triangles, rainbow tetrahedra, Cayley graphs.

2010 Mathematics Subject Classification: 05C15, 05C75, 05C62.

1. Introduction

Cayley graphs are very important because they have many useful applications (cf. [11]) and are related to automata theory (cf. [12, 13]). In the present work, we deal with Cayley graphs of a finite abelian group G with its identity denoted 0. Let S be a subset of G such that $0 \notin S$ and $S = -S$ (that is: $s \in S$ if and only if $-s \in S$). The *Cayley graph* $\Gamma(G, S)$ on G with *connection set* S is a graph that has as its vertices the elements of G and is such that it has an edge e joining vertices g and h if and only if $h = g + s$, for some $s \in S$. In this case, we say that the edge e has *color* s. A concept of "rainbow" has been used in various fashions in a graph theory context, in [1, 2, 3, 8, 9, 10, 14, 15, 16, 17, 18, 19, 20, 21] and related papers. Ours is in relation to edge colors in Cayley graphs of finite cyclic groups. Below, the complete graph $K_n = K_{2k+1}$ will be viewed as the Cayley graph $\Gamma_n = \Gamma(\mathbf{Z}_n, [k])$ of the cyclic group $\mathbf{Z}_n$ of integers modulo n with connecting set $[k] = \{1, 2, \ldots, k\}$. Relations among *rainbow* triangles and tetrahedra in Γ_n (*rainbow* meaning here edges with pairwise different colors) will be shown to yield a family $\mathcal{G}_1$ of connected graphs $G = G_{n,4}$ of largest degree $\Delta(G) = 6$, asymptotic diameter $|V(G)|^{1/3}$ and such that almost all its vertices v have degree: (a) 6 in G; (b) 4 in exactly six connected subgraphs of the $(3, 6, 3, 6)$-semi-regular tessellation ([7], page 43); and (c) 3 in exactly four connected subgraphs of the $\{6, 3\}$-regular hexagonal tessellation ([7], page 43). We refer to each of these ten subgraphs of G as a $\mathcal{D}$- or as an *$\mathcal{H}$-modeled subgraph* of G if it is as in (b) or as in (c) above, respectively. On the other hand, based on rainbow triangles a family $\mathcal{G}_0$ of connected graphs $G = G_{n,3}$ of largest degree $\Delta(G) = 3$ and asymptotic diameter $|V(G)|^{1/2}$ was introduced in [5]. See Section 3 below for a short survey of [5] and for further developments ahead in this paper.

The mentioned asymptotic properties of the families $\mathcal{G}_0$ and $\mathcal{G}_1$ confirm the following conjecture, further discussed in [6].

Conjecture 1. *The asymptotic diameter of a family of graphs G with a common $\Delta(G)$ is a given (radical, logarithmic, ...) function of the vertex number of G.*

2. Main Results

The present paper is devoted to the following results, containing the claimed properties of $\mathcal{G}_1$. (For related properties, see [4] and its references.) The *tessellated neighborhood* of a vertex v in a $\mathcal{D}$- or $\mathcal{H}$-modeled subgraph G is formed by v and its incident edges and faces as well as by the other edges adjacent to those faces and the endvertices of these edges.

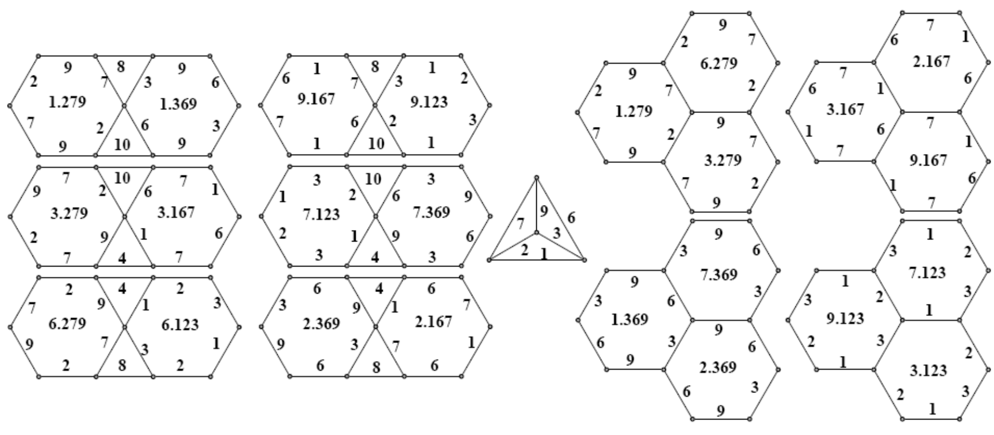

Figure 1. Tessellated neighborhoods of a vertex of $G_{n,4}$ in the subgraphs of Theorem 2.

Theorem 2. *There exists an infinite family $\mathcal{G}_1$ of finite connected graphs $G = G_{n,4}$ with asymptotic diameter $|V(G)|^{1/3}$ such that that the subset V_6 of vertices $v \in V(G)$ with $\deg(v) = \Delta(G) = 6$ has asymptotic order $|V(G)|$. In that case, almost every $v \in V_6$*

1. *is incident to three triangles T_0, T_1, T_2 in G with pairwise intersection $\{v\}$ determining exactly six planar $\mathcal{D}$-modeled subgraphs $D^k_{i,j}$ ($i, j = 0, 1, 2$; $k = 0, 1$) such that $T_i \cup T_j = D^0_{i,j} \cap D^1_{i,j}$ for each pair $\{i, j\} \subset \{0, 1, 2\}$ with $i \neq j$;*
2. *is the intersection of the six $\mathcal{D}$-modeled subgraphs of G above, in which $\deg(v) = 4$, and exactly four $\mathcal{H}$-modeled subgraphs in G, in which $\deg(v) = 3$, and such that the closed neighborhood of v in G is contained in a fixed way in the union of the tessellated neighborhoods of v in the ten cited subgraphs, comprising 43 vertices.*

To give an idea of what is going on locally at almost every vertex in the context of Theorem 2, Figure 1 shows on its left (respectively, right) side the closed (respectively, tessellated) neighborhoods of a particular vertex v—given by the edge-colored copy of K_4 (in $G_{n,4}$ or $G_{\infty,4}$) depicted at the figure center, see Section 5—in each of the ten subgraphs mentioned in the two items of the statement, namely, in the six $\mathcal{D}$-(respectively, four $\mathcal{H}$-) modeled subgraphs of $G_{n,4}$ claimed above, for a value of n sufficiently large, with edges colored via $a = 7$, $b = 9$, $c = 2$, $d = 3$, $e = 1$ and $f = 6$.

Corollary 3. *There is a subfamily $\mathcal{G}'_1$ of $\mathcal{G}_1$ such that any $D^k_{i,j}$ in a member G of $\mathcal{G}'_1$ is a $\mathcal{D}$-modeled subgraph restricted to a 30°-60°-90° triangular region of the Euclidean plane. Moreover, there are $n-1$ pairwise distinct such subgraphs $D^k_{i,h}$ distributed, for $y \geq 1$, into two subsets of size $\frac{n-1}{2}$ composed each by isomorphic subgraphs. By denoting these $\frac{n-1}{2}$-subsets by V^-_y and V^+_y, if $k = 5 + 2y$; respectively, U^-_y and U^+_y if $k = 4 + 2y$, with $|V^-_y| < |V^+_y|$ and $|U^-_y| < |U^+_y|$, then $|V^-_y| = y^2 + y - 1$ and $|V^+_y| = 3y^2 + 3y - 3 - \epsilon(k)$, where $\epsilon(k) = 1$ if $k \equiv 1 \pmod 3$ and $\epsilon(k) = 0$ if $k \not\equiv 1 \pmod 3$; respectively, $|U^-_y| = |V^-_y| - y$ and $|U^-_y| = |V^+_y| - 3y$.*

Figure 9 of [4] illustrates the 30°-60°-90°-triangular regions in Theorem 2; alternatively, see Figures 6 and 7. The proofs of Theorem 2 and Corollary 3 in Section 9 are composed by the arguments presented in Sections 3–9 and, for the $\mathcal{H}$-modeled subgraphs in item 2 of Theorem 2, by Theorem 2 of [4].

