# DENSE ARBITRARILY PARTITIONABLE GRAPHS 

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#### Abstract

A graph $G$ of order $n$ is called arbitrarily partitionable (AP for short) if, for every sequence $\left(n_{1}, \ldots, n_{k}\right)$ of positive integers with $n_{1}+\cdots+n_{k}=n$, there exists a partition $\left(V_{1}, \ldots, V_{k}\right)$ of the vertex set $V(G)$ such that $V_{i}$ induces a connected subgraph of order $n_{i}$ for $i=1, \ldots, k$. In this paper we show that every connected graph $G$ of order $n \geq 22$ and with $\|G\|>$ $\binom{n-4}{2}+12$ edges is AP or belongs to few classes of exceptional graphs. Keywords: arbitrarily partitionable graph, Erdős-Gallai condition, traceable graph, perfect matching.


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## 1. Introduction and Main Result

We use standard notation of graph theory (cf. [8]). In particular, $|G|$ and $\|G\|$ will stand for the order and the size of a graph $G$, respectively. The minimum degree of a vertex in a graph $G$ will be denoted by $\delta(G)$. By $c(G)$ we denote the circumference of a graph $G$, i.e., the length of a longest cycle. If $G$ and $H$ are two graphs with disjoint vertex sets, then the join of $G$ and $H$ is the graph, denoted by $G \vee H$, with the vertex set $V(G \vee H)=V(G) \cup V(H)$ and the edge set

$$
E(G \vee H)=E(G) \cup E(H) \cup\{x y: x \in V(G), y \in V(H)\}
$$

A sequence $\left(n_{1}, \ldots, n_{k}\right)$ of positive integers is called admissible for a graph $G=(V, E)$ of order $n$ if $n_{1}+\cdots+n_{k}=n$. An admissible sequence is said to be realizable in $G$ if there exists a partition of $V$ into $k$ parts $\left(V_{1}, \ldots, V_{k}\right)$ such that $\left|V_{i}\right|=n_{i}$ and the subgraph $G\left[V_{i}\right]$ induced by $V_{i}$ is connected, for every $i=1, \ldots, k$. Such a partition is called a realization of the sequence $\left(n_{1}, \ldots, n_{k}\right)$ in $G$. Note that in fact the ordering of $\left(n_{1}, \ldots, n_{k}\right)$ is irrelevant, i.e., if this sequence is realizable in $G$, then it is also realizable after any permutation of its elements. We say that $G$ is arbitrarily partitionable (AP for short) if every admissible sequence is realizable in $G$.

A simple example of an arbitrarily partitionable graph is a path $P_{n}$. Two obvious and well-known facts play a key role in this paper.

Proposition 1. If $G$ has a spanning subgraph which is $A P$, then $G$ is $A P$ itself.
Proposition 2. Every traceable graph is AP.
The following easy observation sometimes makes proofs shorter and allows us to assume throughout the paper that every admissible sequence has all elements greater than 1.

Proposition 3 [15]. A graph $G$ is AP if and only if every admissible sequence $\left(n_{1}, \ldots, n_{k}\right)$ with $n_{i} \geq 2$ for $i=1, \ldots, k$ is realizable in $G$.

The notion of AP graphs was introduced by Barth, Baudon and Puech [1] (and independently by Horňák and Woźniak [13]) to model a problem in the design of computer networks (see [1] for details). The concept of arbitrarily partitionable graphs, sometimes also called arbitrarily vertex decomposable or fully decomposable or just decomposable, has spawned numerous papers. Some of them investigate AP graphs within some classes of graphs (e.g., [1, 2, 9, 7, 13], KPWZ1). Horňák, Tuza and Woźniak [14] introduced the notion of on-line arbitrarily partionable graphs, and then a few other definitions strengthening the condition for AP graphs appeared (e.g., $[5,6,3,16]$ ). Here we present only those previous results on AP graphs we make use of in the paper.

A sequence $(d, \ldots, d)$ of length $\lambda$ will be denoted by $(d)^{\lambda}$. A caterpillar with three leaves is denoted by $\operatorname{Cat}(a, b)$ if it is obtained from the star $K_{1,3}$ by substituting two of its edges by paths of orders $a$ and $b$, respectively (see Figure $1)$. As $b=n-a$, we will later also use a shorter notation $\operatorname{Cat}(a)$. The following result was proved by Barth et al. [1], and independently by Horñák and Woźniak [13].

Theorem 4. The caterpillar Cat $(a, b)$, with $2 \leq a \leq b$, is AP if and only if a and $b$ are relatively prime. Moreover, each admissible and nonrealizable sequence is of the form $(d)^{k}$, where $a \equiv b \equiv 0(\bmod d)$ and $d>1$.


Figure 1. $\operatorname{Cat}(a, b)$ with $a=5, b=8$.


Figure 2. $\operatorname{Sun}(a, b)$.

A sun with $r$ rays is a graph of order $n \geq 2 r$ with $r$ pendant vertices $u_{1}, \ldots, u_{r}$ whose deletion yields a cycle $C_{n-r}$, and each vertex $v_{i}$ on $C_{n-r}$ adjacent to $u_{i}$ is of degree three. If the sequence of vertices $v_{i}$ is situated on the cycle $C_{n-r}$ in such a way that there are exactly $a_{i} \geq 0$ vertices, each of degree two, between $v_{i}$ and $v_{i+1}, i=1, \ldots, r$ (the indices taken modulo $r$ ), then this sun is denoted by $\operatorname{Sun}\left(a_{1}, \ldots, a_{r}\right)$. Suns with two and three rays are presented in Figures 2 and 3, respectively. Kalinowski, Pilśniak, Woźniak and Zioło characterized all AP suns with at most three rays.


Figure 3. $\operatorname{Sun}(a, b, c)$.
Theorem 5 [15]. A sun with two rays $\operatorname{Sun}(a, b)$ is $A P$ if and only if at most one of the numbers $a$ and $b$ is odd. Moreover, $\operatorname{Sun}(a, b)$ of order $n$ is not AP if and only if $(2)^{n / 2}$ is the unique admissible and nonrealizable sequence.

Theorem 6 [15]. A sun with three rays $\operatorname{Sun}(a, b, c)$ is AP if and only if none of the following three conditions is fulfilled:
(1) at most one of the numbers $a, b, c$ is even,
(2) $a \equiv b \equiv c \equiv 0(\bmod 3)$,
(3) $a \equiv b \equiv c \equiv 2(\bmod 3)$.

Moreover, if $\operatorname{Sun}(a, b, c)$ is not $A P$, then at least one of the following three sequences $(2)^{n / 2},(3)^{n / 3},\left(3,(2)^{(n-3) / 2}\right)$ is admissible and nonrealizable.

In this paper we consider the following question. How many edges in a connected graph $G$ guarantee that a graph is AP or belongs to few families of exceptional graphs?

Dense AP graphs were already investigated in another context. This was initiated by Marczyk who proved in [18], [19] some Ore-type sufficient conditions for a graph to be AP. The best result in this direction is due to Horňák, Marczyk, Schiermeyer and Woźniak.

Theorem 7 [12]. Every connected graph $G$ of order $n \geq 20$ such that the degree sum of each pair of nonadjacent vertices is at least $n-5$ is AP if and only if $G$ admits a perfect matching or a quasi-perfect matching (i.e., a matching omitting exactly one vertex).

