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ON SUPER (a, d)-H-ANTIMAGIC TOTAL COVERING OF STAR RELATED GRAPHS

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Abstract

Let $G = (V(G), E(G))$ be a simple graph and H be a subgraph of G. G admits an H-covering, if every edge in $E(G)$ belongs to at least one subgraph of G that is isomorphic to H. An (a, d)-H-antimagic total labeling of G is a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ such that for all subgraphs H' isomorphic to H, the H' weights

$$wt(H') = \sum\nolimits_{v\in V(H')} \lambda(v) + \sum\nolimits_{e\in E(H')} \lambda(e)$$

constitute an arithmetic progression $a, a + d, a + 2d, \ldots, a + (n - 1)d$ where a and d are positive integers and n is the number of subgraphs of G isomorphic to H. Additionally, the labeling λ is called a super (a, d)-H-antimagic total labeling if $\lambda(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$.

In this paper we study super (a, d)-H-antimagic total labelings of star related graphs $G_u[S_n]$ and caterpillars.

Keywords: super (a, d)-H-antimagic total labeling, star.

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1. Introduction

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be simple and finite graphs. Let $|V(G)| = p, |E(G)| = q$. An edge covering of G is a family of different subgraphs $H_1, H_2, H_3, \dots, H_k$ such that any edge of $E(G)$ belongs to at least one of the subgraphs H_j's, $1 \leq j \leq k$. If the H_j are isomorphic to a given graph H, then G admits an H-covering.

Suppose G admits an H-covering. Gutiérrez and Lladó [1] defined an H-magic labeling which is a generalization of Kotzig and Rosa's edge magic total labeling [5]. A bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \dots, p+q\}$ is called an H-magic labeling of G if there exists a positive integer k such that each subgraph H' isomorphic to H satisfies

$$f(H') = \sum\nolimits_{v \in V(H')} f(v) + \sum\nolimits_{e \in E(H')} f(e) = k.$$

In this case, we say that G is H magic. When $f(V(G)) = \{1, 2, 3, \dots, p\}$, we say that G is H-super magic.

On the other hand, Inayah *et al.* [2] introduced an (a, d)-H-antimagic total labeling of G which is defined as a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \dots, p+q\}$ such that for all subgraphs H' isomorphic to H, the set of H'-weights

$$wt(H') = \sum\nolimits_{v \in V(H')} f(v) + \sum\nolimits_{e \in E(H')} f(e)$$

constitutes an arithmetic progression $a, a+d, a+2d, \dots, a+(n-1)d$ where a and d are some positive integers and n is the number of subgraphs isomorphic to H. In this case we say that G is (a, d)-H-antimagic. When $f(V(G)) = \{1, 2, 3, \dots, p\}$, we say that f is a super (a, d)-H-antimagic total labeling and G is super (a, d)-H-antimagic.

In [1] Gutiérrez and Lladó discussed H-supermagic labelings of stars, complete bipartite graphs, paths and cycles. In [6], Lladó and Moragas studied C_h-supermagic labelings of some graphs, namely, wheels, windmills, prisms and books. In [7], Maryati *et al.* proved that some classes of trees such as subdivisions of stars, shrubs and banana tree graphs are P_h-supermagic for some h. In [2], Inayah *et al.* studied some properties of (a, d)-H-antimagic total labeling for any graph and also discussed the (a, d)-C_h-antimagic total labelings of fans. Recently, Inayah, Simanjuntak and Salman [4] proved that there exists a super (a, d)-H-antimagic total labeling for shackles of a connected graph H.

In this paper we study super (a, d)-H-antimagic total labelings of star related graphs $G_u[S_n]$ and caterpillars.

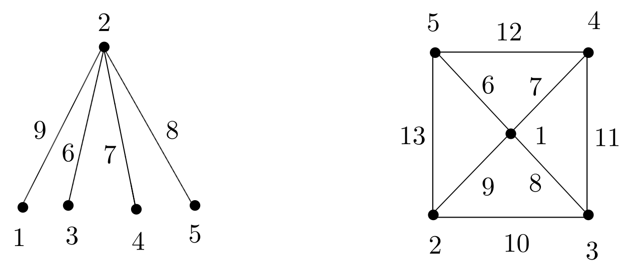

Figure 1. Super $(21, 1)$-P_3-antimagic total labeling
and super $(33, 1)$-C_3-antimagic total labeling.