3. K_3-TYPES AND K_3-TYPE GRAPHS

A triangle in Γ_n has K_3-*type* (a, b, c) if its edges have colors $a, b, c \in [k]$. If no confusion arises, we suppress commas and parentheses, so we write $(a, b, c) = abc$. More generally, a K_3-*type* $abc = acb = bac = bca = cab = cba$ of $\mathbf{Z}_n$ is a 3-multiset $\{a, b, c\}$ of $[k] \cup \{0\}$ such that $a + b \in \{c, -c\} \in [k]$, where $a + b$ is taken modulo

n. (This 3-multiset can be viewed as a class of at most six 3-tuples of colors of $[k] \cup \{0\}$, one of which is abc.)

Example 4. The K_3-types $\{a,b,c\}$ of $\mathbf{Z}_7$ with $\gcd(a,b,c) = 1$ are $\{0,1,1\}$, $\{1,1,2\}$, $\{1,2,3\}$, $\{1,3,-(1+3)=3\}$ and $\{2,3,-(2+3)=2\}$, where the greatest common divisor $\gcd(J)$ of a finite multiset J of nonnegative integers is the largest common divisor of the nonzero integers of J.

Let G_n be the graph whose vertices are the K_3-types of $\mathbf{Z}_n$ and such that any two of them, say v and v', are adjacent via an edge ϵ if and only if v and v' share either two different colors of Γ_n or one color of Γ_n repeated twice, say a and a'; in either case we can consider ϵ as determined by $\{v,v'\}$ or by $\{a,a'\}$. We take $\{a,a'\}$ ($= aa'$, for short) as the *color* of ϵ, so that G_n becomes an edge-*colored* graph. In addition, we assume that G_n does not have multiple edges. In the example above, only 123 is rainbow. Each rainbow triangle t in Γ_n and edge ϵ of t determine exactly one rainbow triangle $t' \neq t$ with the same colors of t and sharing ϵ with t. For $n = 2k+1 \geq 7$, let $G'_n \subseteq G_n$ be the subgraph of G_n induced by the rainbow K_3-types of $\mathbf{Z}_n$. Let $G_{n,3}$ be the component of G'_n containing the K_3-type 123. Then all the remaining components of G'_n are isomorphic to graphs $G_{m,3}$ with $1 < m < n$ and $m|n$. Notice that the vertices of $G_{m,3}$ are 3-sets. Now, consider $\mathbf{N} = \{m \in \mathbf{Z} : m \geq 0\}$ as an infinite *color set*. A K_3-type abc of $\mathbf{Z}$, simply called a K_3-*type*, is a 3-multiset $\{a,b,c\}$ of $\mathbf{N}$ such that the sum of the two least colors equals the greatest one. Let $G_{\infty,3}$ be the graph whose vertices are the K_3-types abc with $\gcd(a,b,c) = 1$ and whose edges are as defined above for G_n. Given $m, m', n \in \mathbf{N}$ with $m' \in [k]$, we say that $m' \equiv m \pmod{n}$ whenever if for $m'' \equiv m \pmod{n}$ with $0 \leq m'' < n$:

(1) if $m'' > n/2$, then $m' = n - m''$;

(2) if not, then $m' = m''$.

Here, m' is said to be the *reduction of* $m \pmod{n}$. It was shown in [5], Proposition 2.16, that for odd $n \geq 7$, $G_{n,3}$ can be obtained, from a connected subgraph F of $G_{\infty,3}$ containing 011, 112, 123 and the remaining K_3-types with colors $\leq n$, by reducing modulo n all the colors of K_3-types of F. Let $\phi(n)$ be the value of Euler's totient function at the positive integer n. It was shown in Theorem 2.17 of [5] that $|V(G_{n,3})| = O(n\phi(n))$ and subsequently, in Theorems 2.20 and 2.21, that the diameter of $G_{n,3}$ is both $\Omega(n)$ and $O(|V(G_{n,3})|^{1/2})$. The family $\mathcal{G}_0$ in the introductory section above is formed by these graphs $G_{n,3}$.

4. K_4-Types and K_4-Type Graphs

A K_4-*type* of $\mathbf{Z}_n$ (respectively, $\mathbf{Z}$) is a maximal class of 6-tuples *abcdef* of colors of $[k]$ (respectively, $\mathbf{N}$) such that abc, cde, aef and bdf are K_3-types of $\mathbf{Z}_n$

(respectively, $\mathbf{Z}$). Such a class has at most twenty-four 6-tuples. A 6-tuple in a K_4-type t is called a *card* of t. If no confusion arises, we represent a K_4-type by one of its cards. A card *abcdef* will be represented

(i) either as a tetrahedron each of whose edges bears a color, as in Figure 2(a);

(ii) or by keeping only the locations of the colors in (i) in an enclosure, as shown in Figure 2(b). The colors in Figure 2(a) split into three different pairs of opposite colors: $\{a,d\}$, $\{b,e\}$, $\{c,f\}$, (opposite in the sense that each pair is held by a corresponding pair of edges of K_4 with no vertices in common, the remaining edges forming a 4-cycle).

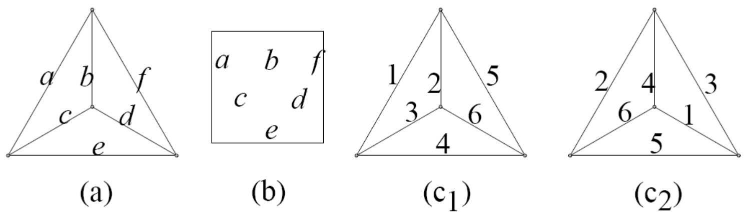

Figure 2. Representing a generic K_4-type *abcdef* and its cases modulo 13.

Any 6-multiset of $\mathbf{N}$ determines *at most* one K_4-type of $\mathbf{Z}$. This is not true for $(\mathbf{Z}_n,[k])$ in place of $(\mathbf{Z},\mathbf{N})$. For example, the two rainbow K_4-types 123645 and 246153 of $\mathbf{Z}_{13}$ represented in Figures 2(c_1) and 2(c_2), respectively, are distinct but have the same underlying multiset.

A *rainbow* K_4-type is one with six different colors. Given $n = 2k+1 \geq 13$, let $G'_{n,4}$ be the graph whose vertices are the rainbow K_4-types *abcdef* of $\mathbf{Z}_n$ with $\gcd(a,b,c,d,e,f,n) = 1$ and such that any two such vertices, say t and t', are adjacent via an edge ϵ if and only if t and t' looked upon as K_4-types share precisely two K_3-types v and v'. In this case, v and v' share exactly one color a of $[k]$. We take a as the (*weak*) *color* of ϵ and this makes $G'_{n,4}$ into an edge-colored graph.

In order to distinguish the $\mathcal{D}$- and $\mathcal{H}$-modeled subgraphs that we claim $G'_{n,4}$ contains, we introduce the graph $G''_{\infty,4}$ as the simple graph (i.e., graph without loops or multiple edges) whose vertices are the K_4-types *abcdef* with $a \neq d$, $b \neq e$ and $c \neq f$ unless *abcdef* $= 011011$ and satisfying $\gcd(a,b,c,d,e,f) = 1$, with two vertices u and v determining an edge if and only if they share precisely two K_3-types in differing locations of the representation of the K_4-types that stand for u and v as in Figure 1.

Figure 3 illustrates $G''_{\infty,4}$ as well as Theorem 5 below. The figure represents a neighborhood N of the K_4-type 123745 in $G''_{\infty,4}$. Notice that the two right-lower K_4-types in Figure 3 (joined by the edge colored with 6) are not rainbow. An edge ϵ joining two vertices t and t' of $G''_{\infty,4}$ with respective cards r and r' determines a

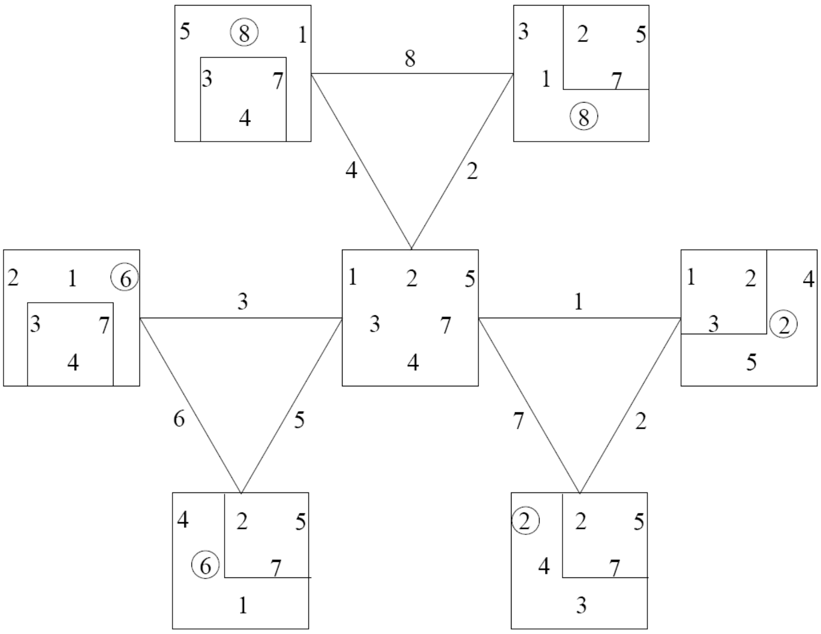

Figure 3. A neighborhood of 123745 in $G''_{\infty,4}$.