$2 \mid n$

$2 \mid n$



Figure 4. Four graphs such that every non-AP graph $G$ with $\|G\|>\binom{n-4}{2}+12$ is a spanning subgraph of one of them (below each graph, requirements on the order $n$ are given).

Let us formulate now our main result.
Theorem 8. If $G$ is a connected graph of order $n \geq 22$ and size

$$
\|G\|>\binom{n-4}{2}+12
$$

then $G$ is AP unless $G$ is a spanning subgraph of one of the graphs depicted in Figure 4.

It is easily seen that none of four graphs in Figure 4 is AP whenever its order $n$ meets the divisibility condition given below the graph. By Proposition 1, every spanning subgraph is non-AP, as well. Observe also that the first two graphs have circumference $c(G)=n-2$ and the other two have $c(G)=n-3$.

It has to be noted that for $n<22$, there are more graphs of order $n$ and size greater than $\binom{n-4}{2}+12$ that are not AP. For example, the graph $G=K_{(n-2) / 2} \vee$ $\bar{K}_{(n+2) / 2}$ has no perfect matching, and its size $\|G\|=\frac{1}{2}\left[\frac{n-2}{2}(n-1)+\frac{n+2}{2} \cdot \frac{n-2}{2}\right]$ is greater than $\binom{n-4}{2}+12$ for every even $n=10, \ldots, 20$. Another example is the graph $G=K_{(n-3) / 2} \vee \bar{K}_{(n+3) / 2}$ which has no realization of the sequence $\left(3,(2)^{\frac{n-3}{2}}\right)$, and its size $\|G\|=\frac{1}{2}\left[\frac{n-3}{2}(n-1)+\frac{n+3}{2} \cdot \frac{n-3}{2}\right]$ is greater than $\binom{n-4}{2}+12$ for every odd $n=11, \ldots, 17$.

## 2. Preliminary Results

This section contains an initial stage of the proof of Theorem 8. We will make use of some classical sufficient conditions for the existence of long cycles in a graph.

Theorem 9 (Erdős, Gallai [11]). Let $G$ be a graph of order $n$. If $\|G\|>\frac{c}{2}(n-1)$, then $c(G)>c$.

Theorem 9 has been extended by Woodall.
Theorem 10 (Woodall [20]). Let $G$ be a graph of order $n=t(c-1)+p$, where $c \geq 2, t \geq 0$ and $1 \leq p \leq c$. If

$$
\|G\|>t\binom{c}{2}+\binom{p}{2},
$$

then $c(G)>c$.
Taking $t=1, c=n-\delta$ and $p=\delta+1$, we obtain the following
Corollary 11. If $n=|G|, \delta=\delta(G)$ and

$$
\|G\|>\binom{n-\delta}{2}+\binom{\delta+1}{2}
$$

then $c(G)>n-\delta$.
The next theorem is the well-known Erdős sufficient condition for hamiltonicity depending on the size and minimum degree.

Theorem 12 (Erdős [10]). Let $G$ be a graph of order $n$ and with minimum degree反. Denote

$$
f(n, \delta)=\max \left\{\binom{n-\delta}{2}+\delta^{2},\binom{n-\left\lfloor\frac{n-1}{2}\right\rfloor}{ 2}+\left\lfloor\frac{n-1}{2}\right\rfloor^{2}\right\}
$$

If $\delta \geq \frac{n}{2}$ or $\|G\|>f(n, \delta)$, then $G$ is Hamiltonian.
We can use Theorem 12 for traceability as follows. Let $H=G \vee K_{1}$. Then $H$ is Hamiltonian if and only if $G$ is traceable. Denote $g(n, \delta)=f(n+1, \delta+1)-n$. Thus

$$
g(n, \delta)=\max \left\{\binom{n-\delta}{2}+(\delta+1)^{2}-n,\binom{n+1-\left\lfloor\frac{n}{2}\right\rfloor}{ 2}+\left\lfloor\frac{n}{2}\right\rfloor^{2}-n\right\}
$$

As $\binom{n-\delta}{2}+(\delta+1)^{2}-n=\binom{n-\delta-1}{2}+\delta(\delta+1)$, this justifies the following result.
Corollary 13. Let $G$ be a graph of order $n$ and with minimum degree $\delta$. If $\delta \geq \frac{n-1}{2}$ or

$$
\|G\|>\max \left\{\binom{n-\delta-1}{2}+\delta(\delta+1),\binom{n+1-\left\lfloor\frac{n}{2}\right\rfloor}{ 2}+\left\lfloor\frac{n}{2}\right\rfloor^{2}-n\right\}
$$

then $G$ is traceable, and hence $A P$.
Suppose $G$ is a graph with minimum degree $\delta$ and with $\|G\|>\binom{n-4}{2}+12$. It follows from Corollary 13 that $G$ is traceable whenever $\delta \geq \frac{n-1}{2}$ or $g(n, \delta) \leq$ $g(n, 3)$. Observe that $\binom{n-\delta-1}{2}+\delta(\delta+1)$ is a quadratic polynomial with respect to $\delta$, so the latter inequality holds unless $g(n, \delta)=\binom{n+1-\left\lfloor\frac{n}{2}\right\rfloor}{ 2}+\left\lfloor\frac{n}{2}\right\rfloor^{2}-n$ and

$$
\binom{n-4}{2}+12<\binom{n+1-\left\lfloor\frac{n}{2}\right\rfloor}{ 2}+\left\lfloor\frac{n}{2}\right\rfloor^{2}-n
$$

We solve this inequality regarding to the parity of the order $n$ of $G$. If $n$ is even, then the inequality is equivalent to $n^{2}-30 n+176<0$, so it holds only if $9 \leq n \leq 21$. If $n$ is odd, then we have $n^{2}-24 n+175<0$, and this does not hold for any $n$.

Obviously, every connected graph $G$ with $c(G)=n-1$ is traceable, and hence AP.

Thus, Corollary 11 and Corollary 13 for $\delta=3$ imply that for the proof of our main result we are left with the following situation

$$
n \geq 22, \quad\|G\|>\binom{n-4}{2}+12, \quad 1 \leq \delta(G) \leq 2, \text { and } n-3 \leq c(G) \leq n-2
$$

The rest of our proof is divided into two parts corresponding to $c(G)=n-2$ (Section 3) and $c(G)=n-3$ (Section 4).

Let us state yet a lemma that follows the approach in [17] and will be used in both sections. First, we introduce some notation. If $C$ is a cycle in a graph $G=(V, E)$, then each vertex of $C$ adjacent to a vertex outside $C$ is called an attachment vertex. Fix an orientation of $C$. For two vertices $x, y \in V(C)$ we denote by $C[x, y]$ the path of $C$ from $x$ to $y$ along this orientation, and by $\overleftarrow{C}[x, y]$ the path from $x$ to $y$ along the reverse orientation of $C$. For a vertex $x \in V(C)$ we denote by $x^{+}, x^{-}$its successor and its predecessor along the orientation of $C$. We also denote $d_{C}(x)=|N(x) \cap V(C)|$. For two sets $A, B \subset V$, let $E(A, B)=$ $\{x y \in E: x \in A, y \in B\}$.

