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DECOMPOSITION OF COMPLETE BIPARTITE MULTIGRAPHS INTO PATHS AND CYCLES HAVING k EDGES

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Abstract

We give necessary and sufficient conditions for the decomposition of complete bipartite multigraph $K_{m,n}(\lambda)$ into paths and cycles having k edges. In particular, we show that such decomposition exists in $K_{m,n}(\lambda)$, when $\lambda \equiv 0 \pmod 2$, $m, n \geq \frac{k}{2}$, $m+n > k$, and $k(p+q) = 2mn$ for $k \equiv 0 \pmod 2$ and also when $\lambda \geq 3$, $\lambda m \equiv \lambda n \equiv 0 \pmod 2$, $k(p+q) = \lambda mn$, $m, n \geq k$, (resp., $m, n \geq 3k/2$) for $k \equiv 0 \pmod 4$ (respectively, for $k \equiv 2 \pmod 4$). In fact, the necessary conditions given above are also sufficient when $\lambda = 2$.

Keywords: path, cycle, graph decomposition, multigraph.

2010 Mathematics Subject Classification: 05C38, 05C51.

1. Introduction

Unless stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology the readers are referred to [8]. A cycle of length m is called an *m-cycle* and it is denoted by C_m and a path of length m is called an *m-path* and it is denoted by P_{m+1}. A circuit (directed cycle) of length m is called an *m-circuit* and it is denoted by $\overrightarrow{C}_m$. Let K_m denote a complete graph on m vertices, $K_{m,n}$ denote a complete bipartite graph with

m and n vertices in the parts, and $K^*_{m,n}$ denote a complete bipartite symmetric directed graph with m and n vertices in the parts. A graph whose vertex set is partitioned into sets $V_1, \ldots, V_m$ such that the edge set is $\bigcup_{i \neq j \in [m]} V_i \times V_j$ is called a *complete m-partite graph* denoted by $K_{n_1,\ldots,n_m}$, where $|V_i| = n_i$ for all i. For any integer $\alpha > 0$, αG denotes a union of α edge-disjoint copies of G. The *λ-multiplication* of G, denoted $G(\lambda)$, is the multigraph obtained from a graph G by replacing each edge with λ edges. For a graph G, $G - I$ denotes the graph G with a 1-factor I removed. Let $x_0x_1 \cdots x_{k-2}x_{k-1}$ and $(x_0x_1 \cdots x_{k-1}x_0)$ respectively denote the path P_k and the cycle C_k with vertices $x_0, x_1, \ldots, x_{k-1}$ and edges $x_0x_1, x_1x_2, \ldots, x_{k-2}x_{k-1}, x_{k-1}x_0$.

By a *decomposition* of the graph G, we mean a list of edge-disjoint subgraphs of G whose union is G (ignoring isolated vertices). For the graph G, if $E(G)$ can be partitioned into $E_1, \ldots, E_k$ such that the subgraph induced by E_i is H_i, for all i, $1 \leq i \leq k$, then we say that $H_1, \ldots, H_k$ *decompose* G and we write $G = H_1 \oplus \cdots \oplus H_k$, since $H_1, \ldots, H_k$ are edge-disjoint subgraphs of G. For $1 \leq i \leq k$, if $H_i \cong H$, we say that G has a *H-decomposition.* If G has a decomposition into p copies of H_1 and q copies of H_2, then we say that G has a *$\{pH_1, qH_2\}$-decomposition.* If such a decomposition exists for all admissible pairs of p and q satisfying trivial necessary conditions, then we say that G has a *full $\{H_1, H_2\}$-decomposition* or G is *fully $\{H_1, H_2\}$-decomposable.*

Study on full $\{H_1, H_2\}$-decomposition of graphs is not new. Abueida, Daven, and Roblee [1, 3] completely determined the values of n for which $K_n(\lambda)$ admits the $\{pH_1, qH_2\}$-decomposition such that $H_1 \oplus H_2 \cong K_t$, when $\lambda \geq 1$ and $|V(H_1)| = |V(H_2)| = t$, where $t \in \{4, 5\}$. Let S_k denotes a star on k vertices, i.e. $S_k = K_{1,k-1}$. Abueida and Daven [2] proved that there exists a $\{pK_k, qS_{k+1}\}$-decomposition of K_n for $k \geq 3$ and $n \equiv 0, 1 \pmod{k}$. Abueida and O'Neil [4] proved that for $k \in \{3, 4, 5\}$, the $\{pC_k, qS_k\}$-decomposition of $K_n(\lambda)$ exists, whenever $n \geq k + 1$ except for the ordered triples $(k, n, \lambda) \in \{(3, 4, 1), (4, 5, 1), (5, 6, 1), (5, 6, 2), (5, 6, 4), (5, 7, 1), (5, 8, 1)\}$. Abueida and Daven [2] obtained necessary and sufficient conditions for the $\{pC_4, q(2K_2)\}$-decomposition of the Cartesian product and tensor product of paths, cycles, and complete graphs. Shyu [17] obtained a necessary and sufficient condition for the existence of a full $\{P_5, C_4\}$-decomposition of K_n. Shyu [18] proved that K_n has a full $\{P_4, S_4\}$-decomposition if and only if $n \geq 6$ and $3(p + q) = \binom{n}{2}$. Also he proved that K_n has a full $\{P_k, S_k\}$-decomposition with a restriction $p \geq k/2$, when k even (resp., $p \geq k$, when k odd). Shyu [19] obtained a necessary and sufficient condition for the existence of a full $\{P_4, K_3\}$-decomposition of K_n. Shyu [20] proved that K_n has a full $\{C_4, S_5\}$-decomposition if and only if $4(p + q) = \binom{n}{2}$, $q \neq 1$, when n is odd and $q \geq \max\{3, \lceil \frac{n}{4} \rceil\}$, when n is even. Shyu [21] proved that $K_{m,n}$ has a full $\{P_k, S_k\}$-decomposition for some m and n and also obtained some necessary and sufficient condition for the existence of a full $\{P_4, S_4\}$-decomposition of

$K_{m,n}$. Sarvate and Zhang [16] obtained necessary and sufficient conditions for the existence of a $\{pP_3, qK_3\}$-decomposition of $K_n(\lambda)$, when $p = q$.