K_3-type s common to t and t' and *equally located* in r and r' in the sense that the component colors of s occupy the same positions in r and r', just as the K_3-type $s = 123$ is not only common to but also equally located in the central card in Figure 3 and the card horizontally located at its right, with s occupying the three upper-left locations in r and r'. The *locations* g_r of the colors in the cards r' of the statement of Theorem 5 obtained from the central card r at the center of Figure 3 are shown encircled. Also, the K_3-type s is highlighted in a sub-enclosure of its own. Observe that in each of the six enclosures representing the neighbors of the central vertex in Figure 3 the two colors outside the sub-enclosure and the encircled color are permuted in their positions.

Theorem 5. *Let $t \in V(G''_{\infty,4})$. Let r be a card of t with color g at location g_r and color g' at the location g'_r opposite to g_r. Then t has a neighbor t' with card r' differing from r just in*
(a) *the color at g_r and*
(b) *a permutation of the colors at the two locations $\neq g'_r$ in just one of the two K_3-types common to r and r' that contain the color at g_r.*

Proof. t' is determined from t as follows. Let s, s' be the two K_3-types not containing g_r in r. Then s and s' contain g'_r. We can assume that s' has its colors equally located in r and r'. Let i, j be the colors of r at the two locations $i_r \neq g'_r$ and $j_r \neq g'_r$ of s. Thus $s = ijg'$. The two other K_3-types in t apart from s and s' are of the form gij' and gji' with $s' = i'j'k$. We take r' as having the colors i, j exchanged with respect to r. So $(i_{r'}, j_{r'}) = (j_r, i_r)$. Let $\nu(a, b) = \{| a - b |\} \cup \{a + b\}$ for each pair of integers $a, b \geq 0$. There is at least one color $h \in \nu(i, j) \cap \nu(i', j') \neq \emptyset$ that yields r' when located at g_r (which should be called $h_{r'}$ in r') so that r' is formed by the K_3-types $s = ijg'$, $s' = i'j'g'$, hii' and hjj'. Moreover, r' does not depend on the selected card r of t. In fact $h = h(r, g_r)$ depends only on r and g_r. If $r = 011011$ and $g = 0$ then h equals either 0, yielding $t' = t$, not a distinct neighbor of t in $G''_{\infty,4}$ so we discard it, or 2, yielding a neighbor t' of t. Otherwise, since no remaining vertex of $G''_{\infty,4}$ is of the form $abcabc \neq 011011$, then $|\nu(i, j) \cap \nu(i', j')| = 1$, even if $(r, g) = (011011, 1)$. Thus, if either $r \neq 011011$ or $(r, g) = (011011, 1)$, then h is unique. ∎

Example 6. In the following special cases, g assumes subsequently colors f, a and d in a K_4-type t of card $r = abcdef$:

(A) applying Theorem 5 to $(r, g) = (112354, 4)$ (so $g = f$) yields $t' = t$ where $g_r = f_r = 4_r$ because exchanging $d_r = 1_r$ and $e_r = 1_r$ does not produce changes from r;

(B) applying Theorem 5 to $(r, g) = (011011, 0)$ (so $g = a, d$) yields, for $h = 2$, neighbors t', t'' with respective cards $r' = 211011$ and $r'' = 011211$ where $g_r = a_r, d_r$ respectively, but observe that $t' = t''$.

5. Canonical Triangles

Let $G_{\infty,4}$ be the supergraph of $G''_{\infty,4}$ obtained by adding to the vertices of $G''_{\infty,4} \backslash \{011011\}$ the loops offered by the method of vertex adjacency in Theorem 5 and Figure 3, taking each maximal set of loops incident to a common vertex and with a common color to have multiplicity 1. Then, a link or loop joining vertices t and t' in $G_{\infty,4}$ has the pair (s, s') in the proof of Theorem 5 as its *strong color* and the only color g' in s and s' that remains at the location $g'_r = g'_{r'}$ both in r and r' as its *weak color.* Let $G'_{\infty,4}$ be the graph obtained from $G_{\infty,4}$ by restriction to the vertices that are rainbow K_4-types.

Applying Theorem 5 to the colors g, g' of a pair of opposite edges of a vertex t of $G_{\infty,4}$ looked upon as a K_4-type with card r yields $h(r, g) = h(r, g')$. This determines in r two corresponding neighboring cards r' and r'' representing respective neighbors t' and t'' of t. The two K_3-types that r' and r'' share and those two that r and r' (respectively, r and r'') share constitute the four K_3-types of r' (respectively, r''). The resulting triangle, whose vertices t, t', t'' have respective cards r, r', r'', is said to be a *canonical triangle*, or *CT*. Since there are three pairs of opposite vertices in the card r associated to the vertex t of $G_{\infty,4}$, then there are at most three CTs incident to t. Since each $G'_{n,4}$ can be obtained from $G'_{\infty,4}$ via reduction modulo n, we have completed the proof of the following corollary.

Corollary 7. *The graphs $G'_{\infty,4}$ and $G'_{n,4}$ are edge-disjoint unions of* CT*s, at most three such* CT*s incident to each vertex.*

When two or three K_4-types in a CT $T = \{t, t', t''\}$ obtained as in Theorem 5 coincide (e.g., either $t = t' \neq t''$ or $t = t'' \neq t'$ or $t \neq t' = t''$ or $t = t' = t''$), then we say that T is a *degenerate CT*.

Example 8. (A) If t has $r = abcdef$ with $a, b > 0$, $c = a + b$, $d = a$, $e = b$, $f = |a - b|$ and $(g_r, g'_r) \in \{(a_r, d_r), (b_r, e_r)\}$, then $t' = t''$. This yields two degenerate CTs with vertices of the form t, t' and $t'' = t'$, where $tt' = tt''$ and $t't''$ is a loop of $G_{\infty,4}$.

(B) Theorem 5 applied to $t = 000111$ yields three degenerate CTs, each representable by: two vertices, namely t (twice) and $t' = 011011$, a link tt' and a loop at t; these three CTs coincide, since edges are assumed to have multiplicity 1.

(C) Theorem 5 applied to $t = 132112$ yields three CTs incident to t, one of which, obtained by making value changes in both cases of color $g = 2$ at opposite locations in t, has its three vertices equal to t, so this CT reduces to a looped vertex in $G_{\infty,4}$. The two remaining CTs incident to t are $\{t, 202111, 132201\}$ and $\{t, 431122, 132421\}$.

