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A NEW CHARACTERIZATION OF UNICHORD-FREE GRAPHS

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Abstract

Unichord-free graphs are defined as having no cycle with a unique chord. They have appeared in several papers recently and are also characterized by minimal separators always inducing edgeless subgraphs (in contrast to characterizing chordal graphs by minimal separators always inducing complete subgraphs). A new characterization of unichord-free graphs corresponds to a suitable reformulation of the standard simplicial vertex characterization of chordal graphs.

Keywords: chordal graph, unichord-free graph, minimal separator.

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1. Chordal Versus Unichord-Free Graphs

A graph is *chordal* if every non-triangular cycle has a *chord* (an edge between nonconsecutive vertices of the cycle)—in other words, if every cycle large enough to have a chord does have a chord. Chordal graphs form an extremely well studied and successfully applied graph class; see [3, 11].

A graph is *unichord-free* if no cycle has a unique chord, which is equivalent to every chord of every cycle having a crossing chord in the cycle (where chords ab and cd of a cycle C are *crossing chords* if their four endpoints come in the order a, c, b, d around C). The unichord-free graphs were introduced (but not called unichord-free) independently in [9, 14]; they have since been studied (and called unichord-free) in additional papers, including [6, 7, 8]. Complete bipartite graphs, the Petersen graph, and the Heawood graph are examples of unichord-free graphs. Reference [14] gives a detailed structural characterization of the unichord-free graphs and observes that 2-connected non-complete unichord-free graphs must be triangle-free.

For nonadjacent vertices v and w in a connected graph G, a *v,w-separator* is a set $S \subseteq V(G) - \{v, w\}$ such that v and w are in different connected components of the subgraph of G induced by $V(G) - S$. A *minimal v,w-separator* is a v,w-separator that is not a proper subset of another v,w-separator. A *minimal separator* of G is a minimal v,w-separator for some $v, w \in V(G)$. For every vertex v of a graph G, let the *neighborhood* $N(v)$ of v be $\{x \in V(G) : xv \in E(G)\}$ and let $N[v] = N(v) \cup \{v\}$. Two vertices v and w are *adjacent twins* if $N[v] = N[w]$.

In 1961, Dirac famously characterized chordal graphs by every minimal separator inducing a complete subgraph; see [3, 11]. Proposition 1, from [9], characterized unichord-free graphs (calling them the *independent separator graphs*) in contrast to Dirac's characterization.

Proposition 1. *A graph is unichord-free if and only if every minimal separator induces an edgeless subgraph.*

It is intriguing that testing whether a graph with n vertices and m edges has a minimal separator that induces an edgeless graph (a "stable cutset" in [2, 4]) is NP-complete, while testing for a minimal separator that induces a complete graph (a "clique cutset" in [13, 15]) can be done by a $O(nm)$ algorithm—and yet testing whether a graph is unichord-free or is chordal can be done by, respectively, $O(n + m)$ and $O(nm)$ algorithms, see [3] and [14].

A *block* of a graph is a either an edge that is in no cycle or a maximal 2-connected subgraph—in other words, a block is a maximal subgraph such that every two edges are in a common cycle. A graph is both chordal and unichord-free if and only if every block is complete (and so every minimal separator is a singleton). Figure 1 illustrates the other combinations of being chordal and being unichord-free. Reference [10] shows that the graphs in which every minimal separator induces either a complete or an edgeless subgraph are precisely the edge-sums of chordal and unichord-free graphs.

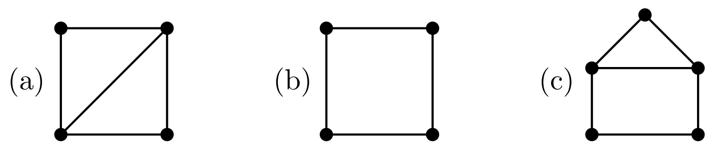

Figure 1. The smallest graphs that are: (a) chordal but not unichord-free, (b) unichord-free but not chordal, (c) neither chordal nor unichord-free.