Chou *et al.* [9] proved that for a given triple (p, q, r) of nonnegative integers, G decompose into p copies of C_4, q copies of C_6, and r copies of C_8 such that $4p + 6q + 8r = |E(G)|$ in the following two cases: (a) $G = K_{m,n}$ with m and n both even and greater than four (b) $G = K_{n,n} - I$, where n is odd. Chou and Fu [10] proved that the existence of a full $\{C_4, C_{2t}\}$-decomposition of $K_{2u,2v}$, where $t/2 \le u, v < t$, when t even (resp., $(t + 1)/2 \le u, v \le (3t - 1)/2$, when t odd) implies such decomposition in $K_{2m,2n}$, where $m, n \ge t$ (resp., m, $n \ge (3t + 1)/2$). The authors [11] reduced the bounds of the sufficient conditions obtained by Chou and Fu [10] for the existence of a full $\{C_4, C_{2t}\}$-decomposition of $K_{2m,2n}$, when $t > 2$. Lee and Chu [13, 14] obtained a necessary and sufficient condition for the existence of a full $\{P_k, S_k\}$-decomposition of $K_{n,n}$ and $K_{m,n}$. Lee and Lin [15] obtained a necessary and sufficient condition for the existence of a full $\{pC_k, qS_{k+1}\}$-decomposition of $K_{n,n} - I$. Abueida and Lian [7] obtained necessary and sufficient conditions for the existence of a $\{pC_k, qS_{k+1}\}$-decomposition of K_n for some n. Recently, the authors [12] obtained some necessary and sufficient conditions for the existence of a full $\{P_{k+1}, C_k\}$-decomposition of K_n and $K_{m,n}$.

In this paper, we study only the existence of a full $\{P_{k+1}, C_k\}$-decomposition of $K_{m,n}(\lambda)$, we abbreviate the notation for such decomposition as $(k; p, q)$-*decomposition* of $K_{m,n}(\lambda)$. The obvious necessary condition for such existence is $k(p + q) = |E(K_{m,n}(\lambda))|$. As we consider only cases where all vertices are of even degree, the case $p \ne 1$ is also obviously necessary, since the presents of a single path in the decomposition would give two vertices of odd degree and the resulting graph is not cycle decomposable. Call the situation with $k(p + q) = |E(K_{m,n}(\lambda))|$, all vertex degrees are even, and $p \ne 1$ the *good case.*

We prove that in the good case $K_{m,n}(\lambda)$ has a $(k; p, q)$-decomposition, when $\lambda \equiv 0 \pmod 2$, $m, n \ge \frac{k}{2}$, $m + n > k$, and $k(p + q) = 2mn$ for $k \equiv 0 \pmod 2$. Further, we show that if $K_{m,n}(\lambda)$, $\lambda \ge 3$, $k \equiv 0 \pmod 4$ (resp., $k \equiv 2 \pmod 4$) has a $(k; p, q)$-decomposition in the good case with $k/2 \le m, n \le k$, (resp., $k/2 \le m, n \le 3k/2$,) then such decomposition also exists in the good case, when $\lambda \ge 3$; $m, n \ge k$ (resp., $m, n \ge 3k/2$).

To prove our results, we use the following:

Theorem 1 [12]**.** *Let p and q be nonnegative integers and k, m, n be positive even integers such that $k \equiv 0 \pmod 4$. For $m \le n$, the graph $K_{m,n}$ has a $(k; p, q)$-decomposition if and only if $m \ge \frac{k}{2}$, $n \ge \lceil \frac{k+1}{2} \rceil$, $k(p + q) = mn$, and $p \ne 1$.*

Theorem 2 [22]**.** *$K^*_{m,n}$ has a $\overrightarrow{C_k}$-decomposition if and only if $m \ge \frac{k}{2}$, $n \ge \frac{k}{2}$, and k divides $2mn$.*

By considering the underlying graph of $K^*_{m,n}$, we have the following from Theorem 2.

Theorem 3. *The graph $K_{m,n}(2)$ has a C_k-decomposition if and only if $m \geq \frac{k}{2}$, $n \geq \frac{k}{2}$, and k divides $2mn$.*

2. $(k; p, q)$-DECOMPOSITION OF $K_{m,n}(\lambda)$ WHEN $k \equiv 0 \pmod 2$

In this section, we investigate the existence of $(k; p, q)$-decomposition of $K_{m,n}(\lambda)$, when $k \equiv 0 \pmod 2$.

Construction 4. *Let $\mathbb{C}_\lambda$ and $\mathbb{C}_\mu$ be two cycles of length k, where $\mathbb{C}_\lambda = (x_1 x_2 \cdots x_k x_1)$ and $\mathbb{C}_\mu = (y_1 y_2 \cdots y_k y_1)$. If v is a common vertex of $\mathbb{C}_\lambda$ and $\mathbb{C}_\mu$ such that at least one neighbour of v from each cycle (say, x_i and y_j) does not belongs to the other cycle, then we have two edge-disjoint paths of length k, say $\mathbb{P}_\lambda$ and $\mathbb{P}_\mu$ from $\mathbb{C}_\lambda$ and $\mathbb{C}_\mu$ as follows (see Figure 1), where $\mathbb{P}_\lambda = (\mathbb{C}_\lambda - vx_i) \cup vy_j, \mathbb{P}_\mu = (\mathbb{C}_\mu - vy_j) \cup vx_i$.*

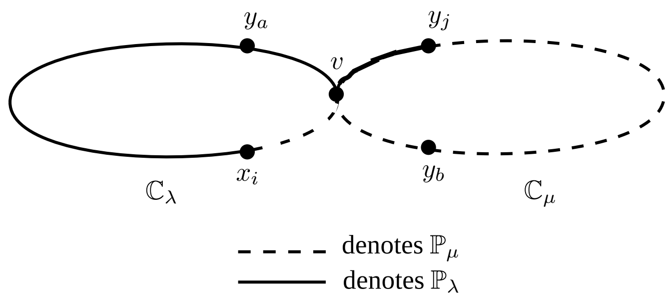

Figure 1. $\mathbb{C}_\lambda \cup \mathbb{C}_\mu = \mathbb{P}_\lambda \cup \mathbb{P}_\mu$.

Remark 5. Let $k \in \mathbb{N}$. If G and H have a $(k; p, q)$-decomposition, then $G \oplus H$ has such a decomposition.

Lemma 6. *Let p, q be nonnegative integers and $\{k, m, n\} \in \mathbb{N}$ such that $k \equiv 0 \pmod 2$ and $m + n > k$. The graph $K_{m,n}(2)$ has a $(k; p, q)$-decomposition if and only if $m, n \geq k/2$, $k(p + q) = 2mn$, and $p \neq 1$.*

Proof. **Necessity**. Conditions $m, n \geq k/2$, $k(p + q) = 2mn$, and $p \neq 1$ are trivial.