Corollary 9. *$G_{\infty,4}$ is connected.*

Proof. Given $t = abcdef$ and $t' = abcydx$ in $G_{\infty,4}$ there exists a 2-path in $G_{\infty,4}$ from t to t' with middle vertex card $abcfxd$ and edge strong colors $\{abc, bdf\}$ and $\{abc, adx\}$. Let cde and cxy be K_3-types of $\mathbf{Z}$ with $\gcd(c, d, e) = \gcd(c, x, y)$. Then there exists a path in $G_{\infty,4}$ whose ends have cards of the form $abcdef$ and $abcxyz$. This uses the fact that if $\gcd(c, d, e) = \gcd(c, x, y)$, then there is a path in $G_{\infty,3}$ from cde to cxy [5]. Thus, if $abcdef \in V(G_{\infty,4})$, then there exist: **(a)** a path in $G_{\infty,4}$ from 110110 to $110aa(a+1)$; **(b)** a path in $G_{\infty,4}$ from $110aa(a+1)$ to $aa0bbc$; **(c)** a path in $G_{\infty,4}$ from $aa0bbc$ to $abcdef$. Hence, every vertex of $G_{\infty,4}$ can be connected to 110110. ∎

6. Generation of $\mathcal{D}$-Modeled Subgraphs

Corollary 10. *The set of* CT*s of $G_{\infty,4}$ is in* 1-1 *correspondence with the family of* 4*-multisets or quadruples $abcd$ of colors of* $\mathbf{N}$ *such that:*

(a) $\nu(a, b) \cap \nu(c, d) \neq \emptyset$ *(or* $\nu(a, c) \cap \nu(b, d) \neq \emptyset$ *or* $\nu(a, d) \cap \nu(b, c) \neq \emptyset$*,*

(b) $\gcd(a, b, c, d) = 1$*, so at least one of a, b, c, d is nonzero.*

Proof. From Theorem 5 and Corollary 7, each CT of $G_{\infty,4}$ has its vertices as K_4-types sharing precisely four colors as in the statement. ■

Example 11. In Figure 3, the upper (respectively, lower-left, lower-right) CT has its vertices sharing the quadruple 1357 (respectively, 1247, 2345).

From now on, each CT will be denoted by its associated multiset in Corollary 10. Given a rainbow K_4-type $t = abcdef$, the CTs incident to t are obtained by deleting from t each one of the three pairs ad, be and cf, which yields respectively $bcef$, $acdf$ and $abde$.

Let $abcdef$ be a vertex of $G_{\infty,4}$ and let $C = acdf$ and $D = abde$ be two CTs in $G_{\infty,4}$ sharing just $abcdef$. Then $C \cup D$ is represented as a colored 5-vertex plane graph $B(t, a, d)$ where C and D participate as respective equilateral triangles $\overline{C}$ and $\overline{D}$, respectively, that share solely a vertex t (i.e., $\overline{C} \cap \overline{D} = \{t\}$) that stands for $abcdef$ and is center of a point symmetry that takes $\overline{C}$ onto $\overline{D}$ and viceversa. Thus, pairs of sides of $\overline{C}$ and $\overline{D}$ incident to t are set collinearly as in Figure 4. We require a to tag the centers of both $\overline{C}$ and $\overline{D}$, and the remaining colors of C and D to tag respectively the vertices of $\overline{C}$ and $\overline{D}$ internally. Then, d is the color tagging t internally in both $\overline{C}$ and $\overline{D}$. We tag each edge of $\overline{C}$ (respectively, $\overline{D}$) with the weak color of the corresponding edge of C (respectively, D), such that the weak color of each edge ϵ of $\overline{C}$ forms: **(a)** a K_3-type $s(\epsilon)$ with the colors tagging the endvertices of ϵ in $\overline{C}$; **(b)** another K_3-type $s'(\epsilon)$, with the central tagging color of $\overline{C}$ and the color tagging the vertex opposite to ϵ in $\overline{C}$. Notice that $\{s(\epsilon), s'(\epsilon)\}$ is the strong color of the image of ϵ in $G_{\infty,4}$. Let ϵ_C and ϵ_D be edges of $\overline{C}$ and $\overline{D}$, respectively, meeting at an angle of 120° at vertex t. Then the color d tagging t in both $\overline{C}$ and $\overline{D}$ forms with the colors tagging ϵ_C and ϵ_D the K_3-type $s(\epsilon_C) = s(\epsilon_D)$.

6.1. Growth of a $\mathcal{D}$-modeled subgraph

The growth of a $\mathcal{D}$-modeled subgraph of $G_{\infty,4}$ sprouting from $B(t, a, d) = \overline{C} \cup \overline{D}$ via Theorem 5 can be performed via the following properties deducible via Theorem 5 and enjoyed by the objects conceived in the previous paragraph with their tagging notation around $r = abcdef$ as shown in Figure 4(c) and illustrated in Figure 4(a)–(b).

(1) Given a CT $C = afgh$, let a be the central tag of $\overline{C}$ and let color f tag a vertex u in $\overline{C}$. Then there is a color i so that **(a)** $\nu(a, h) \cap \nu(f, g) = \{i\}$; **(b)** the edges $\epsilon = uu'$ in $\overline{C}$ with u' having tag g or h in $\overline{C}$ have color i, denoted $\gamma(\epsilon) = i$.

(2) Let ℓ be the line containing u and parallel to the unique edge of $\overline{C} \backslash u$. Then each pair (u, C) determines at most one remaining CT $D \neq C$ sharing u with C, so that $\overline{D} = \rho_\ell(\overline{C})$, where ρ_ℓ is reflection of the plane on ℓ, and having

(a) a as central tag;

(b) the tag f of u in $\overline{C}$ as tag of u in $\overline{D}$;

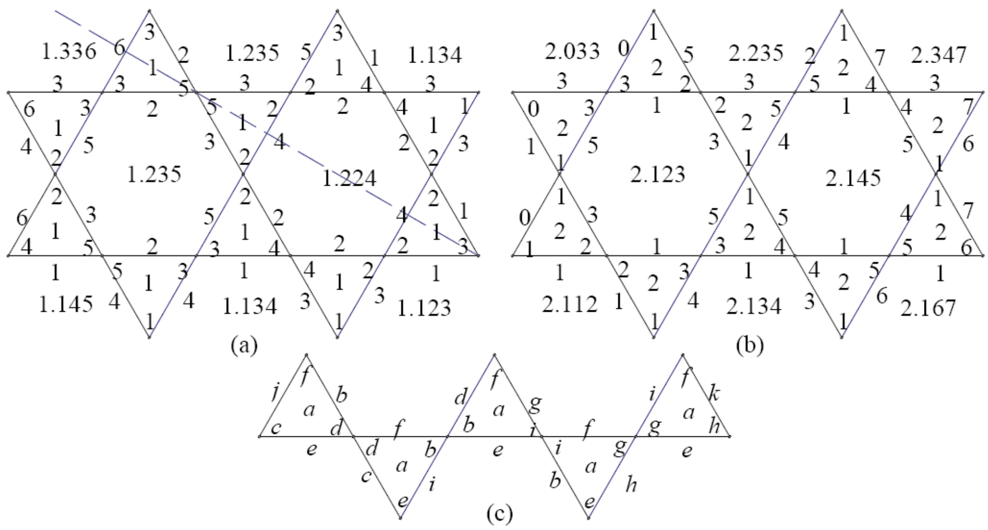

Figure 4. Unfoldings of subgraphs of $G_{\infty,4}$.

(c) for each edge $\epsilon = uu'$ of $\overline{C}$:

(i) $\gamma(\epsilon)$ as the tag of $\rho_\ell(u')$ in $\overline{D}$ and (ii) the tag of u' in $\overline{C}$ as the tag of $\rho_\ell(\epsilon)$.

(3) The vertex u is the K_4-type formed by the K_3-types determined by each edge ϵ of $\overline{D}$ incident to u and formed by:

(a) a and the tags of ϵ and the vertex opposite to ϵ in $\overline{D}$;

(b) the tags of ϵ and the endvertices of ϵ in $\overline{D}$.

The union of two CTs C and D that share precisely one vertex v is said to be a *butterfly* and denoted CvD. In this case, v is called the *central vertex* of CvD. Note that the colors of v in $\overline{C}$ and $\overline{D}$ equal a fixed color d which we call the *butterfly color* of CvD. For example, $B(t,a,d)$ above is a butterfly CtD with central color a and butterfly color d, say with $C = acdf$ and $D = abde$. Given a simple graph G and a pseudograph H (i.e., H is a non-simple graph in which each vertex may be incident to one or more loops), then G is an *unfolding* of H if there exists a surjective map $f : V(G) \to V(H)$ such that for each $v \in V(G)$ there exists a 1-1 correspondence induced by f from the links incident to v in G to the edges incident to $f(v)$ in H.

6.2. Maximal $\mathcal{D}$-modeled graphs

Let $t = abcdef$ be a rainbow K_4-type. A maximal $\mathcal{D}$-modeled graph $H' = H'(t,a) = H'(t,a,d) \supset B(t,a,d)$ that is an unfolding of an edge-disjoint union $H = H(t,a) = H(t,a,d)$ of butterflies in $G_{\infty,4}$ with common central color a is generated by repeated application of item (2), Subsection 6.1, at gradients 0°, 60°, 120°, 180°, 240°, 300° of the line ℓ in the item.