Lemma 14. Let $G=(V, E)$ be a connected graph of order $n \geq 22$ with $\delta(G) \leq 2$ and $\|G\|>\binom{n-4}{2}+12$. Let $C$ be a longest cycle in $G$ such that the set $V \backslash V(C)$ is not a clique.
(1) If $c(G)=n-3$, then each vertex outside $C$ is of degree one.
(2) If $c(G)=n-2$, then each vertex outside $C$ has at most three neighbors on $C$.

Proof. Let $k=k(G)=\max \left\{d_{C}(u): u \in V \backslash V(C)\right\}$, and let $u$ be a vertex outside $C$ with $d_{C}(u)=k$ and $N(u) \cap V(C)=\left\{u_{1}, \ldots, u_{k}\right\}$. Fix an orientation of $C$. Clearly, $k \leq \frac{c(G)}{2}$ and the set $X=\left\{u_{1}^{+}, \ldots, u_{k}^{+}\right\}$is independent since $C$ is a longest cycle in $G$. Moreover, for any pair $u_{i}^{+}, u_{j}^{+}$with $i \neq j$ and any $z \in C\left[u_{i}^{++}, u_{j}\right]$ we have $u_{i}^{+} z^{+} \notin E$ or $z u_{j}^{+} \notin E$, otherwise $C$ would not be a longest cycle. Let $C_{1}=C\left[u_{i}^{++}, u_{j}\right], C_{2}=C\left[u_{j}^{++}, u_{i}\right]$. Then a classical counting argument (cf. [8]) shows that

$$
\begin{aligned}
d_{C}\left(u_{i}^{+}\right)+d_{C}\left(u_{j}^{+}\right) & =d_{C_{1}}\left(u_{i}^{+}\right)+d_{C_{1}}\left(u_{j}^{+}\right)+d_{C_{2}}\left(u_{i}^{+}\right)+d_{C_{2}}\left(u_{j}^{+}\right) \\
& \leq\left|V\left(C_{1}\right)\right|+1+\left|V\left(C_{2}\right)\right|+1=|V(C)|
\end{aligned}
$$

Summing up this inequality for all $\binom{k}{2}$ possible pairs of vertices and dividing by $k-1$ we obtain

$$
\sum_{i=1}^{k} d_{C}\left(u_{i}^{+}\right) \leq \frac{k}{2} c(G)
$$

Now we want to estimate $|\bar{E}(C)|$, i.e., the number of edges within $C$ that are missing in $G$. Since $X$ is independent, all edges incident to vertices from $X$ are contained in $E(X, V(C) \backslash X)$. Hence $|\bar{E}(C)| \geq|\bar{E}(X, V(C) \backslash X)|+|\bar{E}(G[X])| \geq$ $k(c(G)-k)-\frac{k}{2} c(G)+\binom{k}{2}=\frac{k}{2}(c(G)-k-1)$. As $V \backslash V(C)$ is not a clique and each vertex of $V \backslash V(C)$ is connected to $C$ by at most $k$ edges, the number $f(k)=\|\bar{G}\|$ of edges missing in the graph $G$ satisfies the inequality

$$
f(k) \geq 1+(n-c(G))(c(G)-k)+\frac{k}{2}(c(G)-k-1)
$$

However, it is not difficult to see that if $k=k(G)$ and $f(k)=1+(n-c(G))(c(G)-$ $k)+\frac{k}{2}(c(G)-k-1)$ for a graph $G$, then $\delta(G)=d(u)=k+n-c(G)-2$ where $u$ is a vertex outside $C$. Note that $k+n-c(G)-2 \geq k+1$ if $c(G) \leq n-3$. But $\delta(G) \leq 2$ by assumption, hence we have to increase $f(k)$ by $k-1$, so actually

$$
f(k) \geq(n-c(G))(c(G)-k)+\frac{k}{2}(c(G)-k+1)
$$

Note that $\binom{n}{2}-\binom{n-4}{2}-12=4 n-22$, hence $f(k) \leq 4 n-23$ since $\|G\|>\binom{n-4}{2}+12$.
Consider first the case $c(G)=n-3$. Then $f(k) \leq 3(n-k-3)+\frac{k}{2}(n-k-2)$. Suppose, contrary to the claim, that $k \geq 2$. We search for the smallest value of $f(k)$. The derivative $f^{\prime}(k)=\frac{n}{2}-k-4$ is nonnegative for $2 \leq k \leq \frac{n}{2}-4$. Hence $f(k)$ is increasing for $2 \leq k \leq \frac{n}{2}-4$, and decreasing for $\frac{n}{2}-4 \leq k \leq \frac{n-3}{2}$. We have $f(2)=4 n-19>4 n-23$. Also, $f\left(\frac{n-3}{2}\right)=\frac{n^{2}+8 n-33}{8}$, and $f\left(\frac{n-3}{2}\right)-(4 n-23)=$ $\frac{1}{8}\left(n^{2}-24 n+151\right)>0$ for any $n \geq 14$. Thus $f(k)>4 n-23$ for every $k$ with $2 \leq k \leq \frac{n-3}{2}$, a contradiction.

Now, let $c(G)=n-2$. Thus $f(k)=2(n-k-2)+\frac{k}{2}(n-k-1)$ and $f^{\prime}(k)=\frac{n-5}{2}-k$. Hence $f(k)$ is increasing for $2 \leq k \leq \frac{n-5}{2}$, and decreasing for $\frac{n-5}{2} \leq k \leq \frac{n-2}{2}$. Note that $f(4)=\frac{4}{n}-22>4 n-23$, and $f\left(\frac{n-2}{2}\right)=\frac{1}{8}\left(n^{2}+6 n-\right.$ $16)>4 n-23$ because $\frac{1}{8}\left(n^{2}+6 n-16\right)-(4 n-23)=\frac{1}{8}\left(n^{2}-26 n+168\right)>0$. Therefore $k \leq 3$.

In most cases considered in the next two sections, we apply the following strategy. To prove that a graph $G=(V, E)$ satisfying certain conditions has no more than $\binom{n-4}{2}+12$ edges, we choose a graph $G_{0}$ such that $V\left(G_{0}\right)=V$, $\left\|G_{0}\right\| \leq\binom{ n-4}{2}+12$, and there exists an injective mapping of $E \backslash E\left(G_{0}\right)$ into $E\left(G_{0}\right) \backslash E$, whence $\|G\| \leq\left\|G_{0}\right\|$.

## 3. Proof for Circumference $n-2$

To prove that Theorem 8 holds for graphs with circumference $n-2$, it is enough to justify the following.

Proposition 15. If $G=(V, E)$ is a connected graph of order $n \geq 22$ with $c(G)=n-2$ and $\|G\|>\binom{n-4}{2}+12$, then $G$ is $A P$ unless $n$ is even and $G$ is a spanning subgraph of one of two graphs of even order shown in Figure 5.

Proof. Let $C$ be a longest cycle in $G$ and let $u, v$ be the two vertices outside $C$. Clearly, $G$ is traceable if $u v \in E$. Then assume $u v \notin E$.

First suppose that there is only one attachment vertex. If $n$ is even, then the sequence $(2)^{n / 2}$ is not realizable, $G$ is not AP and is a spanning subgraph of the first graph in Figure 5 when $\|G\|>\binom{n-4}{2}+12$ what is possible for $n \geq 10$.


Figure 5. Exceptional supergraphs with circumference $n-2$.

If $n$ is odd, then an admissible sequence contains an element $n_{i} \geq 3$. We take a part $V_{i}$ containing $u, v$ and their common neighbour, and the remaining graph is traceable, so $G$ is AP.