Sufficiency. Let $k \equiv 0 \pmod 2$. In order to have a C_k-decomposition in $K_{m,n}(2)$, we can always find u, v such that $k = 2uv$, $m = ru$, $n = sv$, $r \geq v$, and $s \geq u$, where r and s are positive integers. We denote the vertices of the partite sets of $K_{ru,sv}$ by x_i, $0 \leq i \leq ru - 1$ and y_j, $0 \leq j \leq sv - 1$. By Theorem 3, the graph $K_{ru,sv}(2)$ has a C_{2uv}-decomposition as follows:

$$\mathbb{C}_{\lambda\mu} = \Big(\cdots\big(\cdots x_{(\mu+i)u+j}y_{(\lambda+j)v+i}\cdots\big)_{0\le i\le v-1}\Big)_{0\le j\le u-1},$$
$$0\le\lambda\le s-1; 0\le\mu\le r-1,$$

where the indices of x are to be taken with modulo ru and those of y with modulo sv. Now we construct the required number of P_{k+1} from the C_k-decomposition given above, in two cases.

Case 1: p is even. For a fixed μ and $0\le\lambda\le s-1$, we can have $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{(\lambda+1)\mu}$ as above. Since $x_{\mu u}y_{\lambda v}\in E(\mathbb{C}_{\lambda\mu})$, $x_{\mu u}y_{(\lambda+u+1)v-1}\in E(\mathbb{C}_{(\lambda+1)\mu})$, $y_{\lambda v}\notin V(\mathbb{C}_{(\lambda+1)\mu})$, and $y_{(\lambda+u+1)v-1}\notin V(\mathbb{C}_{\lambda\mu})$, we have two edge-disjoint paths of length k, say $\mathbb{P}_{\lambda\mu}$ and $\mathbb{P}_{(\lambda+1)\mu}$ from $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{(\lambda+1)\mu}$ as follows (see Figure 2).

$$\mathbb{P}_{\lambda\mu} = (\mathbb{C}_{\lambda\mu} - x_{\mu u}y_{\lambda v})\cup x_{\mu u}y_{(\lambda+u+1)v-1},$$
$$\mathbb{P}_{(\lambda+1)\mu} = \big(\mathbb{C}_{(\lambda+1)\mu} - x_{\mu u}y_{(\lambda+u+1)v-1}\big)\cup x_{\mu u}y_{\lambda v}.$$

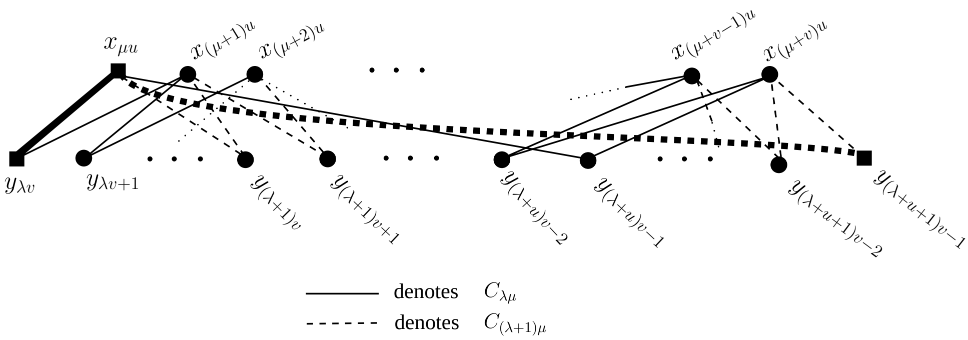

Figure 2. $\mathbb{C}_{\lambda\mu}\cup\mathbb{C}_{(\lambda+1)\mu} = \mathbb{P}_{\lambda\mu}\cup\mathbb{P}_{(\lambda+1)\mu}$.

Similarly, we can find pairs of paths of length k from the pairs of cycles $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{(\lambda+1)\mu}$, where $\lambda = 0, 2, \ldots, s-2$ or $s-1$ and $0\le\mu\le r-1$. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

Now for a fixed λ and $0\le\mu\le r-1$, we can have $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{\lambda(\mu+1)}$ as above. Since $x_{\mu p}y_{(\lambda+p)q-1}\in E(\mathbb{C}_{\lambda\mu})$, $x_{(\mu+q+1)p-1}y_{(\lambda+p)q-1}\in E(\mathbb{C}_{\lambda(\mu+1)})$, $x_{\mu p}\notin V(\mathbb{C}_{\lambda(\mu+1)})$, and $x_{(\mu+q+1)p-1}\notin V(\mathbb{C}_{\lambda\mu})$, we have two edge-disjoint paths of length k, say $\mathbb{P}_{\lambda\mu}$ and $\mathbb{P}_{\lambda(\mu+1)}$ from $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{\lambda(\mu+1)}$ as follows (see Figure 3).

$$\mathbb{P}_{\lambda\mu} = \big(\mathbb{C}_{\lambda\mu} - x_{\mu p}y_{(\lambda+p)q-1}\big)\cup x_{(\mu+q+1)p-1}y_{(\lambda+p)q-1},$$
$$\mathbb{P}_{\lambda(\mu+1)} = \big(\mathbb{C}_{\lambda(\mu+1)} - x_{(\mu+q+1)p-1}y_{(\lambda+p)q-1}\big)\cup x_{\mu p}y_{(\lambda+p)q-1}.$$

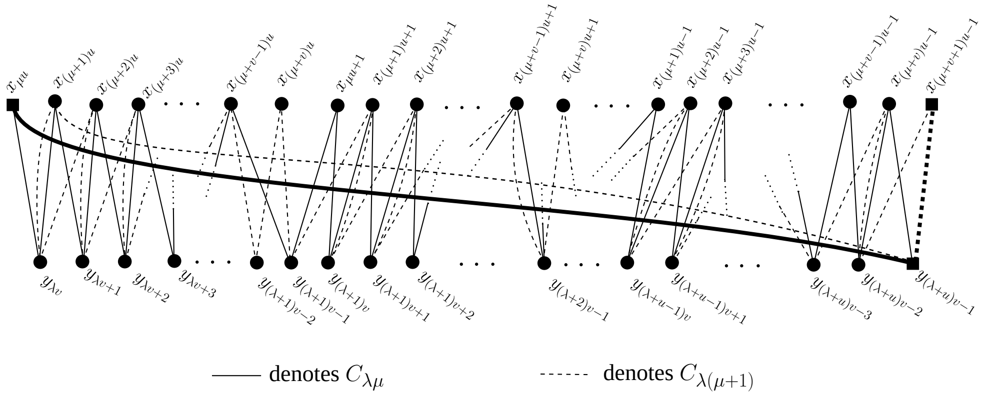

Figure 3. $\mathbb{C}_{\lambda\mu} \cup \mathbb{C}_{\lambda(\mu+1)} = \mathbb{P}_{\lambda\mu} \cup \mathbb{P}_{\lambda(\mu+1)}$.