Example 12. Both Figure 4(a) and 4(b) show parts of an H' as above.

We will see that if such an H' is not a subgraph of $G_{\infty,4}$, then it can be *folded* along at most two *symmetry axes*, or *SA*s, to yield H. The dotted line in Figure 4(a) represents such an SA. In particular, edge colors will coincide by reflection in an SA. The graph obtained from H by removing the resulting loops will be seen to be a subgraph of H' spanning a connected region of the plane delimited by SAs. Edges crossing an SA at 90° will yield loops of H and each CT in H' will be incident to three hexagons.

Observation 13. *Given a vertex t of $H'(t,a,d)$, the three* CT*s incident to t according to Theorem* 5 *are:*

(a) *the two* CT*s incident to t in $H'(t,a,d)$ and*

(b) *the* CT *formed by the colors of the four edges of the two* CT*s in item* (a) *which are incident to t.*

7. Presence and Properties of 6-Cycles

The graph $H'(t,a,d)$ in Subsection 6.2 has two edge-disjoint 6-cycles with just the vertex t in common which are given by regular hexagons in the plane when the CTs of $H'(t,a,d)$ are represented as equilateral triangles as in the discussion after Example 11. This is the specific case in Subsection 7.2 below. If q is any of these 6-cycles, then its edges are colored with the component colors of a K_3-type s. In that case, we denote $q = a.s$, where a is the central color of the six CTs adjacent to q.

7.1. A procedure to determine 6-cycles

Let $bdf = s$ and $cde = s'$ be K_3-types, where $t = abcdef$ is a vertex of $H'(t,a,d)$. We will see that there exists a 6-cycle $(t^0, t^1, t^2, t^3, t^4, t^5)$ in $H'(t,a,d)$ containing $t = t^0$. It will be determined by the following procedure that yields t^i when t^{i-1} is given, successively for $i = 1,2,3,4,5$, (and returns to $t^0 = t^i$ from $t^5 = t^{i-1}$, if $i = 6 \equiv 0$ with indices taken modulo 6.

(a) Declare the card r^i of the K_4-type t^i to have color a (as in Figure 2(b)) fixed in the location a_{r^0} (so that $a_{r^i} = a_{r^0}$) during the entire procedure;

(b) denote locations $b_{r^i} = b_{r^0}$, $c_{r^i} = c_{r^0}$ and $e_{r^i} = e_{r^0}$ regardless of changes in their color values from the initial ones, namely b, c and e respectively along the running of the procedure;

(c) define color $h^i = b$ (respectively, $h^i = f$) if i is even (respectively, odd);

(d) establish a color exchange via a redesignation of locations at the i-th level: $d_{r^i} = h^{i-1}_{r^{i-1}}$ and $h^i_{r^i} = d_{r^{i-1}}$;

(e) the color e_{r^i} (respectively, c_{r^i}) if i is even (respectively, odd) takes the only value from $\nu(a_{r^i},f_{r^i})\cap\nu(c_{r^i},d_{r^i})$ (respectively, $\nu(a_{r^i},b_{r^i})\cap\nu(d_{r^i},e_{r^i})$).
This determines a well-defined card r^i and yields a location instance for the determination of a 6-cycle as claimed.

Example 14. A 6-cycle generated by the procedure in the previous paragraph and starting at $t^0 = 123745$ is

$$a.s = 1.257 = (123745, 123587, 156287, 156712, 176512, 176245).$$

Its accompanying coplanar 6-cycle $a.s'$ is

$$1.347 = (123745, 187345, 187434, 134734, 134376, 123476).$$

An essentially equivalent 6-cycle to this and sharing its first two vertices with $a.s'$ as just given is $7.145 = (123745, 583741, 48C751, 1BC754, 5B6714, 426715)$, where capital hexadecimal notation is used, and its accompanying coplanar 6-cycle is $7.123 = (123745, 321785, 23178A, 13279A, 312796, 213746)$,
sharing its first two vertices with $a.s$.

7.2. On 6-cycles containing specific K_4-types

Each t as above is contained in precisely two 6-cycles $q = a.s$ and $q' = a.s'$ of $H'(t,a,d)$. The edge-color sets of q and q' are respectively $\{b,d,f\}$ and $\{c,d,e\}$, each color tagging opposite edges. Moreover, the color tagging t in its incident CTs in $H'(t,a,d)$ and those tagging the two edges in q (respectively, q') that are incident to t conform s (respectively, s'). Furthermore, d is the color tagging t in its incident CTs in $H'(t,a,d)$ as well as tagging the two parallel edges of $a.bdf$ (respectively, $a.cde$) incident neither to t nor to its corresponding opposite vertex.

Given K_3-types bcd and $bc'd'$ with $b<c<d$ and $b<c'<d'$, define $bcd < bc'd'$ if and only if $c+d < c'+d'$. A graph $H' = H'(t,a,d)$ as in Subsection 6.2 is said to be a *T-subgraph* and denoted $a(s)$, where s is the smallest K_3-type $\neq 000$ coloring a 6-cycle of H' under '$<$', while $H = H(t,a,d)$ is denoted $a[s]$. Hexagons $a.s$ of an $H'(t,s,d)$ and their images in $H(t,a,d)$ are called *canonical hexagons* or *CH*s.

Proposition 15. *Let $H' = H'(t,a,d)$, where $t = abcdef$ is common to $C = acdf$ and $D = abde$, with $\overline{C}\cup\overline{D}\subset H'(t,a,d)$ and d tagging t in both $\overline{C}$ and $\overline{D}$. Then, the T-subgraph $H'' = H'(t,d,a)$ has t common to a flipped copy $\overline{\overline{D}}$ of $\overline{D}$ and a direct copy $\overline{\overline{C}}$ of $\overline{C}$. As a result, $d.caf$ and $d.bae$ contain the colors of the* CT*s incident to t in H''. Moreover, $H'' = H'$ if and only if $f = c$ and $e = b$.*

Proof. $H'' = H'(t,d,a)$ is established as follows:

(1) represent H'' as a temporarily uncolored T-subgraph and set t as one of its vertices;

(2) represent $\overline{\overline{C}}$ and $\overline{\overline{D}}$ in H'' as the respective CTs $\overline{C}$ and $\overline{D}$ of H' with common vertex t but set the locations of a and d in $\overline{\overline{C}}$ and $\overline{\overline{D}}$, instead, as those of d and a in $\overline{C}$ and $\overline{D}$, respectively;

(3) the vertex colors c and f in $\overline{\overline{C}}$ are exchanged with respect to their locations in $\overline{C}$ while the two vertex colors b and e in $\overline{\overline{D}}$ are left as in $\overline{D}$.

The remaining colors of H'' can be set uniquely as in Subsection 6.1 above. If $H'' \neq H'$, then reflection with respect to the line perpendicular to the line ℓ in Subsection 6.1 through t takes each edge color of $\overline{\overline{D}}$ in H'' to its location in $\overline{D}$, while the edge colors of $\overline{\overline{C}}$ remain as in $\overline{C}$. The statement follows immediately, as illustrated in Figure 4, where (b), at right, represents part of the T-subgraph H'' corresponding to the T-subgraph H', partly represented itself in (a), with $t = 235142$ at the center in both representations. ■

8. FROM $\mathcal{D}$-MODELED SUBGRAPHS TO CHARTS

Local plane representations of some subgraphs $a[s] = a[bcd]$ of $G_{\infty,4}$ are provided in Figure 5 with notation given before Proposition 15, $a = 10, d = 13, g = 16$ and thin (respectively, thick) edges for links (respectively, loops). In fact, the subgraphs induced by the set of links of these $a[s]$ yield subgraphs of the corresponding graphs $a(s) = a(bcd)$. Concretely, Figure 5 upper-left (respectively, upper-right) shows a plane region delimited by two dotted lines ℓ and ℓ' that form an internal angle of 30° (respectively, 90°) and determine a partial representation of $H'(s,1) = 1(011)$ (respectively, $H'(s,2) = 2(011)$), where $s = 110001$ (respectively, $s = 211011$). This representation can be identified with $H(s,1) = 1[011]$ (respectively, $H(s,2) = 2[011]$) by interpreting as a loop each thick edge interrupted perpendicularly by some dotted line ℓ. Moreover, $H'(s,1)$ (respectively, $H'(s,2)$) is obtained by unfolding $H(s,1)$ (respectively, $H(s,2)$) along the SAs formed by the lines in the finite sequence $\ell_0 = \ell$, $\ell_1 = \ell', \ldots, \ell_i =$ reflected line of ℓ_{i-2} on the line ℓ_{i-1}, for $i = 2, \ldots, k-1$, where additionally $\ell_{k-1} =$ reflected line of ℓ_1 on the line ℓ_0, with $k = 360°/30° = 12$ (respectively, $k = 360°/90° = 4$).