2. Sum Set Partitions

As in [1, 3, 8], the proofs of our main results are based on the use of sum set partitions. We recall in this section some useful facts on this concept.

Let $x < y$ be positive integers. Throughout the paper we denote by $[x, y]$ to mean $\{i \in N : x \leq i \leq y\}$. Given a set X of integers and a partition $\mathcal{P} = \{X_1, X_2, \ldots, X_k\}$ of X into k parts. We denote by $\sum(\mathcal{P}) = (\sum X_1, \sum X_2, \ldots, \sum X_k)$, the sum set partition of $\mathcal{P}$ where $\sum X_i = \sum_{x \in X_i} x$. We will always order the partition in such a way that the sequence of subset sums $\sum X_1 \leq \sum X_2 \leq \cdots \leq \sum X_k$ is non decreasing.

When all sets in $\mathcal{P}$ have the same cardinality then we say that $\mathcal{P}$ is an equipartition of X or k-equipartition or a k-balanced multisets of X.

We have the following lemmas.

Lemma 1 [8]**.** *Let x and y be nonnegative integers. Let $X = [x + 1, x(y + 1)]$ with $|X| = xy$ and $Y = [x(y+2), 2x(y+1) - 1]$ with $|Y| = xy$. Then, there exists a partition K of $X \cup Y$ such that $\sum(K)$ is an arithmetic progression starting at $x(y + 3) + 1$ with common difference 2 and hence K is xy-balanced with all its subsets being 2-sets.*

Proof. For each $i \in [1, xy]$, define $K_i = \{a_i, b_i\}$ such that $a_i = x + i$, $b_i = x(y + 2) + i - 1$. Thus $\sum K_i = x(y + 3) + 2i - 1$, for all $i \in [1, xy]$.

Hence, the sum set partition of K, $\sum(K) = (\sum K_1, \sum K_2, \ldots, \sum K_{xy})$ forms an arithmetic progression with common difference 2. Therefore, K is xy-balanced with all its subsets being 2-sets. ■

Lemma 2. *Let x, y and z be nonnegative integers. Let $X = [x + 1, x + y]$ with $|X| = y$ and $Y = [x + y + z + 1, x + 2y + z]$ with $|Y| = y$. Then, there exists a partition K of $X \cup Y$ such that*

(i) *$\sum(K)$ is an arithmetic progression starting at $2x + 2y + z + 1$ with common difference 0, and*

(ii) $\sum(K)$ *is an arithmetic progression starting at* $2x+y+z+2$ *with common difference* 2 *and hence* K *is* y*-balanced with all its subsets being* 2*-sets.*

Proof. (i) For each $i \in [1, y]$ define the sets $K_i = \{a_i, b_i\}$ such that $a_i = x + i$, $b_i = x + 2y + z - i + 1$. Then $\sum K_i = 2x + 2y + z + 1$, for each $i \in [1, y]$.

Hence, the sum set partition of K, $\sum(K) = (\sum K_1, \sum K_2, \ldots, \sum K_y)$ forms an arithmetic progression with common difference 0. Therefore, K is y-balanced with all its subsets being 2-sets.

(ii) For each $i \in [1, y]$, we take the sets $K_i = \{a_i, b_i\}$ such that: $a_i = x + i$, $b_i = x + y + z + i$. Then $\sum K_i = 2x + y + z + 2i$, for each $i \in [1, y]$.

Hence, the sum set partition of K, $\sum(K) = (\sum K_1, \sum K_2, \ldots, \sum K_y)$ forms an arithmetic progression with common difference 2. Therefore, K is y-balanced with all its subsets being 2-sets. ■

Lemma 3. *Let* x, y *and* z *be nonnegative integers. Let* $X = \{1, 3, 5, \ldots, 2y - 1\}$ *with* $|X| = y$ *and* $Y = [x + y + z + 1, x + 2y + z]$ *with* $|Y| = y$*. Then, there exists a partition* K *of* $X \cup Y$ *such that*

(i) $\sum(K)$ *is an arithmetic progression starting at* $x+2y+z+1$ *with common difference* 1*, and*

(ii) $\sum(K)$ *is an arithmetic progression starting at* $x+y+z+2$ *with common difference* 3 *and hence* K *is* y*-balanced with all its subsets being* 2*-sets.*

Proof. (i) For each $i \in [1, y]$, we define $K_i = \{a_i, b_i\}$ where $a_i = 2i - 1$, $b_i = x + 2y + z - i + 1$. Then $\sum K_i = x + 2y + z + i$, for each $i \in [1, y]$.