Now assume that there are at least two attachment vertices. For every pair of independent edges $u u^{\prime}, v v^{\prime}$ with $u^{\prime}, v^{\prime} \in V(C)$, the deletion of $u^{\prime}, v^{\prime}$ from $C$ yields two paths of orders $a$ and $b$ such that $a+b=n-4$ and $0 \leq a \leq b \leq n-4$. Thus $\operatorname{Sun}(a, b)$ is a spanning subgraph of $G$. By Theorem 5, the graph $G$ is AP when at most one of the numbers $a, b$ is odd (in particular when $n$ is odd). Henceforth, we assume that $n$ is even and both $a$ and $b$ are odd for any pair of independent edges $u u^{\prime}, v v^{\prime}$. Again, Theorem 5 implies that to prove that $G$ is AP, it suffices to show that the sequence $(2)^{n / 2}$ is realizable in $G$, i.e., $G$ admits a perfect matching. Choose edges $u u^{\prime}, v v^{\prime}$ such that $a$ is as large as possible (and not greater than $b$ ), and denote the vertices of $C$ by $u^{\prime}, x_{1}, \ldots, x_{a}, v^{\prime}, y_{1}, \ldots, y_{b}$ according to the orientation of $C$. Suppose that $G$ is not AP.

Case $a=1$. Suppose first that there are only two attachment vertices. Then $d(u) \leq 2$ and $d(v) \leq 2$. Let $d\left(x_{1}\right)=2$. If $n$ is even, then $G$ has no perfect matching and is a spanning subgraph of the second graph in Figure 5 whenever $\|G\|>\binom{n-4}{2}+12$, and the latter inequality may hold for $n \geq 12$.

Then assume that $d\left(x_{1}\right) \geq 3$, i.e., $C$ has at least one chord incident to $x_{1}$. We will show that in this case there does not exist a non-AP graph satisfying our assumptions. Indeed, suppose there exists such a graph $G$. First observe that $x_{1}$ cannot be adjacent to any vertex $y_{2 l-1}$ since otherwise $G$ would have a perfect matching: $\left\{u u^{\prime}, v v^{\prime}, x_{1} y_{2 l-1}\right\} \cup\left\{x_{2 i-1} x_{2 i}: i=1, \ldots, l-1\right\} \cup\left\{x_{2 i} x_{2 i+1}\right.$ : $\left.i=l, \ldots, \frac{b-1}{2}\right\}$. Suppose $l$ is the smallest positive integer such that $x_{1} y_{2 l} \in E$. Without loss of generality, we may assume that $2 l<\frac{b}{2}$ (we can change the orientation of $C$, if necessary), i.e., $l \leq \frac{n-4}{4}$. For any $i \leq l$ and $j \geq l$, an edge
$y_{2 i-1} y_{2 j+1}$ would give a perfect matching if it appeared in $G$. The number of these edges equals $l\left(\frac{n-4}{2}-l\right) \geq l^{2}$, and they are missing in $G$. Moreover, for every $p>l$, an edge $x_{1} y_{2 p} \in E$ creates a new missing edge $y_{2 p-1} y_{2 p+1}$, and an edge $y_{1} y_{2 p} \in E$, except for $p=\frac{n-6}{2}$, creates another missing edge $y_{2 p-1} y_{2 p+3}$. Hence $\|G\| \leq\binom{ n-4}{2}+8+2 l-2-l\left(\frac{n-4}{2}-l\right)+1 \leq\binom{ n-4}{2}+9+2 l-l^{2} \leq\binom{ n-4}{2}+9$, a contradiction.


Figure 6. Three attachment vertices for $a=1$.
It is easily seen that the number of attachment vertices can be at most three as $a=1$ was chosen greatest possible. Then one of the vertices outside $C$, say $u$, is a pendant vertex and $v$ is adjacent to $y_{b-1}$ (see Figure 6). Suppose that $G$ satisfies our assumptions and has no perfect matching. Then clearly, $x_{1} y_{b}$, as well as $x_{1} y_{2 i+1}$ and $y_{b} y_{2 i+1}$ cannot belong to $E$. Consider a graph $G_{0}$ of size $\binom{n-4}{2}+10$ such that $V\left(G_{0}\right)=V$, the set $V \backslash\left\{u, v, x_{1}, y_{b}\right\}$ is a clique, and $E\left(G_{0}\right)$ contains also the edges $u u^{\prime}, v v^{\prime}, v u^{\prime}, v y_{b-1}, v^{\prime} x_{1}, x_{1} u^{\prime}, u^{\prime} y_{b}, y_{b} y_{b-1}, x_{1} y_{b-1}, y_{b} v^{\prime}$. For every $l=1, \ldots, \frac{b-3}{2}$, whenever $x_{1} y_{2 l}$ belonged to $E$, the edge $y_{2 l-1} y_{2 l+1}$ would create a perfect matching in $G$, thus it is missing in $G$, and whenever $y_{b} y_{2 l} \in E$ (except $2 l=b-3$ ), then $y_{2 l-1} y_{2 l+3}$ is missing in $G$. Therefore $\|G\| \leq\left\|G_{0}\right\|+1=$ $\binom{n-4}{2}+11<\binom{n-4}{2}+12$, a contradiction.

Case $a \geq 3$. It follows from Lemma 14 that the vertices $u, v$ are of degree at most three, since the total number of vertices incident to them cannot be greater than six. Let $G_{1}$ be a graph of size $\binom{n-4}{2}+12$ containing these six edges, six edges $x_{1} x_{2}, x_{1} u^{\prime}, x_{1} v^{\prime}, x_{a} x_{2}, x_{a} u^{\prime}, x_{a} v^{\prime}$ and $\binom{n-4}{2}$ edges of the clique $V \backslash\left\{u, v, x_{1}, x_{a}\right\}$. If $a=3$, we set $G_{0}=G_{1}$. For $a \geq 5$, we define $G_{0}$ as follows. We add to $G_{1}$ the edges $x_{1} x_{2 j}, x_{a} x_{2 j}, j=2, \ldots, \frac{a-1}{2}$, and delete the edges $x_{2 i-1} y_{2 j-1}, i=2, \ldots, \frac{a-1}{2}, j=1, \ldots, \frac{b+1}{2}$. Thus the number of added edges equals $a-3$, and the number of deleted ones equals $\frac{a-1}{2} \cdot \frac{b+1}{2}$ and is not smaller than $\frac{a^{2}-1}{4}$ since $b \geq a$. Therefore $\left\|G_{0}\right\|<\left\|G_{1}\right\|$. Observe that the set $\left\{x_{2 i}: i=\right.$
$\left.1, \ldots, \frac{a-1}{2}\right\} \cup\left\{y_{i}: i=1, \ldots, b\right\} \cup\left\{u^{\prime}, v^{\prime}\right\}$ forms a clique in $G_{0}$, and the only edges that may appear in $G$ and not in $G_{0}$ are of the form $x_{\nu} x_{2 j-1}$ or $x_{\nu} y_{2 l}$, where $\nu \in\{1, a\}$.