The extensions of these partial pictures to the plane will be referred to as *charts*. Observe that the two charts in the previous paragraph are the only charts of the form $H'(t,a)$ with $a = 1, 2$. However, no remaining value of a produces just one chart. For example, there are two charts $H'(s,3)$, one of which is $3(112)$, with $3[112]$ partially shown in the bottom of Figure 5, where two straight lines ℓ_0 and ℓ_1 at an angle of 60° delimit its representation, and with finite sequence $\ell_0, \ell_1, \ldots$, as above, of length $k = 360°/60° = 6$. The remaining $H'(s,3)$ is $3(011)$, with $3[011]$ having exactly one SA, delimiting a semi-plane representation. As a

increases its value, the first chart H not having an SA is $H = 6(123) = 6[123]$.

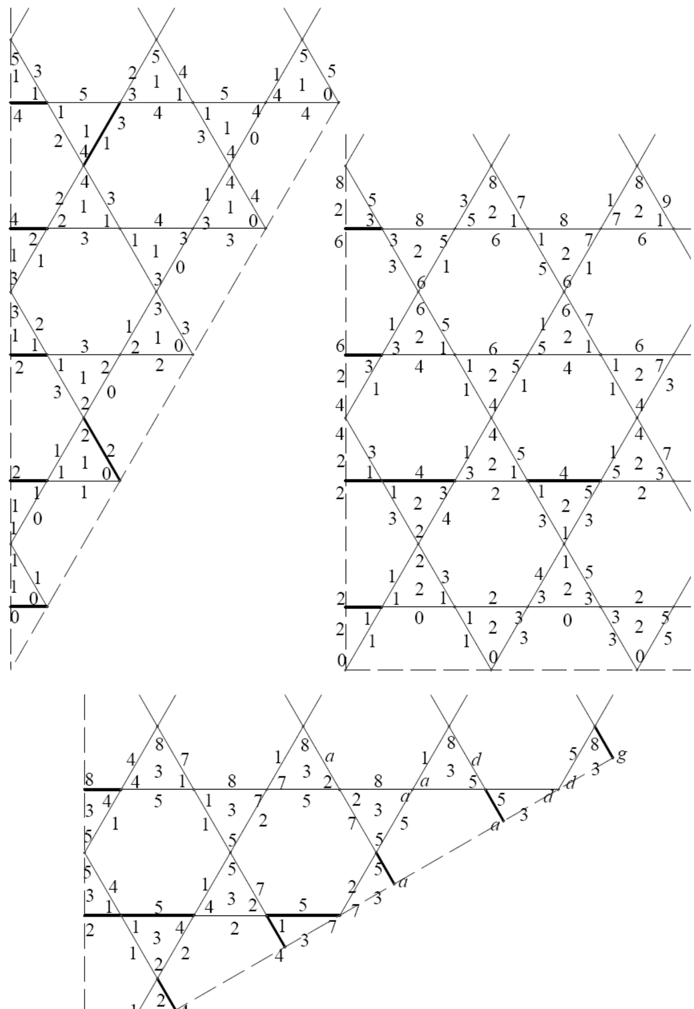

Figure 5. Charts for 1[011], 2[011] and 3[112].

8.1. Unfolding charts

To see how the unfolding of a graph $a(bcd)$ onto its corresponding $a[bcd]$ takes place, we observe that if $H(t,a) \neq H'(t,a)$, then $H(t,a)$ is obtained by folds of $H'(t,a)$ along SAs of two types:

1. SAs dividing all CHs of the form $a.0cc$ in symmetric halves through vertices colored with 0 in CTs of the form $a0cd$, i.e., through all vertices of the forms

$0bbcca$ and $0ccdda$;

2. SAs dividing all CHs of the form $a.0cc$ in symmetric halves through vertices colored with 0 in CTs of the form $a0cd$, i.e., through all vertices of the forms $0bbcca$ and $0ccdda$; the form $a.bbc$ in symmetric halves and passing at 90° through the midpoints of their edges colored with c (which are thick edges that yield loops) and through the vertices opposite to them in corresponding CTs.

In a chart H', a thick edge halved perpendicularly in its middle point by some SA yields a *half-edge* of H, and a CT that contains a half-edge yields a *half-CT* of H. Degenerate CT 1113, shown in the lower-left corner of the chart 3[112] in Figure 5, has its center as the intersection of two SAs (and three SAs in 3(112)) and constitutes the only *one-sixth-CT* of any chart of $G_{\infty,4}$. See also the example (C) before Corollary 9 in Section 5, where the CTs in their shown order are 1113, 1122 and 1123, the first two present in 3[112]. The following properties are observed:

1. A maximal connected region of an $H'(t,a)$ delimited by SAs but with its interior not intersecting any remaining SA yields a chart of $H(t,a)$.

2. Charts $a(bcd)$ and $a[bcd]$ exist, for $b \le c \le d$, if and only if $c+d \le a$.

3. Every loop of $G_{\infty,4}$ not in CTs $0011, 1111, 0112, 1113$ appears as a half-edge in two different charts and as a thick edge in a different one. The CT that contains such a loop: **(a)** is of the form $aabc$, where a,b,c are pairwise different and $(2a,b,c)$ is a K_3-type; **(b)** appears as a half-CT obtained by halving a degenerate CT as in the example (A) in Section 5 by means of an SA in $b[112]$ or $c[112]$, and as a 3-cycle in $a[011]$.

Two edges in a butterfly $B(t,a,d)$ are said to be *opposite* if none has t as an endvertex. Each butterfly has just one pair of opposite edges.

8.2. Color-alternating infinite paths

Any infinite path of $H' = H'(t,a) = a(bcd)$ contained in a line has successive edge tags in alternating colors f and g either differing in or adding up to a, the latter occurring precisely if both $f \le a$ and $g \le a$.

Denoting a path H' as above by $L(f,g,a)$, we have:

1. $f = g$ whenever $f = a/2 \in \mathbf{Z}$ or $g = a/2 \in \mathbf{Z}$; in this case, $d = a/2$ if $d \ge b,c$;

2. the edges colored $2a$ in $L(a,2a,a)$ are thick.

If two such paths are parallel and contiguous in H' then they are expressible as $L(f,g,a)$ and $L(h,f,a)$, with $|g-h|=2a$ or $g+h=2a$, the latter occurring precisely if both $g\leq 2a$ and $h\leq 2a$. Here, g,h are the edge colors opposite in the butterflies taking place between $L(f,g,a)$ and $L(h,f,a)$. The edges of $L(f,g,a)$ and $L(h,f,a)$ colored with f are divided into pairs of opposite edges of the CHs lying between $L(f,g,a)$ and $L(h,f,a)$.

Observation 16. *Given a vertex v of $H(t,a)$, let f,g,h,i be the colors of the edges incident to an unfolding vertex of v in $H'(t,a)$. If a is odd or if v is not in an $L(a/2,a/2,a)$ then there is exactly one other vertex u of H such that the edges incident to any unfolding vertex of u in H' have colors f,g,h,i. In this case u and v belong to $s=fghi$ and the edge uv has color a.*

We may assume that v is shared in $H(t,a)$ by $a.fgj$ and by $a.hij$ so that the edge of s having v as an endvertex but not having u as an endvertex is colored with j, and j colors v in s.