Similarly, we can find pairs of paths of length k from the pairs of cycles $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{\lambda(\mu+1)}$, where $\mu = 0, 2, \ldots, r-2$ or $r-1$. Hence we have the desired paths.

Case 2: p is odd. Fixing $v = \gcd(n, k/2)$, we have $u = k/2v$, $s = n/v$. Since k divides $2mn$, i.e. $2uv$ divides $2mn$ and v divides n, we have $r = m/u$.

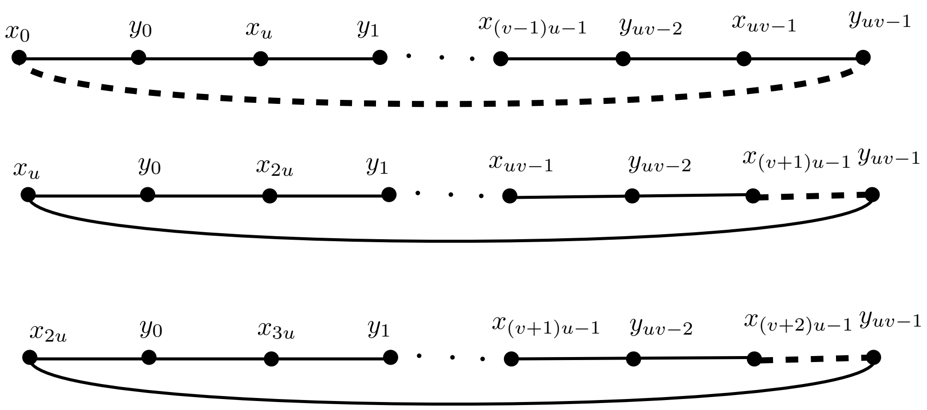

Figure 4. $\mathbb{C}_{00} \cup \mathbb{C}_{01} \cup \mathbb{C}_{02}$.

Subcase 2a: $(v+2)u - 1 \leq m$ and $v + 2 \leq r$. Since $r \geq 3$ and $s \geq 1$, we can have $\mathbb{C}_{00}$, $\mathbb{C}_{01}$, and $\mathbb{C}_{02}$ (see Figure 4). By applying a procedure similar to Construction 4, we have three edge-disjoint paths of length k, say $\mathbb{P}_{00}$, $\mathbb{P}_{01}$, and $\mathbb{P}_{02}$ from $\mathbb{C}_{00}$, $\mathbb{C}_{01}$, and $\mathbb{C}_{02}$ as follows (see Figure 5).

$$\mathbb{P}_{00} = (\mathbb{C}_{00} - x_0 y_{uv-1}) \cup x_{(v+1)u-1} y_{uv-1},$$
$$\mathbb{P}_{01} = \big(\mathbb{C}_{01} - x_{(v+1)u-1} y_{uv-1}\big) \cup x_{(v+2)u-1} y_{uv-1},$$
$$\mathbb{P}_{02} = \big(\mathbb{C}_{02} - x_{(v+2)u-1} y_{uv-1}\big) \cup x_0 y_{uv-1}.$$

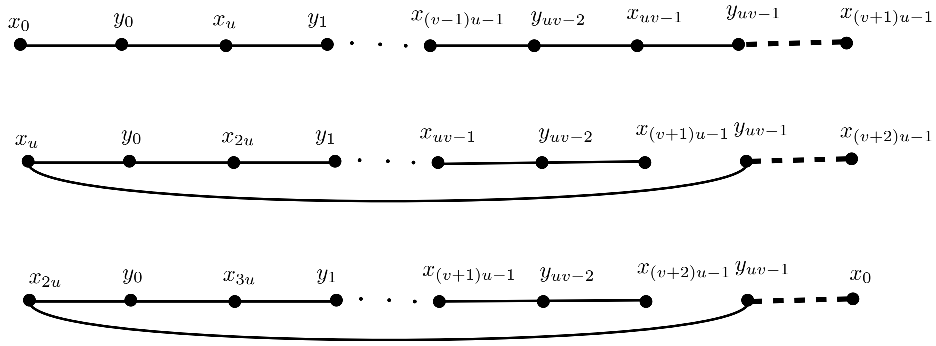

Figure 5. $\mathbb{P}_{00} \cup \mathbb{P}_{01} \cup \mathbb{P}_{02}$.

By applying a procedure similar to Case 1, the remaining pairs of cycles $\mathbb{C}_{\lambda\mu} \oplus \mathbb{C}_{\lambda(\mu+1)}$, (λ, μ), $(\lambda, \mu + 1) \neq (0, 0), (0, 1), (0, 2)$ decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

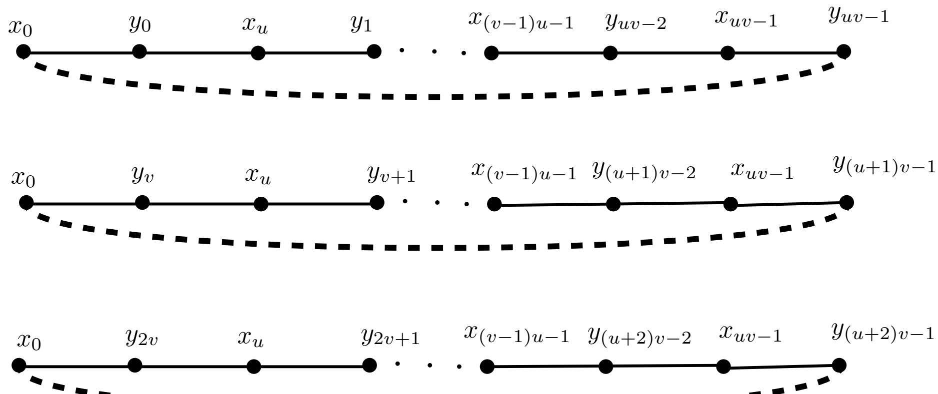

Figure 6. $\mathbb{C}_{00} \cup \mathbb{C}_{10} \cup \mathbb{C}_{20}$.