For any $i=1, \ldots, \frac{a-1}{2}$, if $x_{1} x_{2 i+1} \in E$, then $y_{1} x_{2 i} \notin E$, and if $x_{a} x_{2 i-1} \in E$, then $y_{b} x_{2 i} \notin E$, otherwise $G$ has a perfect matching. For any $j=1, \ldots, \frac{b-1}{2}$, if $x_{1} y_{2 j} \in E$, then $y_{1} y_{2 j+1} \notin E$, and if $x_{a} y_{2 j} \in E$, then $y_{b} y_{2 j-1} \notin E$. It is easy to see that we have just defined an injective mapping of $E \backslash E\left(G_{0}\right)$ into $E\left(G_{0}\right) \backslash E$ unless the edge $y_{1} y_{b}$ was counted twice as a missing edge in $E$. This means that either $\|G\| \leq\left\|G_{0}\right\| \leq\binom{ n-4}{2}+12$ or $\|G\|=\left\|G_{0}\right\|+1=\binom{n-4}{2}+13$. But it is easy to see that in the latter case $\delta(G)=3$, so $G$ is traceable by Corollary 13 . We thus obtained a contradiction in both cases.

## 4. Proof for Circumference $n-3$

In this section we accomplish the proof of our main result by showing that Theorem 8 holds for graphs with circumference $n-3$. Let us introduce some additional notation first.

For any two vertices $x$ and $y$ of a cycle $C$ of a sun $S$ with a fixed orientation, we denote by $x C y$ the caterpillar consisting of a path $C[x, y]$ together with the leaves of $S$ if the corresponding attachment vertex belongs to $C[x, y]$. By $y \overleftarrow{C} x$ we denote the same caterpillar but in the reverse order.

Let $G=(V, E)$ be a connected graph of size $\|G\|>\binom{n-4}{2}+12$, and let $C$ be a longest cycle of $G$ of length $n-3$. Lemma 14 states that each of three vertices $u, v, w$ outside $C$ has at most one neighbor on $C$. The attachment vertices of $C$ adjacent to $u, v, w$ are denoted by $u^{\prime}, v^{\prime}, w^{\prime}$, respectively (some of the vertices $u^{\prime}, v^{\prime}, w^{\prime}$ may coincide or do not exist if there are less than three attachment vertices). If $C$ has three attachment vertices, denote the vertices of $C$ by $u^{\prime}, x_{1}, \ldots, x_{a}, v^{\prime}, y_{1}, \ldots, y_{b}, w^{\prime}, z_{1}, \ldots, z_{c}$ according to a fixed orientation of $C$. Let $X=\left\{x_{i}: i=1, \ldots, a\right\}, Y=\left\{y_{i}: i=1, \ldots, b\right\}, Z=\left\{z_{i}: i=1, \ldots, c\right\}$. If $C$ has only two attachment vertices, then we assume that $Z$ is empty, and $C$ is the sequence $u^{\prime}, x_{1}, \ldots, x_{a}, v^{\prime}, y_{1}, \ldots, y_{b}$.

If $a \geq 1$ and $b \geq 2$, then two edges of the form $x_{1} y_{i+1}, x_{a} y_{i}$ are said to be a good couple from $X$ to $Y$. The case $a=1$ is allowed. Analogously we define good couples from $Y$ to $X$, from $X$ to $Z$ and so on (see Figure 7).
Lemma 16. Let $C$ be a longest cycle of a connected graph $G=(V, E)$ of size $\|G\|>\binom{n-4}{2}+12$ and circumference $c(G)=n-3$. If the number of attachment vertices of $C$ is two and their distance on $C$ is at least three, then there exists a good couple of edges in $G$.
Proof. Using the notation from the beginning of this section, we may assume without loss of generality that $w$ is a vertex adjacent to $v$ or $v^{\prime}$. Then $G-w$ is


Figure 7. A good couple of edges from $X$ to $Z$.
spanned by $\operatorname{Sun}(a, b)$ where $2 \leq a \leq b$ and $a+b=n-5$. There are three or four edges outside $C$, therefore the number of chords of $C$ missing in the graph $G$ is less than $n+12$. Indeed, $\binom{n-3}{2}+4-\binom{n-4}{2}-12=n-12$.

Suppose that there is no good couple of edges in $G$. Then for every $i=$ $1, \ldots, b-1$, if $x_{1} y_{i+1} \in E$, then $x_{a} y_{i} \notin E$. It follows that the number of missing edges between $\left\{x_{1}, x_{a}\right\}$ and $Y$ is at least $b-1$. Moreover, for every $i=2, \ldots, a-1$, if $y_{1} x_{i+1} \in E$, then $y_{b} x_{i} \notin E$. Therefore the total number of missing chords between $X$ and $Y$ is at least $a-3+b-1=n-9>n-12$, a contradiction.

Lemma 17. Let $C$ be a longest cycle of a connected graph $G=(V, E)$ of size $\|G\|>\binom{n-4}{2}+12$ and circumference $c(G)=n-3$. If $C$ has three attachment vertices and no two of them are consecutive vertices on $C$, then there exists a good couple of edges in $G$.

Proof. By assumptions, the graph $G$ is spanned by $\operatorname{Sun}(a, b, c)$ where $1 \leq a \leq$ $b \leq c$ and $a+b+c=n-6$. Lemma 14 implies that there are exactly three edges outside $C$. Hence, there are less than $n-12$ chords of $C$ missing in $G$. Assume that $G$ has no good couple of edges.

Suppose first that $a=b=1$. Then $x_{1} z_{i+1} \in E$ implies $x_{1} z_{i} \notin E, i=$ $1, \ldots, c-1$, otherwise these two edges would be a good couple. Therefore, there are at least $\frac{c-1}{2}$ missing edges from $x_{1}$ to $Z$. Analogously, the number of missing edges between $y_{1}$ and $Z$ is not less than $\frac{c-1}{2}$. Altogether, we get at least $c-1=$ $n-9>n-12$ missing chords of $C$, a contradiction.

Suppose now that $a=1$ and $b \geq 2$. We analogously infer that there are at least $\frac{c-1}{2}$ missing edges from $x_{1}$ to $Z$. For any $i=1, \ldots, c-1$, whenever $y_{1} z_{i+1}$ is an edge in $G$, then $y_{b} z_{i}$ is not, for, otherwise these two edges would be a good couple. Thus there are at least $c-1$ chords of $C$ between $\left\{y_{1}, y_{b}\right\}$ and $Z$ missing in $G$. Furthermore, $z_{1} y_{i+1} \in E$ implies $z_{c} y_{i} \notin E$ for $i=2, \ldots, b-1$, so we get additional $b-3$ missing edges between $Y$ and $Z$. Hence there are at least
$\frac{c-1}{2}+c-1+b-3 \leq \frac{3}{2} c+b-5=n-12+\frac{c}{2}>n-12$, again a contradiction.
Finally, let $a \geq 2$. Denote by $\rho$ the number of edges joining $x_{a}$ with the set $Z$. Then $c-\rho$ edges between $x_{a}$ and $Z$ are missing. Moreover, for each edge $x_{a} z_{i}$, the edge $x_{1} z_{i+1}$ is missing. Therefore, at least $\rho$ edges between $x_{1}$ and $Z$ are missing. So, since there is no good couple from $X$ to $Z$, at least $(c-\rho)+\rho=c$ edges joining the vertices $x_{1}$ and $x_{a}$ with $Z$ are missing. Analogously, since there is no good couple from $X$ to $Y$ as well as $Y$ to $Z$ we can show that there are at least $b$ missing edges joining the vertices $x_{1}$ and $x_{a}$ with $Y$ and at least $c$ missing edges joining the vertices $y_{1}$ and $y_{b}$ with $Z$. Therefore, there are at least $c+(b+c)$ missing edges, and since $b+c \geq \frac{2}{3}(n-6)$ and $c \geq \frac{1}{3}(n-6)$, we have at least $n-6>n-12$ missing edges, a contradiction.