Hence, the sum set partition of K, $\sum(K) = (\sum K_1, \sum K_2, \ldots, \sum K_y)$ forms an arithmetic progression with common difference 1. Therefore, K is y-balanced and all its subsets are 2-sets.

(ii) For each $i \in [1, y]$, we define $K_i = \{a_i, b_i\}$ where $a_i = 2i - 1$, $b_i = x + y + z + i$. Then $\sum K_i = x + y + z + 3i - 1$, for each $i \in [1, y]$.

Hence, the sum set partition of K, $\sum(K) = (\sum K_1, \sum K_2, \ldots, \sum K_y)$ forms an arithmetic progression with common difference 3. Therefore, K is y-balanced and all its subsets are 2-sets. ■

3. Main Results

Let G be a (p, q) graph and S_n be a star with n edges. Fix a vertex u of G. Then $G_u[S_n]$ is the graph obtained by identifying the vertex u with the centre of S_n. Let w be any vertex of S_n. Then $G + e$, $e = uw$, is a subgraph of $G_u[S_n]$. In this section, we consider graphs G for which $G_u[S_n]$ contains exactly n subgraphs isomorphic to $G + e$.

Let $G' \cong G_u[S_n]$. Let $v_1, v_2, \dots, v_p$ and $w_1, w_2, \dots, w_n$ be the vertices of G and S_n respectively. Let $e_1, e_2, \dots, e_q$ and $e_{q+1}, e_{q+2}, \dots, e_{q+n}$ be the edges of G and S_n respectively. Then $|V(G')| = p+n$ and $|E(G')| = q+n$.

Lemma 4. *If the graph $G_u[S_n], n \geq 2$, admits a super (a,d)-$(G+e)$-antimagic total labeling, then $d \leq p+q+2$.*

Proof. Let $G' \cong G_n[S_n]$. Suppose there exists a bijection $f : V(G') \cup E(G') \to \{1, 2, 3, \dots, p+q+2n\}$ which is a super (a,d)-$(G+e)$-antimagic total labeling of G'. Let $wt(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ be the weights of the subgraph H' isomorphic to $G+e$ and let $W = \{w(H') : H' \cong G+e\} = \{a, a+d, a+2d, \dots, a+(t-1)d\}$ be the set of H' weights and t be the number of subgraphs. Here $t = n$. Now, it is easy to see that the minimum possible weight of H' is at least $(p+1)(p+2)/2 + (q+1)(p+n) + (q+1)(q+2)/2$ i.e., $a \geq (p+1)(p+2)/2 + (q+1)(p+n) + (q+1)(q+2)/2$. Also the maximum possible weight of H' is not more than $(p+1)(p+n) - p(p+1)/2 + (q+1)(p+q+2n) - q(q+1)/2$, i.e., $a+(t-1)d \leq (p+1)(p+n) - p(p+1)/2 + (q+1)(p+q+2n) - q(q+1)/2$, $(n-1)d \leq (n-1)(p+q+2)$, thus $d \leq p+q+2$. ∎

Theorem 5. *The graph G' admits a super $\left(\frac{1}{2}(p+q)(p+q+3) + n(q+2) + p + 1, 0\right)$-$(G+e)$-antimagic total labeling.*

Proof. Let $Z = [1, p+q+2n]$ and partition Z into four sets such that $Z = A \cup B \cup C \cup D$ where $A = [1,p], B = [p+1, p+n], C = [p+n+1, p+q+n]$ and $D = [p+q+n+1, p+q+2n]$. Let $K = B \cup D$ and let $x = p, y = n$ and $z = q$. Then by Lemma 2(i), K is n-balanced multisets with all its subsets being 2-sets and $\sum K_i = 2p+q+2n+1$, for each $i \in [1,n]$.