9. K_4-Types of $\mathbf{Z}_n$

Proposition 17. *Let $0<n=2k+1\in\mathbf{Z}$. There is a colored supergraph $G_{n,4}$ of the graph $G'_{n,4}$ introduced in* Section 4 *and a well-defined transformation Φ_n from $G_{\infty,4}$ onto $G_{n,4}$ that operates by replacing all colors of $\mathbf{N}$ tagging the objects, e.g. vertices, edges,* CT*s and* CH*s of $G_{\infty,4}$, by their image colors under reduction modulo n in the sense that all vertices* (*respectively, edges*) *with a common image modulo n color disposition can be identified to a corresponding vertex* (*respectively, edge*)*.*

Proof. Let A be the subset of vertices of the graph $G_{\infty,4}$ introduced in Section 5 whose colors have exclusively constituents $\leq k$ and let B be the set of neighbors of vertices of A in $G_{\infty,4}$. Let F be the graph induced by $A\cup B$ in $G_{\infty,4}$. By reducing modulo n all the colors tagging objects of F, the resulting color identifications in F yield $G_{n,4}$. Note that the reduction modulo n for vertices happens solely for the vertices of B. Once these vertices are reduced modulo n, they have the same colors as some vertices of A, so they must be identified correspondingly, and the edges from A to B are then transformed into edges joining vertices of A which were not originally induced by A in $G_{\infty,4}$. Now, Φ_n is defined by replacing the colors of the objects in $G_{\infty,4}$ (vertices, edges, CTs and CHs) by their reductions modulo n, which yields the corresponding objects in $G_{n,4}$. ∎

Observation 18. *The graph $G_{n,4}$ is an edge-disjoint union of possibly degenerate* CT*s, at most three incident to each vertex.*

Corollary 19. *$G_{n,4}$ is connected, for any odd positive integer n.*

Proof. Apply Corollary 9 and Proposition 17 to the (continuous) map Φ_n : $G_{\infty,4} \to G_{n,4}$. ■

Application of Φ_n to the charts of $G_{\infty,4}$ yields charts of $G_{n,4}$. The collection of charts of $G_{n,4}$, ($G_{\infty,4}$), whose CT centers are colored i, for each $i \in \{1, \dots, n/2\}$, is called an *i-atlas.*

Corollary 20. *Let $\rho_n : [k] \to \{atlases\ of\ G_{n,4}\}$ be the assignment given by $\rho_n(i) = i$-atlas of $G_{n,4}$, for each $i \in [k]$. If $\gcd(n,i) = 1 < i < n/2$, then $\rho_n(i)$ is obtained from $\rho_n(1)$ by replacing each color c tagging a vertex, edge,* CT *or* CH *of $\rho_n(1)$ by the reduction modulo n of $c.i$. If n is prime, applying Φ_n to the i-atlases of $G_{\infty,4}$ yields $\lfloor n/2 \rfloor$ i-atlases of $G_{n,4}$, which are graph isomorphic.*

Proof. The given reduction modulo n identifies oppositely signed colors modulo n. ■

Chart $\rho_{13}(1)$, depicted in Figure 6 (where a superposition of part of the $\{6,3\}$-regular hexagonal tessellation $\mathcal{H}$ with its edges intersecting at 90 deg some of the edges of $\rho_{13}(1)$ is shown in relation to Figure 7 below) exemplify the following properties, which follow by combining the images of the subgraphs 1[011], 2[011], 3[112] under the isomorphisms $\rho_n(1) \to \rho_n(i)$:

1. Chart $\rho_n(1)$ is representable in a plane triangle $T(n,1)$ whose sides are SAs of the subgraph $1[011] \subset G_{\infty,4}$, namely two SAs of type (2) and one of type (1), as in Subsection 8.1.

2. The internal angle between the SAs of type (2) is 60°. The internal angles between each of these and the SA of type (1) are 30° and 90°. The angle of 30° has its vertex at the center v of the CH 1.000 so $\rho_n(1)$ is represented as a twelfth part of the total angle of 360° at v. The angle of 90° has its vertex at $0jj1jj$, where $j = (n-1)/2$.

3. There is only one maximal path $L_{n,1}$ of $\rho_n(1)$ passing through $0jj1jj$ with its edges having color j and cutting the opposite side of $T(n,1)$ at 90° on a thick edge.

4. The angle of 60° has its vertex at the center of the CT $1hhh$, where $h = (n-5)/2$.

Proposition 21. *The diameter of $G_{n,4}$ is both $\Omega(n)$ and $O(|V(G_{n,4})|^{1/3})$, so that the asymptotic diameter of $G_{n,4}$ is $|V(G_{n,4})|^{1/3}$.*

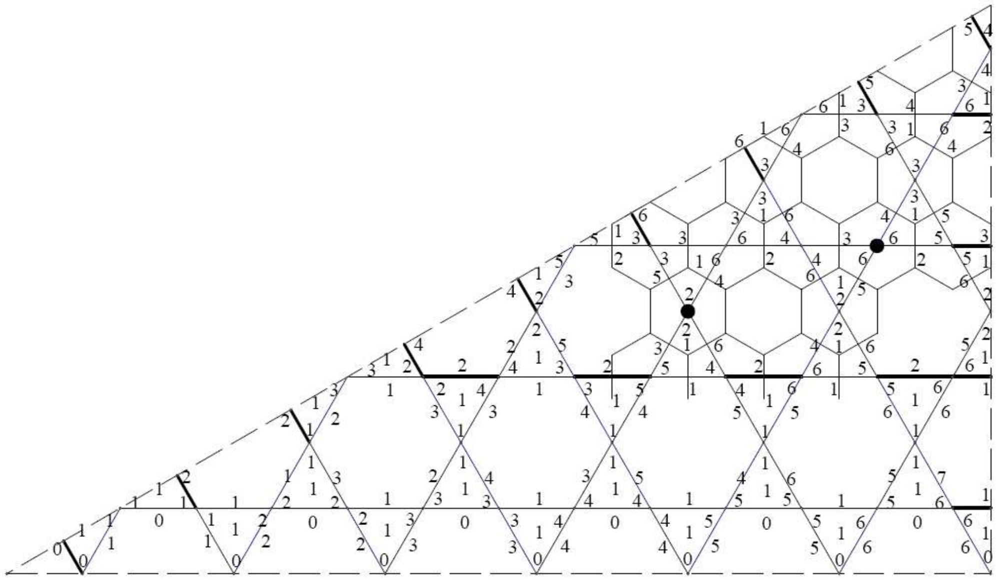

Figure 6. Superposition of drawings for $\sigma_n(1)$ and $\tau_n(1)$.

Proof. First, we claim that $|V(G_{n,3})| = O(n\phi(n))$, where $\phi(n)$ = Euler characteristic of n. Every $aa0$, where $\gcd(a,n) = 1$, belongs to $G_{n,3}$. Thus, there are $\lfloor \phi(n)/2 - 1 \rfloor$ paths whose ends are 011 and $0aa$, with $0 < a \leq \lfloor n/2 \rfloor$ and $\gcd(a,n) = 1$. But the distance from $0aa$ to 011 in $G_{n,3}$ is no more than a, yielding our claim. If we fix a K_3-type of $abcdef \in G_{n,4}$, say abc, then for each color d modulo n there are at most two different values for e but a unique value for f. This way, there are at most $n\phi(n)(2\lfloor n/2 \rfloor)$ different K_4-types modulo n. Thus, $|V(G_{n,4})| = O(n^2\phi(n))$. Let us see now that the diameter of $G_{n,4}$ is $\Omega(n)$. A path of length $n+1$ between 110110 and $112(n-1)nn$ happens along the image of $L(1,2,2)$. Thus, the diameter of $G_{n,4}$ is both $\Omega(n)$ and $O(|V(G_{n,4})|^{1/3})$. ■

A representation of the charts of $G'_{n,4}$ leading to the connectedness of $G'_{n,4}$ for n large is introduced. Let $\sigma_n(1)$ be the restriction of $\rho_n(1)$ induced by the rainbow K_4-types. We superpose the T-subgraph representation of $\sigma_n(1)$ with a $\{6,3\}$-regular hexagonal tessellation $\mathcal{H} = \tau_n(1)$ ([7], page 43) such that:

(a) each edge ϵ of $\sigma_n(1)$ is traversed by an edge ϵ' of $\tau_n(1)$ at 90° at the common midpoint of ϵ and ϵ';

(b) each CH of $\sigma_n(1)$ contains in its interior a regular hexagon of $\tau_n(1)$. Figure 6 contains a superposition of a representation of $\sigma_{13}(1)$, with the two rainbow K_4-types indicated as bullets • and the part of $\tau_{13}(1)$ used to represent $\sigma_{13}(1)$ in Figure 7.