Let $C$ have only one attachment vertex $u^{\prime}$. If the subgraph $G[\{u, v, w\}]$ induced by $u, v, w$ is traceable, then $G$ is traceable itself. If all three vertices $u, v, w$ are pendant in $G$, then $G$ is a spanning subgraph of the third graph in Figure 4 and is not AP for any $n$ because either $(2)^{n / 2}$ or $\left(3,(2)^{(n-3) / 2}\right)$ is an admissible and nonrealizable sequence. Otherwise, $G[\{u, v, w\}]$ has exactly one edge, say $u v$, and $G$ is a spanning subgraph of the fourth graph in Figure 4. Then $G$ is not AP if and only if the order $n$ of $G$ is a multiple of three since the sequence $(3)^{n / 3}$ cannot be realized. For any other $n$, every admissible sequence $\left(n_{1}, \ldots, n_{k}\right)$ either has an element $n_{i}=2$ or $n_{i} \geq 4$. If $n_{i}=2$ we take a corresponding part $V_{i}=\{u, v\}$ and if $n_{i} \geq 4$ we take $V_{i} \supseteq\left\{u, v, w, u^{\prime}\right\}$. Then $G-V_{i}$ is traceable, and hence AP.

Suppose that $C$ has two attachment vertices $u^{\prime}, v^{\prime}$ with $u u^{\prime}, v v^{\prime} \in E$. As before, we assume that $w$ is adjacent $v$ or $v^{\prime}$. Observe that the subgraph $G^{\prime}=G-w$ of size $\left\|G^{\prime}\right\| \geq\binom{ n-4}{2}+11$ is spanned by a sun with two rays $\operatorname{Sun}(a, b)$ with $0 \leq a \leq b$. We will first show that $G^{\prime}$ is traceable. This is clear for $a=0$. If $a \geq 2$ then $G$ has a good couple of edges. Without loss of generality, we may assume that $x_{1} y_{i+1}, x_{a} y_{i}$ is a good couple. Then $v v^{\prime} y_{1} \cdots y_{i} x_{a} \cdots x_{1} y_{i+1} \cdots y_{b} u^{\prime} u$ is a Hamiltonian path of $G^{\prime}$. If $a=1$, suppose that $G^{\prime}$ is not traceable and consider the graph $G_{0}$ such that $V\left(G_{0}\right)=V\left(G^{\prime}\right)$ and $E\left(G_{0}\right)$ consists of $\binom{n-4}{2}+4$ edges: $u u^{\prime}, v v^{\prime}, u^{\prime} x_{1}, v^{\prime} x_{1}$ and all edges of the clique induced by $V(C) \backslash\left\{x_{1}\right\}$. Hence $G^{\prime}$ has at least seven chords incident to $x_{1}$. However, if $E\left(G^{\prime}\right)$ contained $x_{1} y_{1}$ or $x_{1} y_{b}$, then it is easy to see that $G^{\prime}$ would be traceable. Moreover, for $i=2, \ldots, b-1$, if $x_{1} y_{i} \in E\left(G^{\prime}\right)$ then $y_{i-1} y_{b} \notin E\left(G^{\prime}\right)$ since otherwise $v v^{\prime} y_{1} \cdots y_{i-1} y_{b} \cdots y_{i} x_{1} u^{\prime} u$ would be a Hamiltonian path of $G^{\prime}$. Thus $\left\|G^{\prime}\right\| \leq\left\|G_{0}\right\|<\binom{n-4}{2}+11$, a contradiction.

It follows that $G$ is traceable whenever $v w \in E$. Then assume $v w \notin E$. Let $\left(n_{1}, \ldots, n_{k}\right)$ be an admissible sequence for $G$ ordered decreasingly: $n_{1} \geq \cdots \geq$ $n_{k} \geq 2$. If $n_{1} \geq 3$, then we put $w, v, v^{\prime}$ to $V_{1}$ and continue a partition of $V$ along the Hamiltonian path of $G^{\prime}$. Otherwise, $\left(n_{1}, \ldots, n_{k}\right)=(2)^{n / 2}$ and $n$ is even.

Then $G$ is a spanning subgraph of the second graph in Figure 4 without a perfect matching.

To end the proof of Theorem 8 it suffices to settle the case when a longest cycle $C$ has three attachment vertices.

Lemma 18. Let $G=(V, E)$ be a graph of size $\|G\|>\binom{n-4}{2}+12$ and circumference $c(G)=n-3$. If a longest cycle $C$ has three attachment vertices, then $G$ is $A P$.

Proof. It follows from Lemma 14, that there are exactly three independent edges outside $C$, namely $u u^{\prime}, v v^{\prime}, w w^{\prime}$, and $G$ is spanned by $\operatorname{Sun}(a, b, c)$ where $0 \leq a \leq$ $b \leq c \leq n-6$. To show that $G$ is AP, we consider three cases depending on admissible sequences.

Case 1: Sequence $(2)^{n / 2}$. Suppose that the sequence $(2)^{n / 2}$ is admissible but not realizable in the graph $G$. Hence exactly two of the numbers $a, b, c$ are odd, say $a$ and $b$ with $a \leq b$.

Let $a=1$. Consider a graph $G_{0}$ of size $\left\|G_{0}\right\|=\binom{n-4}{2}+6$ containing all edges of the clique $V \backslash\left\{u, v, w, x_{1}\right\}$ and the edges $u u^{\prime}, v v^{\prime}, w w^{\prime}, u^{\prime} x_{1}, x_{1} v^{\prime}, x_{1} w^{\prime}$. It follows that $x_{1}$ is adjacent to at least seven vertices of $Y \cup Z$. However, any edge of the form $x_{1} y_{2 l-1}$ would give a perfect matching in $G$. Moreover, if $x_{1} y_{2 l} \in E$, then $y_{1} y_{2 l+1} \notin E$. Furthermore, if $x_{1} z_{2 l} \in E$, then $y_{1} z_{2 l-1} \notin E$, and if $x_{1} z_{2 l-1} \in E$, then $y_{1} z_{2 l} \notin E$, otherwise $G$ would have a perfect matching. Therefore $\|G\| \leq\left\|G_{0}\right\|$, a contradiction.