Now we define a total labeling f on G' as follows:

Label the vertices $v_i, 1 \leq i \leq p$ by the elements of A and label the edges $e_i, 1 \leq i \leq q$ by the elements of C in any manner. Next use the elements of K to label all the vertices and edges of the star, use the smaller labels for the vertices and bigger labels for the edges in reverse order. Then for each $i, 1 \leq i \leq n$,

$$\begin{aligned}
wt(G+e_{q+i}) &= (1+2+3+\cdots+p) + (p+n+1+p+n+2, \dots, p+q+n) \\
&\quad + \sum K_i \\
&= \frac{p(p+1)}{2} + q(p+n) + \frac{q(q+1)}{2} + 2p+q+2n+1 \\
&= \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + (p+n)(q+2) + q + 1 \\
&= \frac{1}{2}(p+q)(p+q+3) + n(q+2) + p + 1.
\end{aligned}$$

Hence G' has a super $(\frac{1}{2}(p+q)(p+q+3) + n(q+2) + p + 1, 0)$-$(G+e)$-antimagic total labeling. ∎

Theorem 6. *The graph G' has a super* $\left(\frac{1}{2}[(p+q)^2+(p+q)(2n+3)+5n-n^2]+1,1\right)$*-$(G+e)$-antimagic total labeling.*

Proof. Let $Z=[1,p+q+2n]$ and partition Z into four sets such that $Z=A\cup B\cup C\cup D$ where $A=\{2,4,\ldots,2n,2n+1,2n+2,\ldots,p+n\}$, $B=\{1,3,5,\ldots,2n-1\}$, $C=[p+n+1,p+q+n]$ and $D=[p+q+n+1,p+q+2n]$. Let $K=B\cup D$ and let $x=p,y=n$ and $z=q$. Then by Lemma 3(i), K is n-balanced multisets with all its subsets being 2-sets and $\sum K_i=p+q+2n+i$, for each $i\in[1,n]$.

Now we define a total labeling f on G' as follows:

Label the vertices $v_i, 1\le i\le p$ with the elements of A and label the edges $e_i, 1\le i\le q$ with the elements of C in any order. Next use the elements of K to label all the vertices and edges of the star, use the smaller labels for the vertices and bigger labels for the edges in reverse order. Then for each $i, 1\le i\le n$,

$$\begin{aligned} wt(G+e_{q+i}) &= 2+4+6+\cdots+2n+2n+1+2n+2+\cdots+2n+p-n \\ &\quad +p+n+1+p+n+2+\cdots+p+n+q+\sum K_i \\ &= n(n+1)+(p-n)2n+\frac{(p-n)(p-n+1)}{2} \\ &\quad +q(p+n)+\frac{q(q+1)}{2}+p+q+2n+i \\ &= \frac{1}{2}\left((p+q)^2+(p+q)(2n+3)+5n-n^2\right)+i. \end{aligned}$$

Hence G' has a super $(\frac{1}{2}[(p+q)^2+(p+q)(2n+3)+5n-n^2]+1,1)$-$(G+e)$-antimagic total labeling. ∎

Theorem 7. *The graph G' has a super* $\left(\frac{1}{2}(p+q)(p+q+3)+(q+1)n+p+2,2\right)$*-$(G+e)$-antimagic total labeling.*

Proof. Consider the partition of $[1,p+q+2n]$ introduced in the proof of Theorem 5. By Lemma 2(ii), $\sum K_i=2p+q+n+2i$, for each $i\in[1,n]$.

$$\begin{aligned} wt(G+e_{q+i}) &= \frac{p(p+1)}{2}+q(p+n)+\frac{q(q+1)}{2}+2p+q+n+2i \\ &= \frac{1}{2}(p+q)(p+q+3)+(q+1)n+p+2i. \end{aligned}$$

Hence G' has a super $(\frac{1}{2}(p+q)(p+q+3)+(q+1)n+p+2,2)$-$(G+e)$-antimagic total labeling. ∎

Theorem 8. *The graph G' has a super* $\left(\frac{1}{2}[(p+q)^2+(p+q)(2n+3)-(n-1)(n-2)]+3,3\right)$*-$(G+e)$-antimagic total labeling.*