Subcase 2b: $(u + 2)v - 1 \leq n$ and $u + 2 \leq s$. Since $r \geq 1$ and $s \geq 3$, we can have $\mathbb{C}_{00}$, $\mathbb{C}_{10}$, and $\mathbb{C}_{20}$ (see Figure 6). By applying a procedure similar to

Construction 4, we have three edge-disjoint paths of length k, say $\mathbb{P}_{00}$, $\mathbb{P}_{10}$, and $\mathbb{P}_{20}$ from $\mathbb{C}_{00}$, $\mathbb{C}_{10}$, and $\mathbb{C}_{20}$ as follows (see Figure 7).

$$\begin{aligned}\mathbb{P}_{00} &= (\mathbb{C}_{00} - x_0y_{uv-1}) \cup x_0y_{(u+1)v-1},\\ \mathbb{P}_{10} &= \big(\mathbb{C}_{10} - x_0y_{(u+1)v-1}\big) \cup x_0y_{(u+2)v-1},\\ \mathbb{P}_{20} &= \big(\mathbb{C}_{20} - x_0y_{(u+2)v-1}\big) \cup x_0y_{uv-1}.\end{aligned}$$

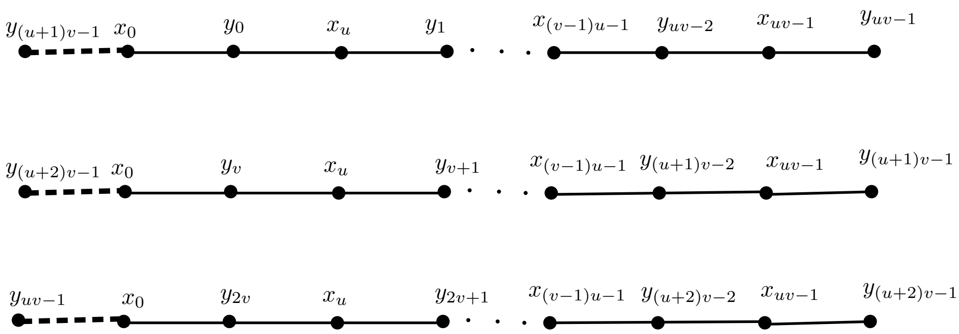

Figure 7. $\mathbb{P}_{00} \cup \mathbb{P}_{10} \cup \mathbb{P}_{20}$.

By applying a procedure similar to Case 1, the remaining pairs of cycles $\mathbb{C}_{\lambda\mu} \oplus \mathbb{C}_{(\lambda+1)\mu}$ (λ, μ), $(\lambda + 1, \mu) \neq (0, 0), (1, 0), (2, 0)$ decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

Subcase 2c: $(v+1)u-1 \leq m$, $(u+1)v-1 \leq n$, $u+1 \leq s$, and $v+1 \leq r$, m or $n \neq k/2$. Since $r, s \geq 2$ we can have $\mathbb{C}_{00}$, $\mathbb{C}_{10}$, and $\mathbb{C}_{11}$. By applying a procedure similar to Case 1, we have two edge-disjoint paths of length k, say $\mathbb{P}_{10}$ and $\mathbb{P}_{11}$ from $\mathbb{C}_{10}$ and $\mathbb{C}_{11}$ as follows:

$$\begin{aligned}\mathbb{P}_{10} &= \big(\mathbb{C}_{10} - x_0y_{(u+1)v-1}\big) \cup x_{(v+1)u-1}y_{(u+1)v-1},\\ \mathbb{P}_{11} &= \big(\mathbb{C}_{11} - x_{(v+1)u-1}y_{(u+1)v-1}\big) \cup x_0y_{(u+1)v-1}.\end{aligned}$$

Now consider $\mathbb{C}_{00}$ and $\mathbb{P}_{11}$ (see Figure 8); since $x_0y_{uv-1} \in E(\mathbb{C}_{00})$, $x_{(v+1)u-2}y_{uv-1} \in E(\mathbb{P}_{11})$, $x_{(v+1)u-2} \notin V(\mathbb{C}_{00})$, and $x_0 \in V(\mathbb{P}_{11})$, we have two edge-disjoint paths of length k, say $\mathbb{P}_{00}$ and $\hat{\mathbb{P}}_{11}$ from $\mathbb{C}_{00}$ and $\mathbb{P}_{11}$ as follows (see Figure 9).

$$\begin{aligned}\mathbb{P}_{00} &= (\mathbb{C}_{00} - x_0y_{uv-1}) \cup x_{(v+1)u-2}y_{uv-1},\\ \hat{\mathbb{P}}_{11} &= \big(\mathbb{P}_{11} - x_{(v+1)u-2}y_{uv-1}\big) \cup x_0y_{uv-1}.\end{aligned}$$

By applying a procedure similar to Case 1, the remaining pairs of cycles both $\mathbb{C}_{\lambda\mu} \oplus \mathbb{C}_{(\lambda+1)\mu}$ and $\mathbb{C}_{\lambda\mu} \oplus \mathbb{C}_{\lambda(\mu+1)}$, (λ, μ), $(\lambda+1, \mu)$ $(\lambda, \mu+1) \neq (0, 0), (0, 1), (1, 1)$

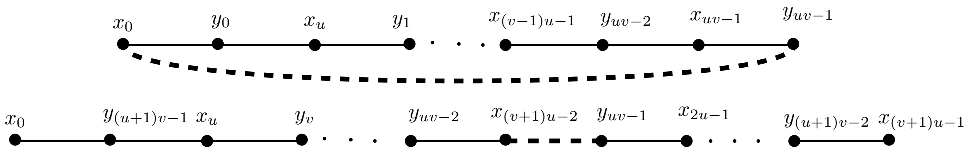

Figure 8. $\mathbb{C}_{00}$ and $\mathbb{P}_{11}$.

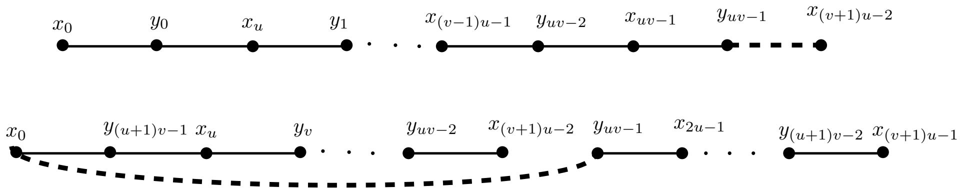

Figure 9. $\mathbb{P}_{00}$ and $\hat{\mathbb{P}}_{11}$.

decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

Subcase 2d: $m = k/2 + 1$ and $n = k/2$. When $m = k/2 + 1$ and $n = k/2$, we have $s = p = 1$ and $r = q + 1$. Since $\lambda = 2$ and $0 \le \mu \le r - 1$, we can have $\mathbb{C}_{00}$ and $\mathbb{C}_{01}$ (see Figure 10). By applying a procedure similar to Case 1, we have two edge-disjoint paths of length k, say $\mathbb{P}_{00}$ and $\mathbb{P}_{01}$ from $\mathbb{C}_{00}$ and $\mathbb{C}_{01}$ as follows (see Figure 11).

$$\mathbb{P}_{00} = (\mathbb{C}_{00} - x_0 y_{2a-3}) \cup x_{2a-2} y_{2a-3},$$
$$\mathbb{P}_{01} = (\mathbb{C}_{01} - x_{2a-2} y_{2a-3}) \cup x_0 y_{2a-3}.$$

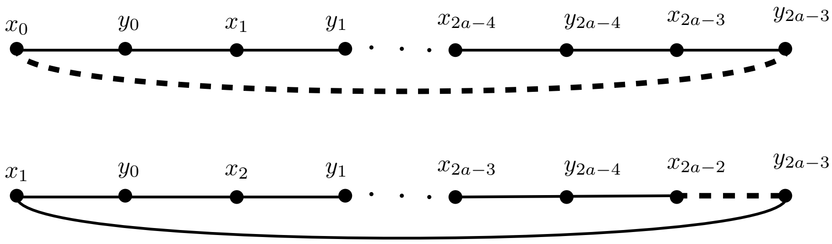

Figure 10. $\mathbb{C}_{00} \cup \mathbb{C}_{01}$.