Let $a \geq 3$. Here we argue similarly as in Section 3 for $a \geq 3$. Let $G_{1}$ be a graph of size $\binom{n-5}{2}+11$ containing the edges $u u^{\prime}, v v^{\prime}, w w^{\prime}, u^{\prime} x_{1}, x_{1} x_{2}, x_{1} v^{\prime}, x_{1} w^{\prime}$, $x_{a} x_{2}, x_{a} u^{\prime}, x_{a} v^{\prime}, x_{a} w^{\prime}$ and all edges of the clique formed by $V \backslash\left\{u, v, w, x_{1}, x_{a}\right\}$. If $a=3$, we set $G_{0}=G_{1}$. For $a \geq 5$, we define $G_{0}$ as follows. We add to $G_{1}$ the edges $x_{1} x_{2 j}, x_{a} x_{2 j}, j=2, \ldots, \frac{a-1}{2}$, and delete the edges $x_{2 i-1} y_{2 j-1}, i=$ $2, \ldots, \frac{a-1}{2}, j=1, \ldots, \frac{b+1}{2}$. Thus the number of added edges equals $a-3$, and the number of deleted ones equals $\frac{a-1}{2} \cdot \frac{b+1}{2}$ and is not smaller than $\frac{a^{2}-1}{4}$ since $b \geq a$. Therefore $\left\|G_{0}\right\|<\left\|G_{1}\right\|$. To avoid a perfect matching, the only edges that may appear in $G$ and are not in $G_{0}$ are of the form $x_{\nu} x_{2 i-1}$ or $x_{\nu} y_{2 j}$ or $x_{\nu} z_{l}$ where $\nu \in\{1, a\}$.

For any $i=1, \ldots, \frac{a-1}{2}$, if $x_{1} x_{2 i+1} \in E$, then $y_{1} x_{2 i} \notin E$, and if $x_{a} x_{2 i-1} \in E$, then $y_{b} x_{2 i} \notin E$, otherwise $G$ admits a perfect matching. For any $j=1, \ldots, \frac{b-1}{2}$, if $x_{1} y_{2 j} \in E$, then $y_{1} y_{2 j+1} \notin E$, and if $x_{a} y_{2 j} \in E$, then $y_{b} y_{2 j-1} \notin E$ (here the edge $y_{1} y_{b}$ may be counted twice as missing in $E$ ). For any $j=1, \ldots, \frac{c-1}{2}$, if $x_{1} z_{2 j} \in E$, then $y_{1} z_{2 j-1} \notin E$, and if $x_{a} z_{2 j} \in E$, then $y_{b} z_{2 j-1} \notin E$. Finally, for any $j=1, \ldots, \frac{c+1}{2}$, if $x_{1} z_{2 j-1} \in E$, then $y_{1} z_{2 j} \notin E$, and if $x_{a} z_{2 j-1} \in E$, then $y_{b} z_{2 j} \notin E$. Whence, $\|G\| \leq\left\|G_{0}\right\|+1 \leq\binom{ n-4}{2}+12$, a contradiction.

Case 2: Sequence $(3)^{n / 3}$. Suppose that the sequence $(3)^{n / 3}$ is admissible but
is not realizable in $G$. It easily follows from Theorem 6 that either $a \equiv b \equiv c \equiv$ $0(\bmod 3)$ or $a \equiv b \equiv c \equiv 2(\bmod 3)$.

Assume first that $a \equiv b \equiv c \equiv 0(\bmod 3)$. Let $0 \leq a \leq b \leq c$. Put $V_{1}=\left\{u, u^{\prime}, z_{c}\right\}$ and $V_{2}=\left\{w, w^{\prime}, z_{1}\right\}$. Let $G_{0}$ be a subgraph of $G$ obtained by deleting all chords of $C$ incident to $v^{\prime}$ except $v^{\prime} z_{1}$ and $v^{\prime} z_{c}$. Then $\left\|G_{0}\right\| \leq\binom{ n-4}{2}+7$, and there are another chords of $C$ incident to $v^{\prime}$ in $G$.

Suppose $a=b=0$. If $v^{\prime} z_{3 l+2}$ was an edge of $G$ for some $l \geq 0$, then the sequence $(3)^{n / 3}$ would have a realization in $G$. Indeed, we put $V_{3}=\left\{v, v^{\prime}, z_{3 k+2}\right\}$ and observe that the cycle $C$ splits into at most four paths of order divisible by 3 after removing the vertices of $V_{1} \cup V_{2} \cup V_{3}$. Also, if $v^{\prime} z_{3 l} \in E$ with $1 \leq l \leq$ $\frac{c}{3}-1$, then $z_{3 l-1} z_{3 l+1} \notin E$ otherwise $G$ has a realization of $(3)^{n / 3}$. Similarly, if $v^{\prime} z_{3 l+1} \in E$ with $1 \leq l \leq \frac{c}{3}-1$, then $z_{3 l} z_{3 l+2} \notin E$.

Suppose $b>0$. If $v^{\prime} y_{3 k} \in E$, then $y_{3 k-1} z_{2} \notin E$, since otherwise we would have a realization of $(3)^{n / 3}$ by taking $V_{3}=\left\{y_{3 k-2}, y_{3 k-1}, z_{2}\right\}$. Analogously, if $v^{\prime} y_{3 k+1} \in E$, then $y_{3 k+2} z_{2} \notin E$ because we could take $V_{3}=\left\{y_{3 k+2}, y_{3 k}, z_{2}\right\}$. Again, if both edges $v^{\prime} y_{3 k+2}$ and $y_{3 k+1} z_{1}$ appeared in $G$, then we could redefine $V_{2}=\left\{w, w^{\prime}, y_{b}\right\}$ and put $V_{3}=\left\{v, v^{\prime}, y_{3 k+2}\right\}, V_{4}=\left\{y_{3 k+1}, z_{1}, z_{2}\right\}$ to obtain a realization of $(3)^{n / 3}$.

If also $a>0$, then the same arguments as in the previous paragraph for $b>0$ can be applied to justify the assertion: every new chord from $E \backslash E\left(G_{0}\right)$ causes the absence of another chord in $G$. It follows that $\|G\| \leq\left\|G_{0}\right\|<\binom{n-4}{2}+12$.

Now, assume that $a \equiv b \equiv c \equiv 2(\bmod 3)$. Let $V_{1}=\left\{u, u^{\prime}, x_{1}\right\}, V_{2}=$ $\left\{v, v^{\prime}, x_{a}\right\}, V_{3}=\left\{w, w^{\prime}, y_{b}\right\}$. Remember that the number of chords of $C$ missing in $G$ is at most $n-12$. For every $i=1, \ldots, \frac{b}{3}$ and $j=1, \ldots, \frac{c}{3}$, the vertex $y_{3 i-2}$ cannot be a neighbor neither of $z_{3 j-2}$ nor of $z_{3 j-1}$ because then we could take $V_{4}=\left\{y_{3 i-2}, z_{3 j-2}, z_{3 j-1}\right\}$, and $C$ would split into paths of orders being multiples of three after removing $V_{1} \cup V_{2} \cup V_{3} \cup V_{4}$. Analogously, $y_{3 i-1} z_{3 j-2}, y_{3 i-1} z_{3 j-1} \notin E$. Thus, there are $4 \cdot \frac{b}{3} \cdot \frac{c}{3}$ missing chords of $C$. As $b+c \geq \frac{2}{3}(n-6)$, we have $\frac{4}{9} b c \geq \frac{4}{81}(n-6)^{2}>n-12$ for any $n$, a contradiction.