Proof. Consider the partition of $[1, p+q+2n]$ introduced in the proof of Theorem 6. By Lemma 3(ii), $\sum K_i = p+q+n-1+3i$, for each $i \in [1, n]$.

$$\begin{aligned} wt(G+e_{q+i}) &= \frac{n(n+1)}{2} + (p-n)(2n) + \frac{(p-n)(p-n+1)}{2} q(p+n) \\ &+ \frac{q(q+1)}{2} + p+q+n-1+3i \\ &= \frac{1}{2}[(p+q)^2 + (p+q)(2n+3) - (n-1)(n-2)] + 3i. \end{aligned}$$

Hence G' has a super $\left(\frac{1}{2}[(p+q)^2 + (p+q)(2n+3) - (n-1)(n-2)] + 3, 3\right)$-$(G+e)$-antimagic total labeling. ∎

Theorem 9. *The graph $G_u[S_2]$ admits a super (a, d)-$(G+e)$-antimagic total labeling if and only if $d \in \{0, 1, 2, \ldots, p+q+2\}$.*

Proof. By Theorems 5–8, we have $d \in \{0, 1, 2, 3\}$. The weight of G is the same for all the weights of the subgraphs $(G+e_i), i = 1, 2$. So it is enough to find the labels of vertices and edges of the star S_2. Now, for each $i, 1 \leq i \leq p-2$ we define the labeling f_i as follows.

$$\begin{aligned} f_i(w_1) &= p-i, \qquad 1 \leq i \leq p-2, \\ f_i(e_{q+1}) &= p+q+3, \\ f_i(w_2) &= p+2, \text{ and} \\ f_i(e_{q+2}) &= p+q+4. \end{aligned}$$

Thus, the induced sums of the labels of vertices and edges of S_2 are $2p+q+3-i$ and $2p+q+6$. Hence, $d = 3+i, 1 \leq i \leq p-2$. Therefore, $d = 4, 5, \ldots, p+1$.

Also for each $i, 1 \leq i \leq q+1$, we define the labeling f_i as follows

$$\begin{aligned} f_i(w_1) &= 1, \\ f_i(e_{q+1}) &= p+q+4-i, 1 \leq i \leq q+1, \\ f_i(w_2) &= p+2, \text{ and} \\ f_i(e_{q+2}) &= p+q+4. \end{aligned}$$

Thus, the induced sums of the labels of vertices and edges of S_2 are $p+q+5-i, 2p+q+6$. Hence $d = p+1+i, 1 \leq i \leq q+1$. Therefore, $d = p+2, p+4, \ldots, p+q+2$.

Hence the results follows. ∎

Open Problem 10. *For each $d, 4 \leq d \leq p+q+2$, either find the super (a, d)-$(G+e)$-antimagic total labeling of the graph $G_u[S_n], n \geq 3$, or prove that this labeling does not exist.*

4. CATERPILLAR

Definition 11. *The backbone of a caterpillar is the graph obtained from it by removing its pendant edges.*

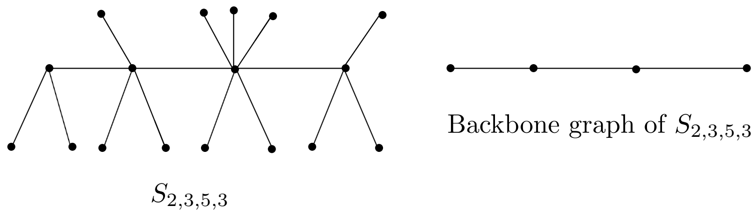

Figure 2

Theorem 12. *A caterpillar* $S_{n_1,n_2,\ldots,n_k}$ *has a super* $\big(2(k+2)n^2+7kn+2k+1+\lceil\frac{k}{2}\rceil, 4n^2\big)$*-*$S_{n,n}$*-antimagic total labeling for* $n_1=n_2=\cdots=n_k=n$.

Proof. As in [8], let $G \cong S_{n_1,n_2,\ldots,n_k}$ with $n_1=n_2=\cdots=n_k=n$. Then $|V(G)|=k(n+1)$ and $|E(G)|=k(n+1)-1$.