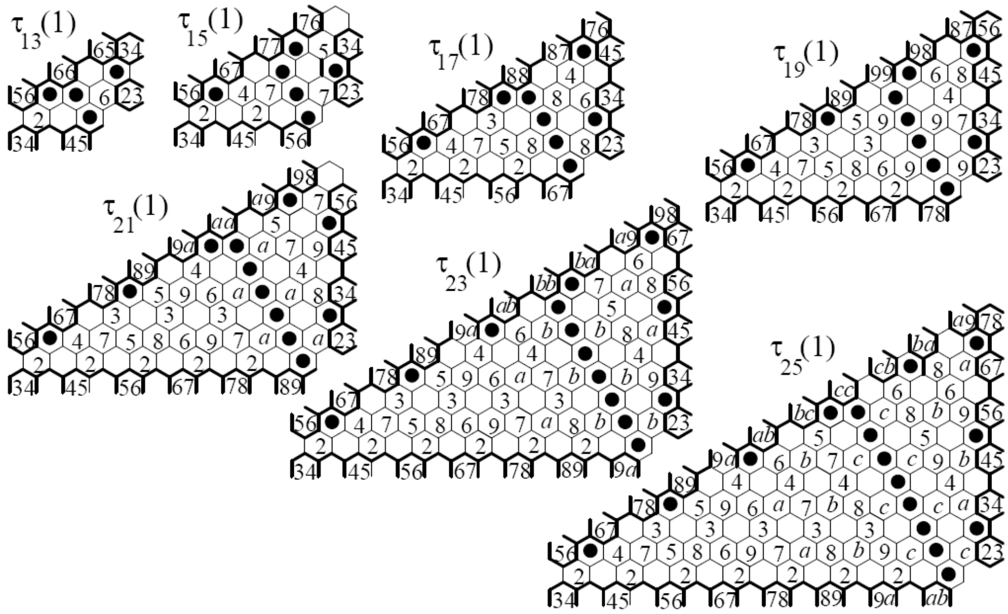

Figure 7. The representations $\tau_n(1)$, for $n = 13, \ldots, 25$.

In Figure 7, representing $\tau_n(1)$ for odd $n = 13, \ldots, 25$, each rainbow K_4-type of $\sigma_n(1)$ is given by an hexagon of $\tau_n(1)$ tagged by a positive integer, as suggested in Figure 6 for $n = 13$ by the indicated superposition. Each tagged hexagon representing a vertex of $\sigma_n(1)$ is the intersection of two tagged-hexagon sequences in $\tau_n(1)$. There are three directions of parallelism for existing tagged-hexagon sequences: one horizontal and the other two at angles of $\pm 60°$ from the horizontal. Each such sequence is headed on the boundary of $\tau_n(1)$ by a partially-drawn thick-trace hexagon tagged by a pair of integers. Assume the integer tagging an hexagon ζ of $\tau_n(1)$ is i and the integer pairs heading its two tagged-hexagon sequences are (p, q) and (r, s). Then the K_3-types composing ζ are: $1pq$, $1rs$ and either ipr and iqs or ips and iqr. Here, an hexagon is tagged with a bullet $\bullet$ instead of an integer if it represents a non-rainbow K_4-type. Each remaining (non-tagged) hexagon stands for a corresponding CH. It follows that each $\sigma_n(1)$ has at least two isolated vertices, represented in $\tau_n(1)$ by:

(1) the hexagon tagging 2 at the lower-left corner of $\tau_n(1)$ (that is the K_4-type 134265);

(2) the hexagon tagged by $\lfloor n/2 \rfloor$, at the lower-right corner of $\tau_n(1)$ (that is the K_4-type $123k(k-2)(k-1)$, where $n = 2k+1$).

If $n \neq 0 \pmod 3$ then these are the only two isolated vertices of $\sigma_n(1)$. Otherwise, there is exactly one more isolated vertex in $\sigma_n(1)$ and this is determined by the hexagon tagged by $n/3$ at the upper-right corner of $\tau_n(1)$ (that is the

K_4-type $1(k-2)(k-1)k(k+1)(k+2)$).

For $n \geq 17$, the isolated vertices of $\sigma_n(1)$ are nonisolated in the remaining charts $\sigma_n(i)$, where $i \neq 1$ ranges over the units modulo n from 2 to $\lfloor n/2 \rfloor$. This suggests the following conjecture.

Conjecture 22. *$G'_{n,4}$ is a connected graph, for $n \geq 17$.*

The six charts $\tau_{13}(i)$, for $i = 1, \ldots, 6$, represent the same pair of isolated vertices shown in Figure 2(c$_1$) and 2(c$_2$), which are thus the only components of $G'_{13,4}$. In addition, the four charts $\tau_{15}(i)$, for $i = 1, 2, 4, 7$, represent only a CT and four isolated vertices.

10. Proofs of the Main Results

***Proof of Theorem* 2.** By Proposition 21, the asymptotic diameter of $G_{n,4}$ is $|V(G_{n,4})|^{1/3}$. The vertices $v \in V_6$ in any member $G = G_{n,4}$ of $\mathcal{G}_1$ are the rainbow K_4-types in G. The four K_3-types of each such rainbow K_4-type form three distinct pairs of K_3-types, each corresponding to a respective triangle of G. This yields three triangles T_0, T_1, T_2 almost always distinct as in the statement, so that each pair $\{T_i, T_j\}$ with $i \neq j$ determines two different butterflies at v and respective charts $D^0_{i,j}$ and $D^1_{i,j}$. Let $S \subseteq V_6$ be composed by these vertices v. Clearly, $|S|$ is asymptotically $|V_6|$. Now, $V(G) \setminus V_6$ has its vertices at distance no more than 2 both from the boundary of charts $\tau_n(i)$ and from the diagonal paths $\eta(i)$ in them, with these paths departing from boundary vertices realizing angles of 90° as in the upper right representation in Figure 5 and as in Figure 6. This insures that $|V(G) \setminus V_6|$ grows linearly as n increases, while $|V_6|$ has a quadratic growth with respect to n, so V_6 has asymptotic order $|V(G)|$. Each of the four K_3-types composing the K_4-type associated with a vertex of S offers three positive integers that color the edges of a corresponding chart modeled on $\mathcal{H}$ as in [4], Theorem 2. Each of these three integers colors the edges of a parallel class of edges in that chart. These completes the proof of Theorem 2. ∎

***Proof of Theorem* 3.** Let $\mathcal{G}'_1 \subset \mathcal{G}_1$ be formed by the $G_{n,4}$ with n an odd prime. Then, the charts $\tau_n(i)$ are pairwise isomorphic. They are related with the graphs $D^k_{i,j}$ as follows, for $i = 1, \ldots, \frac{n}{2}$. Each $\tau_n(i)$ has two components formed by vertices representing rainbow K_4-types. These components are: **(a)** contained in a 30°-60°-90° triangle R (formed by the three delimiting SAs); **(b)** separated by the path $\eta(i)$ in $\tau_n(i)$. The union of the two 30°-60°-90° triangles delimited by the SAs and $\eta(i)$ yields $\tau_n(i)$. By Corollary 20, there are $\lfloor n/2 \rfloor$ charts $\tau_n(i)$. We consider stripping bands of the delimiting SAs in the 30°-60°-90° triangles in order to get rid of loops. This reduces the resulting $(n-1)$ 30°-60°-90° triangles. The stripped triangles are split into two halves by the paths $\eta(i)$, each half leading to isomorphic $\mathcal{D}$-modeled subgraphs, with the vertex numbers in the two halves, for $y \geq 1$, equal to: $|V_y'^-| = 2\sum_{i=1}^{y} i$ and $|V_y'^+| = -2 + 6\sum_{i=1}^{y} i$, if $k = 5 + 2y$; respectively, $|U_y'^-| = |V_y'^-| - y$ and $|U_y'^-| = |V_y'^+| - 3y$, if $k = 4 + 2y$. By removing from $U_y'^{\pm}$ (respectively, $V_y'^{\pm}$) the isolated vertices in lower-left (respectively, lower-; upper-right) corners in the $\tau_n(1)$ in Figure 7, tagged 2 (respectively, k; $n/3$ if $n \equiv 0 \pmod 3$), a maximal connected $\mathcal{D}$-modeled subgraph $U_y^{\pm}$ (respectively, $V_y^{\pm}$) is obtained. ■

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