Case 3: Sequences different from (2) ${ }^{n / 2}$ and $(3)^{n / 3}$. Consider first the case $a \geq 1$. Then, we can apply Lemma 17. Without loss of generality, we may assume that the good couple is from $X$ to $Z$, i.e., there is an $i$ such that $x_{1} z_{i+1}, x_{a} z_{i} \in E$ and $1 \leq i \leq c$. Then, observe that the subgraph of $G$ induced by the vertex sequence $x_{0} \overleftarrow{C} z_{i+1} x_{1} C x_{a} z_{i} \overleftarrow{C} v^{\prime}$ contains a caterpillar $\operatorname{Cat}(b+3)$. So, by Theorem 4, we are able to realize all admissible sequences except, maybe, for sequences of the form $(d)^{n / d}$ for $d \mid(b+3)$. If $a=1$, then a part of such a sequence could be realized on the caterpillar $\operatorname{Cat}(2)=x_{1} C y_{b}$ of order $b+3$ because $d \neq 2$ by assumption, and the rest of it on the path $w w^{\prime} C u^{\prime} u$. If $a \geq 2$, then a part of this sequence could be realized either on $\operatorname{Cat}(2)=x_{1} C y_{b}$ or on $\operatorname{Cat}(2)=y_{1} C z_{1}$, and the rest of the sequence either on $\operatorname{Cat}(a)=x_{a-1} \overleftarrow{C} w^{\prime}$ or on $\operatorname{Cat}(a+3)=v^{\prime} \overleftarrow{C} z_{2}$, respectively. If none of the two latter caterpillars admits a realization of the sequence $(d)^{n / d}$
that means that $d \mid a$ and $d \mid(a+3)$. This implies that $d=3$.
Let $a=0$ and $b \geq 1$. Then $G$ contains a caterpillar $\operatorname{Cat}(b+3)=v v^{\overleftarrow{C}} u^{\prime} u$. So, by Theorem 4, any admissible and nonrealizable sequence should be of the form $(d)^{n / d}$ for $d \mid(b+3)$. As $d \neq 2$, then a part of such a sequence could be realized on $\operatorname{Cat}(2)=y_{1} C z_{1}$, and the rest of the sequence on $\operatorname{Cat}(3)=v v^{\prime} \overleftarrow{C} z_{2}$, except for the case where $d=3$.

If $a=b=0$ then it is easy to see that $G$ is spanned by a caterpillar Cat(3), so only the sequence (3) $)^{n / 3}$ may not be realizable.


Figure 8. A non-AP graph of size $\binom{n-4}{2}+12$.

## 5. Final Remarks

The following is an easily seen consequence of Corollary 13.
Proposition 19. If $G$ is a connected graph of order $n$ and size $\|G\|>\binom{n-2}{2}+2$, then $G$ is traceable.

Clearly, the bound $\binom{n-2}{2}+2$ is sharp for every $n \geq 4$ since the first graph shown in Figure 4 (a clique $K_{n-2}$ with two pendant edges attached to it in one vertex) is not traceable. The difference between $\binom{n-2}{2}+2$ and the lower bound $\binom{n-4}{2}+12$ in our main result equals $2 n-17$.

Observe that there are quite many connected nontraceable graphs $G$ with more than $\binom{n-4}{2}+12$ edges, which are AP by Theorem 8. In particular, if the order $n$ of $G$ is not divisible neither by two nor by three, then $G$ is AP unless it is a spanning subgraph of the third graph in Figure 4 (a clique $K_{n-3}$ with three pendant edges attached in one and the same vertex). Moreover, for every $n$ if
$c(G)=n-3$ and $G$ has three independent pendant edges, then $G$ is AP, and clearly nontraceable.

It has to be noted that if we decrease the bound $\binom{n-4}{2}+12$ even by one, then we obtain new exceptional graphs that are not AP. For example, the graph in Figure 8 has $\binom{n-4}{2}+12$ edges and is not AP for even $n$.

## References

[1] D. Barth, O. Baudon and J. Puech, Network sharing: a polynomial algorithm for tripodes, Discrete Appl. Math. 119 (2002) 205-216. doi:10.1016/S0166-218X(00)00322-X
[2] O. Baudon, J. Bensmail, R. Kalinowski, A. Marczyk, J. Przybyło and M. Woźniak, On the Cartesian product of an arbitrarily partitionable graph and a traceable graph, Discrete Math. Theor. Comput. Sci. 16 (2014) 225-232.
[3] O. Baudon, J. Bensmail, J. Przybyło and M. Woźniak, Partitioning powers of traceable or Hamiltonian graphs, Theoret. Comput. Sci. 520 (2014) 133-137. doi:10.1016/j.tcs.2013.10.016
[4] O. Baudon, F. Foucaud, J. Przybyło and M. Woźniak, On the structure of arbitrarily partitionable graphs with given connectivity, Discrete Appl. Math. 162 (2014) 381385. doi:10.1016/j.dam.2013.09.007
[5] O. Baudon, F. Gilbert and M. Woźniak, Recursively arbitrarily vertex decomposable graphs, Opuscula Math. 32 (2012) 689-706. doi:10.7494/OpMath.2012.32.4.689
[6] O. Baudon, F. Gilbert and M. Woźniak, Recursively arbitrarily vertex decomposable suns, Opuscula Math. 31 (2011) 533-547. doi:10.7494/OpMath.2011.31.4.533
[7] O. Baudon, J. Przybyło and M. Woźniak, On minimal arbitrarily partitionable graphs, Inform. Process. Lett. 112 (2012) 697-700. doi:10.1016/j.ipl.2012.06.010
[8] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer, 2008).
[9] H.J. Broersma, D. Kratsch and G.J. Woeginger, Fully decomposable split graphs, Lecture Notes in Comput. Sci. 5874 (2009) 4105-4112. doi:10.1007/978-3-642-10217-2_13
[10] P. Erdős, Remarks on a paper of Pósa, Magyar Tud. Akad. Mat. Kutató Int. Kőzl. 7 (1962) 227-229.
[11] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10 (1959) 337-356. doi:10.1007/BF02024498
[12] M. Horňák, A. Marczyk, I. Schiermeyer and M. Woźniak, Dense arbitrarily vertex decomposable graphs, Graphs Combin. 28 (2012) 807-821.
doi:10.1007/s00373-011-1077-3
[13] M. Horňák and M. Woźniak, Arbitrarily vertex decomposable trees are of maximum degree at most six, Opuscula Math. 23 (2003) 49-62.
[14] M. Horňák, Zs. Tuza and M. Woźniak, On-line arbitrarily vertex decomposable trees, Discrete Appl. Math. 155 (2007) 1420-1429. doi:10.1016/j.dam.2007.02.011
[15] R. Kalinowski, M. Pilśniak, M. Woźniak and I.A. Zioło, Arbitrarily vertex decomposable suns with few rays, Discrete Math. 309 (2009) 3726-3732. doi:10.1016/j.disc.2008.02.019
[16] R. Kalinowski, M. Pilśniak, M. Woźniak and I.A. Zioło, On-line arbitrarily vertex decomposable suns, Discrete Math. 309 (2009) 6328-6336. doi:10.1016/j.disc.2008.11.025
[17] A. Kemnitz and I. Schiermeyer, Improved degree conditions for Hamiltonian properties, Discrete Math. 312 (2012) 2140-2145. doi:10.1016/j.disc.2011.07.013
[18] A. Marczyk, A note on arbitrarily vertex decomposable graphs, Opuscula Math. 26 (2006) 109-118.
[19] A. Marczyk, An Ore-type condition for arbitrarily vertex decomposable graphs, Discrete Math. 309 (2009) 3588-3594.
doi:10.1016/j.disc.2007.12.066
[20] D.R. Woodall, Maximal circuits of graphs I, Acta Math. Acad. Sci. Hungar. 28 (1976) 77-80.
doi:10.1007/BF01902497

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