Let $V(G)=\{c_i : 1\le i\le k\}\cup\{v_{ij} : 1\le i\le k, 1\le j\le n\}$ and
$E(G)=\{c_ic_{i+1} : 1\le i\le k-1\}\cup\{c_iv_{ij} : 1\le i\le k, 1\le j\le n\}$.

Let $Z=[1,2k(n+1)-1]$ and partition Z into four sets such that $Z=A\cup B\cup C\cup D$, where $A=[1,k]$, $B=[k+1,k(n+1)]$, $C=[k(n+1)+1,k(n+1)+k-1]$ and $D=[k(n+2),2k(n+1)-1]$. Let us take $A=\{x_i : 1\le i\le k\}$ such that

$$x_i=\begin{cases}\lfloor\frac{i}{2}+1\rfloor & \text{for odd } i,\\ \lceil\frac{k}{2}\rceil+\frac{i}{2} & \text{for even } i.\end{cases}$$

Let $K=B\cup D$ and let $x=k, y=n$. Then by Lemma 1, K is kn-balanced with all its subsets being 2-sets and $\sum K_i=k(n+3)+2i-1$, for each $i\in[1,k_n]$.

Now we define a total labeling f on G as follows:

Label the vertices of the backbone by the elements of A with the ordering from left to right and label the backbone edges by the elements of C from right to left. Next we use the elements of K to label all the remaining edges and vertices, use the smaller labels for the vertices.

Now for each $1\le h\le k-1$, we have

$$wt(S^h_{n,n})=\sum\nolimits_{j=nh-n+1}^{(h+1)n}[k(n+3)+2j-1]+\frac{h+1}{2}+\left\lceil\frac{k}{2}\right\rceil+\frac{h+1}{2}$$
$$+k(n+1)+k-h=2kn^2+7kn+2k+1+\left\lceil\frac{k}{2}\right\rceil+4hn^2.$$

In particular, we obtain that $a=wt(S^1_{n,n})=2(k+2)n^2+7kn+2k+1+\lceil\frac{k}{2}\rceil$ and $d=wt(S^{h+1}_{n,n})-wt(S^h_{n,n})=4n^2$, then G has a super $(2(k+2)n^2+7kn+2k+1+\lceil\frac{k}{2}\rceil, 4n^2)$-$S_{n,n}$-antimagic total labeling. ∎

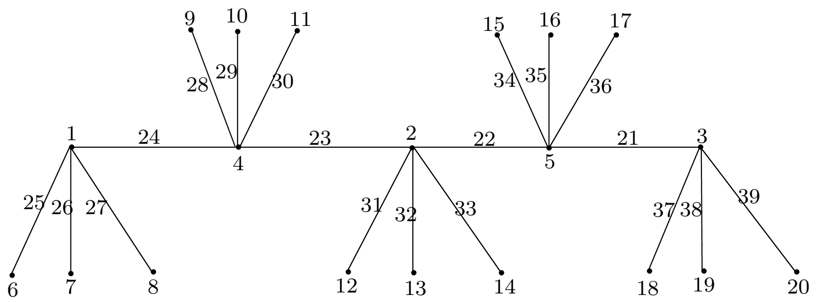

Figure 3. Super (245, 36)-$S_{3,3}$-antimagic total graph.

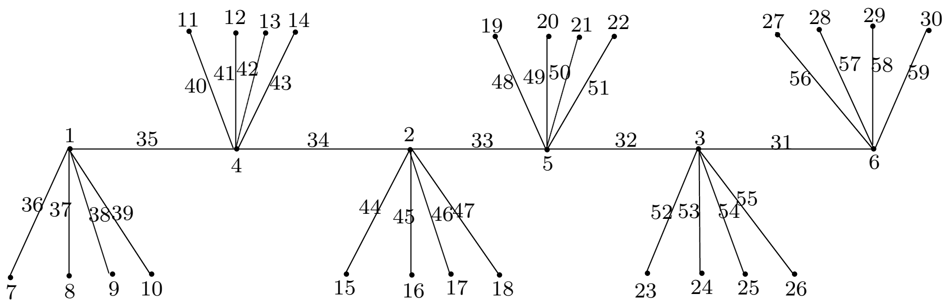

Figure 4. Super (440, 64)-$S_{4,4}$-antimagic total graph.

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