A graph is unichord-free [or, respectively, chordal] if and only each of its blocks is unichord-free [respectively, chordal], since cycles are always contained inside blocks. In terms of minimal separators, a graph is unichord-free if and only if every minimal x, y-separator S for which x and y are in different blocks must have $|S| = 1$, making the subgraph induced by S be simultaneously edgeless and complete. This motivates the 2-connectedness assumption in the theorems in the present paper.

Section 2 contains a new characterization of unichord-free graphs that somewhat evokes the often-used characterization that G is chordal if and only if every induced subgraph of G contains a *simplicial vertex*—a vertex v such that $N(v)$ induces a complete subgraph; see [3, 11]. Section 3 then discusses the correspondence between the two characterizations in detail.

2. A New Unichord-Free Graph Characterization

Theorem 2. *A graph is unichord-free if and only if, in every* 2*-connected induced subgraph* H *that is not complete, every* $N(v)$ *is a minimal separator of* H *that induces an edgeless subgraph of* H*.*

Proof. First suppose G is a unichord-free graph with an induced subgraph H that is 2-connected and not complete, say containing nonadjacent vertices v and w. Since H is also unichord-free, H is triangle-free and so $N(v)$ is an independent set.

Suppose for the moment that $w \in V(H) - N[v]$ and the v, w-separator $N(v)$ is not a minimal separator. Let $S \subset N(v)$ be a minimal v, w-separator and $x \in N(v) - S$. Since H is 2-connected, H has a cycle C that contains the edge vx and the vertex w. Let π be the v-to-x-to-w subpath of C, and let y be the first vertex in $V(\pi) \cap S$ met when traveling along π from v toward w (such a y exists since S is a minimal v, w-separator). Let τ be a v-to-x-to-y path with $V(\tau) \subseteq V(\pi)$ whose x-to-y subpath is chordless. Since H is 2-connected and S is a minimal v, w-separator, $|S| \geq 2$ and so H has a cycle C' that contains vy and w but not x, with no vertex of τ adjacent to a vertex in $V(C') - (S \cup \{v\})$ (again since S is a minimal v, w-separator). Let τ' be a chordless v-to-y path in $H - vy$ with $V(\tau') \subseteq V(C')$. But now, since $N(v)$ is independent, vy would be the unique chord of the cycle $\tau \cup \tau'$ (contradicting that H is unichord-free).

Therefore, $N(v)$ must be a minimal separator of H that induces an edgeless subgraph.

Conversely, suppose G is not unichord-free. Thus G contains an induced subgraph H that consists of a cycle C of length 4 or more with a unique chord xy (and so H is 2-connected and not complete). But then $N(x)$ would not even be a minimal separator of H. ■

The condition that H is 2-connected is necessary in Theorem 2, since trees are unichord-free without $N(v)$ being a minimal separator whenever v is a non-leaf vertex. The condition that H is not complete is also necessary, since complete graphs are unichord-free without any $N(v)$ being a minimal separator.

3. Comparison with Chordal Graph Results

It is natural to wonder about a chordal graph analog to the characterization of unichord-free graphs in Theorem 2, given the similarity between Proposition 1 and Dirac's complete minimal separator characterization of chordal graphs. To this end, Theorem 3 can be viewed as a mixture of Dirac's characterization and the familiar characterization of chordal graphs by every induced subgraph containing a simplicial vertex (see [3, 11]).

Define a *maxclique* of a graph G to be a maximal complete subgraph of G (and so a vertex is simplicial if and only if it is in a unique maxclique). Following [12], a *simplicial clique* is a maxclique that contains at least one simplicial vertex of G, and a *boundary clique* is a simplicial clique Q such that either every vertex in Q is simplicial (which makes G complete) or there is a second maxclique Q' of G that contains all the nonsimplicial vertices in Q. Equivalently, a simplicial clique Q is a boundary clique if and only if all the nonsimplicial vertices in Q have a common neighbor outside of Q (and inside Q'). Every chordal graph G that is not complete will have such maxcliques Q and Q', and $V(Q) \cap V(Q')$ will always be a minimal separator of G; see [3, 11, 12] for details (including that the boundary cliques Q of a chordal graph G will correspond to the leaf nodes of a clique tree T for G, with edges QQ' corresponding to the pendant edges of T).