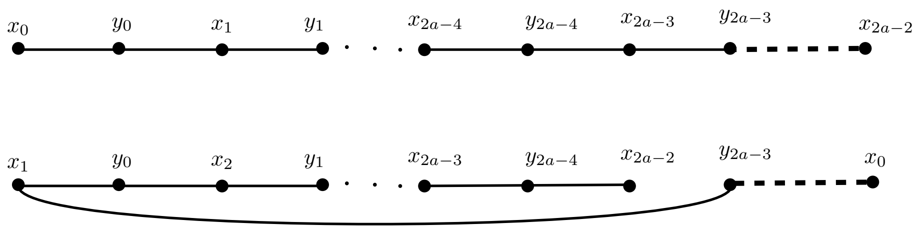

Figure 11. $\mathbb{P}_{00} \cup \mathbb{P}_{01}$.

Let $a = r + 1/2$. Now we consider $\mathbb{P}_{00}$ and $\mathbb{C}_{0a}$ (see Figure 12). Since $x_{2a-1}y_{a-2} \in E(\mathbb{C}_{a0})$, $x_{a-1}y_{a-2} \in E(\mathbb{P}_{00})$, and $x_{a-1} \notin V(\mathbb{C}_{a0})$ we have two edge-disjoint paths of length k, say $\mathbb{P}_{0a}$ and $\hat{\mathbb{P}}_{00}$ from $\mathbb{C}_{0a}$ and $\mathbb{P}_{00}$ as follows (see Figure 13).

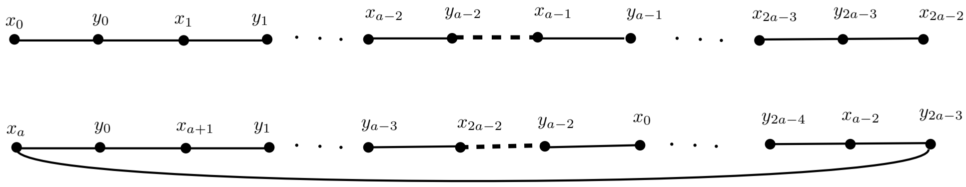

Figure 12. $\mathbb{C}_{00} \cup \mathbb{C}_{01}$.

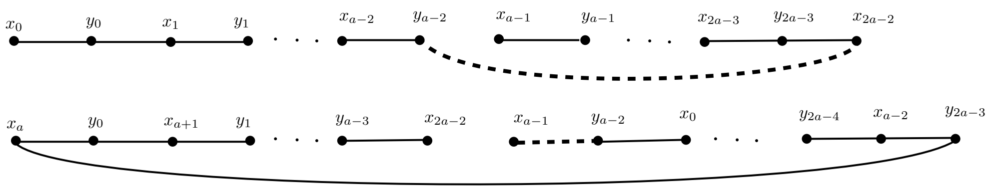

Figure 13. $\mathbb{P}_{00} \cup \mathbb{P}_{01}$.

By applying a procedure similar to Case 1, the remaining pairs of cycles $\mathbb{C}_{0\mu}$ and $\mathbb{C}_{0(\mu+1)}$, $2 \leq \mu \neq a \leq r-1$ decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition. ■

Theorem 7. *Let p, q be nonnegative integers and $\{k, m, n, \lambda\} \in \mathbb{N}$ such that $k \equiv \lambda \equiv 0 \pmod 2$, $m+n > k \geq 4$, and k divides $2mn$. If $m, n \geq k/2$, $k(p+q) = \lambda mn$, and $p \neq 1$, then the graph $K_{m,n}(\lambda)$ has a $(k; p, q)$-decomposition.*

Proof. When $\lambda \geq 2$, we can write $K_{m,n}(\lambda) = (\lambda/2)\, K_{m,n}(2)$. By Lemma 6 and Remark 5, the graph $(\lambda/2)\, K_{m,n}(2)$ has a $(k; p, q)$-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition. ■

Remark 8.

1. Let k, m, n be positive even integers such that $k \geq 4$. If the graph $K_{m,n}(\lambda)$ has a $(k; p, q)$-decomposition, then for every positive integer x, the graph $K_{m,n}(x\lambda)$ has a $(k; p, q)$-decomposition.

2. Let k, m, n be positive even integers such that $k \geq 4$. If the graph $K_{m,n}(\lambda)$ has a $(k; p, q)$-decomposition, then for all positive integers r and s, the graph $K_{rm,sn}(\lambda)$ has a $(k; p, q)$-decomposition.

3. Let k, $n_1, n_2, \ldots, n_m$ be positive even integers such that $k \geq 4$. If the graph $K_{n_i,n_j}(\lambda)$, for $1 \leq i \neq j \leq m$ has a $(k; p, q)$-decomposition, then the graph $K_{n_1,n_2,\ldots,n_m}(\lambda)$ has a $(k; p, q)$-decomposition.

3. $(k; p, q)$-Decomposition of $K_{m,n}(\lambda)$, when $\lambda \geq 3$

In this section, we investigate the existence of a $(k; p, q)$-decomposition of $K_{m,n}(\lambda)$, when $\lambda \geq 3$ and $\lambda m \equiv \lambda n \equiv k \equiv 0 \pmod 2$.

Theorem 9. *Let $\{k, m, n, \lambda\} \in \mathbb{N}$ and i, j be nonnegative integers such that $\lambda \geq 3$, $\lambda m \equiv \lambda n \equiv 0 \pmod 2$, and $k \equiv 0 \pmod 4$. If $K_{\frac{k}{2}+i,\frac{k}{2}+j}(\lambda)$, $0 \leq i, j \leq k/2$ has a $(k; p, q)$-decomposition, then the graph $K_{m,n}(\lambda)$, where $m, n \geq k$, has a $(k; p, q)$-decomposition.*

Proof. By the hypothesis, let $m = tk + x$ and $n = sk + y$, where t and s are positive integers, x and y are nonnegative integers such that $0 \leq x, y < k$.