Theorem 3. *A graph is chordal if and only if, in every* 2*-connected induced subgraph H that has no adjacent twins, some $N(v)$ is a minimal separator of H that induces a complete subgraph of H.*

Proof. First suppose G is a chordal graph with an induced subgraph H that is 2-connected without adjacent twins. Thus H is also chordal and is not complete, so H has a boundary clique Q and a second maxclique Q' such that Q contains a unique simplicial vertex v (since distinct simplicial vertices would be adjacent twins) and $V(Q) \cap V(Q') = V(Q) - \{v\} = N(v)$ is a minimal separator of H. Therefore, $N(v)$ is a minimal separator that induces a complete subgraph.

Conversely, suppose G is not chordal. Thus G contains an induced subgraph H that consists of a chordless cycle C of length 4 or more (and so H is 2-connected without adjacent twins). But then each vertex v in $H = C$ would have $N(v)$ be a minimal separator of H that would not induce a complete subgraph of H. ■

The condition that H is 2-connected is necessary in Theorem 3, since trees are chordal without $N(v)$ being a minimal separator whenever v is a non-leaf vertex. The condition that H has no adjacent twins is also necessary, since the

2-connected graph shown in Figure 2 is chordal, with (three pairs of) adjacent twins without any $N(v)$ being a minimal separator.

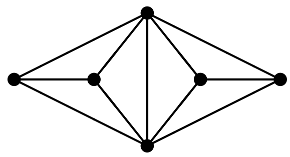

Figure 2. A chordal graph that is not complete and has adjacent twins in which no neighborhood is a minimal separator.

Theorem 4 will be a combined—but more awkward—form of Theorems 2 and 3, replacing the condition “not complete” and the stronger condition “no adjacent twins” in those theorems with the logically intermediate condition “every boundary clique contains a unique simplicial vertex”.

Theorem 4. *A graph is unichord-free* [*or, respectively, chordal*] *if and only if, in every* 2*-connected induced subgraph* H *in which every boundary clique contains a unique simplicial vertex, every* [*respectively, some*] $N(v)$ *is a minimal separator of* H *that induces an edgeless* [*respectively, a complete*] *subgraph of* H.

Proof. Theorem 4 has essentially the same proofs as do Theorems 2 and 3 after the substitution of “every boundary clique contains a unique simplicial vertex”. This follows from the four observations below for, respectively, the “only if” direction ($\Rightarrow$) and the “if” direction ($\Leftarrow$) of Theorems 2 and 3.

($\Rightarrow$, Theorem 2): If a unichord-free graph G has a 2-connected induced subgraph H in which every boundary clique contains a unique simplicial vertex, then H is not complete.

($\Leftarrow$, Theorem 2): If H consists of a cycle C of length 4 or more with a unique chord xy, then H is 2-connected. When $|V(C)| = 4$, H has two boundary cliques: its two triangles, each containing a unique simplicial vertex. When $|V(C)| \geq 5$, H has no boundary cliques, and so every boundary clique of H would, vacuously, contain a unique simplicial vertex of H.

($\Rightarrow$, Theorem 3): If a chordal graph G has a 2-connected induced subgraph H in which every boundary clique contains a unique simplicial vertex, then H is also chordal and so has Q, Q', and v as in the proof of Theorem 3.

($\Leftarrow$, Theorem 3): If H consists of a cycle of length 4 or more with no chord, then H is 2-connected and has no boundary cliques, and so every boundary clique of H would, vacuously, contain a unique simplicial vertex of H. ■

In closing, observe how the nearly parallel characterizations of unichord-free and chordal in Theorem 4 highlight the unexpected quantifier switch occuring between “every” and “some.” This switch apparently combines with the conflicting properties of "edgeless" and "complete" to cause—somehow—the intriguing resemblance of the cycle-and-chord definitions of unichord-free graphs and chordal graphs in Section 1.

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