When $x = y = 0$, we can write $K_{m,n}(\lambda) = K_{tk,sk}(\lambda) = \lambda ts K_{k,k}$. When $x = y = k/2$, we can write

$$\begin{aligned} K_{m,n}(\lambda) &= K_{(t-1)k+\frac{3k}{2},(s-1)k+\frac{3k}{2}}(\lambda) \\ &= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,\frac{3k}{2}}(\lambda) \oplus K_{\frac{3k}{2},(s-1)k}(\lambda) \oplus K_{\frac{3k}{2},\frac{3k}{2}}(\lambda) \\ &= ((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)\lambda K_{k,\frac{3k}{2}} \oplus (s-1)\lambda K_{\frac{3k}{2},k} \oplus \lambda K_{\frac{3k}{2},\frac{3k}{2}}. \end{aligned}$$

Since $k \equiv 0 \pmod 4$, by Theorem 1 the graphs $K_{k,k}$, $K_{k,\frac{3k}{2}}$, and $K_{\frac{3k}{2},\frac{3k}{2}}$ have a $(k; p, q)$-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 1: $x = 0$ and $0 < y < k$. When $0 < y < k/2$, we can write

$$\begin{aligned}K_{m,n}(\lambda) &= K_{tk,(s-1)k+\frac{k}{2}+\frac{k}{2}+y}(\lambda) = K_{tk,(s-1)k+\frac{k}{2}}(\lambda) \oplus K_{tk,y+\frac{k}{2}}(\lambda)\\ &= (t\lambda)K_{k,(s-1)k+\frac{k}{2}} \oplus tK_{k,y+\frac{k}{2}}(\lambda)\\ &= (t(s-1)\lambda)K_{k,k} \oplus (t\lambda)K_{k,\frac{k}{2}} \oplus tK_{k,y+\frac{k}{2}}(\lambda).\end{aligned}$$

By Theorem 1, the graphs $K_{k,k}$, $K_{k,\frac{k}{2}}$ both have a $(k;p,q)$-decomposition and by the hypothesis, the graph $K_{k,y+\frac{k}{2}}(\lambda)$ has a $(k;p,q)$-decomposition.

When $k/2 \leq y < k$, we can write

$$\begin{aligned}K_{m,n}(\lambda) &= K_{tk,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda)\\ &= (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda).\end{aligned}$$

By Theorem 1, the graph $K_{k,k}$ has a $(k;p,q)$-decomposition and by the hypothesis, the graph $K_{k,y}(\lambda)$ has a $(k;p,q)$-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 2: $k/2 < x < k$ and $k/2 \leq y < k$. We can write

$$\begin{aligned}K_{m,n}(\lambda) &= K_{tk+x,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) \oplus K_{x,sk}(\lambda) \oplus K_{x,y}(\lambda)\\ &= (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda) \oplus sK_{x,k}(\lambda) \oplus K_{x,y}(\lambda),\end{aligned}$$

and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda(tsk+sx+ty) + \lambda xy/k$. By Theorem 1, the graph $K_{k,k}$ has a $(k;p,q)$-decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{x,k}(\lambda)$ both have a $(k;p,q)$-decomposition. Since k divides λmn, we have k divides λxy and also $k/2 \leq x, y < k$, then by the hypothesis, $K_{x,y}(\lambda)$ has a $(k;p,q)$-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 3: $0 < x, y \leq k/2$. We can write

$$\begin{aligned}K_{m,n}(\lambda) &= K_{(t-1)k+(k+x),(s-1)k+(k+y)}(\lambda)\\ &= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,k+y}(\lambda) \oplus K_{k+x,(s-1)k}(\lambda) \oplus K_{k+x,k+y}(\lambda)\\ &= (t-1)(s-1)K_{k,k}(\lambda) \oplus (t-1)K_{k,k+y}(\lambda) \oplus (s-1)K_{k+x,k}(\lambda)\\ &\oplus K_{k/2,k+y}(\lambda) \oplus K_{k/2+x,k+y}(\lambda)\\ &= \lambda(t-1)(s-1)K_{k,k} \oplus (t-1)K_{k,k/2}(\lambda) \oplus (t-1)K_{k,k/2+y}(\lambda)\\ &\oplus (s-1)K_{k/2,k}(\lambda) \oplus (s-1)K_{k/2+x,k}(\lambda) \oplus K_{k/2,k+y}(\lambda)\\ &\oplus K_{k/2+x,k/2}(\lambda) \oplus K_{k/2+x,k/2+y}(\lambda),\end{aligned}$$

and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda k(t-1)(s-1) + \lambda(t-1)(k+y) + \lambda(k+x)(s-1) + \lambda(k+x+y) + (\lambda xy)/k$. By Theorem 1, the graphs $K_{k,k}$ and $K_{k/2,k}$ both have a $(k;p,q)$-decomposition and by the hypothesis, the graphs $K_{k,k/2+y}(\lambda)$, $K_{k/2+x,k}(\lambda)$, both have a $(k;p,q)$-decomposition. Since k divides λmn and $k \equiv 0 \pmod 4$, we have k divides $\lambda(k/2+x)(k/2+y)$, 2 divides λx, and 2 divides λy and $k/2 \leq (k/2+x),(k/2+y) \leq k$. Then by the hypothesis, the graphs $K_{k/2+x,k/2+y}(\lambda)$, $K_{k/2+x,k/2}(\lambda)$, and $K_{k/2,k/2+y}(\lambda)$ have a $(k;p,q)$-decomposition. The graph $K_{k/2,k+y}(\lambda)$ can be viewed as $K_{k/2,k/2}(\lambda)\oplus K_{k/2,k/2+y}(\lambda) = \lambda K_{k/2,k/2}\oplus K_{k/2,k/2+y}(\lambda)$. By Theorem 2, the graph $K_{k/2,k/2}$ has a C_k-decomposition and by the hypothesis, the graph $K_{k/2,k/2+y}(\lambda)$ has a $(k;p,q)$-decomposition. Now for any pair of cycles of length k, one from the graph $\lambda K_{k/2,k/2}$, say $\mathbb{C}_\alpha$ and the other from the graph $K_{k/2,k/2+y}(\lambda)$, say $\mathbb{C}_\beta$, we have a common vertex in $\mathbb{C}_\alpha \oplus \mathbb{C}_\beta$, say v, such that at least one neighbor of v from each cycle does not belongs to the other cycle. Then by the Construction 4 we have two edge-disjoint paths of length k from $\mathbb{C}_\alpha$ and $\mathbb{C}_\beta$. By applying a similar procedure to the remaining pairs of cycles, we have edge-disjoint pairs of paths. Hence the graph $K_{k/2,k+y}(\lambda)$ has a $(k;p,q)$-decomposition. Therefore, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 4: $0 < x \leq k/2$ and $k/2 < y < k$. We can write

$$\begin{aligned} K_{m,n}(\lambda) &= K_{(t-1)k+(k+x),sk+y}(\lambda) \\ &= K_{(t-1)k,sk}(\lambda) \oplus K_{(t-1)k,y}(\lambda) \oplus K_{k+x,sk}(\lambda) \oplus K_{k+x,y}(\lambda) \\ &= ((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k+x,k}(\lambda) \oplus K_{k+x,y}(\lambda) \\ &= ((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k/2,k}(\lambda) \oplus sK_{k/2+x,k}(\lambda) \\ &\oplus K_{k/2,y}(\lambda) \oplus K_{k/2+x,y}(\lambda), \end{aligned}$$

and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda((t-1)sk+(t-1)y+sk/2+s(k/2+x))+\lambda(k+x)y/k$. By Theorem 1, the graphs $K_{k,k}$ and $K_{k/2,k}$ both have a $(k;p,q)$-decomposition. Since k divides λmn, we have 2 divides λy, k divides $xy\lambda$ and also $k/2 \leq (k/2+x), y \leq k$, then by the hypothesis, the graphs $K_{k,y}(\lambda)$, $K_{k/2+x,k}(\lambda)$, and $K_{k/2+x,y}(\lambda)$ have a $(k;p,q)$-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition. ∎

Theorem 10. *Let* $\{k,m,n,\lambda\} \in \mathbb{N}$ *and* i, j *be nonnegative integers such that* $\lambda \geq 3$, $\lambda m \equiv \lambda n \equiv 0 \pmod 2$, *and* $k \equiv 2 \pmod 4$. *If* $K_{\frac{k}{2}+i,\frac{k}{2}+j}(\lambda)$, $0 \leq i,j \leq k$ *has a* $(k;p,q)$*-decomposition, then the graph* $K_{m,n}(\lambda)$, *where* $m,n \geq 3k/2$, *has a* $(k;p,q)$*-decomposition.*

Proof. By the hypothesis, let $m = tk+x$ and $n = sk+y$, where t and s are positive integers, x and y are nonnegative integers such that $0 \leq x,y < k$.

When $x = y = k/2$, we can write

$$\begin{aligned} K_{m,n}(\lambda) &= K_{(t-1)k+\frac{3k}{2},(s-1)k+\frac{3k}{2}}(\lambda) \\ &= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,\frac{3k}{2}}(\lambda) \oplus K_{\frac{3k}{2},(s-1)k}(\lambda) \oplus K_{\frac{3k}{2},\frac{3k}{2}}(\lambda) \\ &= ((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)\lambda K_{k,\frac{3k}{2}} \oplus (s-1)\lambda K_{\frac{3k}{2},k} \oplus \lambda K_{\frac{3k}{2},\frac{3k}{2}}. \end{aligned}$$

By Theorem 1, the graph $K_{k,k}$, has a $(k;p,q)$-decomposition and by the hypothesis, the graphs $K_{k,\frac{3k}{2}}$, and $K_{\frac{3k}{2},\frac{3k}{2}}$ both have a $(k;p,q)$-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 1: $0 \leq x, y < k/2$. When $0 \leq x, y < k/2$, we have $t, s \geq 2$. We can write

$$\begin{aligned} K_{m,n}(\lambda) &= K_{(t-1)k+(k+x),(s-1)k+(k+y)}(\lambda) \\ &= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,k+y}(\lambda) \oplus K_{k+x,(s-1)k}(\lambda) \oplus K_{k+x,k+y}(\lambda) \\ &= ((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)K_{k,k+y}(\lambda) \oplus (s-1)K_{k+x,k}(\lambda) \\ &\oplus K_{k+x,k+y}(\lambda), \end{aligned}$$

and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda((t-1)(s-1)k + (s-1)(k+x) + (t-1)(k+y)) + \lambda(k+x)(k+y)/k$.

By Theorem 1, the graph $K_{k,k}$ has a $(k;p,q)$-decomposition and by the hypothesis, the graphs $K_{k,k+y}(\lambda)$ and $K_{k+x,k}(\lambda)$ both have a $(k;p,q)$-decomposition. Since k divides λmn, we have k divides $\lambda(k+x)(k+y)$ and also $k/2 \leq (k+x), (k+y) \leq 3k/2$, then by the hypothesis, the graph $K_{k+x,k+y}(\lambda)$ has a $(k;p,q)$-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 2: $k/2 \leq x < k$ and $k/2 < y < k$. We can write $K_{m,n}(\lambda) = K_{tk+x,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) \oplus K_{x,sk}(\lambda) \oplus K_{x,y}(\lambda) = (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda) \oplus sK_{x,k}(\lambda) \oplus K_{x,y}(\lambda)$, and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda(tsk+sx+ty)+\lambda xy/k$. By Theorem 1, the graph $K_{k,k}$ has a $(k;p,q)$-decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{x,k}(\lambda)$ both have a $(k;p,q)$-decomposition. Since k divides λmn, we have k divides λxy and also $k/2 \leq x, y < k$, then by the hypothesis, the graph $K_{x,y}(\lambda)$ has a $(k;p,q)$-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 3: $0 \leq x < k/2$ and $k/2 \leq y < k$. When $0 \leq x < k/2$ and $k/2 \leq y < k$, we have $t \geq 2$ and $s \geq 1$. We can write

$$\begin{aligned} K_{m,n}(\lambda) &= K_{(t-1)k+(k+x),sk+y}(\lambda) \\ &= K_{(t-1)k,sk}(\lambda) \oplus K_{(t-1)k,y}(\lambda) \oplus K_{k+x,sk}(\lambda) \oplus K_{k+x,y}(\lambda) \\ &= ((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k+x,k}(\lambda) \oplus K_{k+x,y}(\lambda), \end{aligned}$$

and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda((t-1)sk + s(k+x) + (t-1)y) + \lambda(k+x)y/k$. By Theorem 1, the graph $K_{k,k}$ has a $(k;p,q)$-decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{k+x,k}(\lambda)$ both have a $(k;p,q)$-decomposition. Since k divides λmn, we have k divides $\lambda(k+x)y$ and also $k/2 \leq (k+x), y \leq 3k/2$, then by the hypothesis, the graph $K_{k+x,y}(\lambda)$ has a $(k;p,q)$-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition. ■

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