Discussiones Mathematicae Graph Theory 35 (2015) 715–731 doi:10.7151/dmgt.1830

DECOMPOSITION OF COMPLETE BIPARTITE MULTIGRAPHS INTO PATHS AND CYCLES HAVING k EDGES

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Abstract

We give necessary and sufficient conditions for the decomposition of complete bipartite multigraph $K_{m,n}(\lambda)$ into paths and cycles having k edges. In particular, we show that such decomposition exists in $K_{m,n}(\lambda)$, when $\lambda \equiv 0$ (mod 2), $m, n \geq \frac{k}{2}, m+n > k$, and k(p+q) = 2mn for $k \equiv 0 \pmod{2}$ and also when $\lambda \geq 3$, $\lambda m \equiv \lambda n \equiv 0 \pmod{2}$, $k(p+q) = \lambda mn, m, n \geq k$, (resp., $m, n \geq 3k/2$) for $k \equiv 0 \pmod{4}$ (respectively, for $k \equiv 2 \pmod{4}$). In fact, the necessary conditions given above are also sufficient when $\lambda = 2$.

Keywords: path, cycle, graph decomposition, multigraph.

2010 Mathematics Subject Classification: 05C38, 05C51.

1. INTRODUCTION

Unless stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology the readers are referred to [8]. A cycle of length m is called an m-cycle and it is denoted by C_m and a path of length m is called an m-path and it is denoted by P_{m+1} . A circuit (directed cycle) of length m is called an m-circuit and it is denoted by \overrightarrow{C}_m . Let K_m denote a complete graph on m vertices, $K_{m,n}$ denote a complete bipartite graph with m and n vertices in the parts, and $K_{m,n}^*$ denote a complete bipartite symmetric directed graph with m and n vertices in the parts. A graph whose vertex set is partitioned into sets V_1, \ldots, V_m such that the edge set is $\bigcup_{i \neq j \in [m]} V_i \times V_j$ is called a *complete m-partite graph* denoted by K_{n_1,\ldots,n_m} , where $|V_i| = n_i$ for all i. For any integer $\alpha > 0$, αG denotes a union of α edge-disjoint copies of G. The λ -multiplication of G, denoted $G(\lambda)$, is the multigraph obtained from a graph Gby replacing each edge with λ edges. For a graph G, G - I denotes the graph Gwith a 1-factor I removed. Let $x_0x_1\cdots x_{k-2}x_{k-1}$ and $(x_0x_1\cdots x_{k-1}x_0)$ respectively denote the path P_k and the cycle C_k with vertices $x_0, x_1, \ldots, x_{k-1}$ and edges $x_0x_1, x_1x_2, \ldots, x_{k-2}x_{k-1}, x_{k-1}x_0$.

By a decomposition of the graph G, we mean a list of edge-disjoint subgraphs of G whose union is G (ignoring isolated vertices). For the graph G, if E(G) can be partitioned into E_1, \ldots, E_k such that the subgraph induced by E_i is H_i , for all $i, 1 \leq i \leq k$, then we say that H_1, \ldots, H_k decompose G and we write G = $H_1 \oplus \cdots \oplus H_k$, since H_1, \ldots, H_k are edge-disjoint subgraphs of G. For $1 \leq i \leq k$, if $H_i \cong H$, we say that G has a H-decomposition. If G has a decomposition into p copies of H_1 and q copies of H_2 , then we say that G has a $\{pH_1, qH_2\}$ decomposition. If such a decomposition exists for all admissible pairs of p and qsatisfying trivial necessary conditions, then we say that G has a full $\{H_1, H_2\}$ decomposition or G is fully $\{H_1, H_2\}$ -decomposable.

Study on full $\{H_1, H_2\}$ -decomposition of graphs is not new. Abueida, Daven, and Roblee [1, 3] completely determined the values of n for which $K_n(\lambda)$ admits the $\{pH_1, qH_2\}$ -decomposition such that $H_1 \oplus H_2 \cong K_t$, when $\lambda \geq 1$ and $|V(H_1)| = |V(H_2)| = t$, where $t \in \{4, 5\}$. Let S_k denotes a star on k vertices, i.e. $S_k = K_{1,k-1}$. Abueida and Daven [2] proved that there exists a $\{pK_k, qS_{k+1}\}$ decomposition of K_n for $k \geq 3$ and $n \equiv 0, 1 \pmod{k}$. Abueida and O'Neil [4] proved that for $k \in \{3, 4, 5\}$, the $\{pC_k, qS_k\}$ -decomposition of $K_n(\lambda)$ exists, whenever $n \geq k+1$ except for the ordered triples $(k, n, \lambda) \in \{(3, 4, 1), (4, 5, 1), ($ (5, 6, 1), (5, 6, 2), (5, 6, 4), (5, 7, 1), (5, 8, 1). Abueida and Daven [2] obtained necessary and sufficient conditions for the $\{pC_4, q(2K_2)\}$ -decomposition of the Cartesian product and tensor product of paths, cycles, and complete graphs. Shyu [17] obtained a necessary and sufficient condition for the existence of a full $\{P_5, C_4\}$ decomposition of K_n . Shyu [18] proved that K_n has a full $\{P_4, S_4\}$ -decomposition if and only if $n \ge 6$ and $3(p+q) = \binom{n}{2}$. Also he proved that K_n has a full $\{P_k, S_k\}$ -decomposition with a restriction $p \ge k/2$, when k even (resp., $p \ge k$, when k odd). Shyu [19] obtained a necessary and sufficient condition for the existence of a full $\{P_4, K_3\}$ -decomposition of K_n . Shyu [20] proved that K_n has a full $\{C_4, S_5\}$ -decomposition if and only if $4(p+q) = \binom{n}{2}, q \neq 1$, when n is odd and $q \geq \max\{3, \lceil \frac{n}{4} \rceil\}$, when n is even. Shyu [21] proved that $K_{m,n}$ has a full $\{P_k, S_k\}$ -decomposition for some m and n and also obtained some necessary and sufficient condition for the existence of a full $\{P_4, S_4\}$ -decomposition of $K_{m,n}$. Sarvate and Zhang [16] obtained necessary and sufficient conditions for the existence of a $\{pP_3, qK_3\}$ -decomposition of $K_n(\lambda)$, when p = q.

Chou et al. [9] proved that for a given triple (p, q, r) of nonnegative integers, G decompose into p copies of C_4 , q copies of C_6 , and r copies of C_8 such that 4p + 6q + 8r = |E(G)| in the following two cases: (a) $G = K_{m,n}$ with m and n both even and greater than four (b) $G = K_{n,n} - I$, where n is odd. Chou and Fu [10] proved that the existence of a full $\{C_4, C_{2t}\}$ -decomposition of $K_{2u,2v}$, where $t/2 \le u, v < t$, when t even (resp., $(t+1)/2 \le u, v \le (3t-1)/2$, when t odd) implies such decomposition in $K_{2m,2n}$, where $m, n \ge t$ (resp., $m, n \ge (3t+1)/2$). The authors [11] reduced the bounds of the sufficient conditions obtained by Chou and Fu [10] for the existence of a full $\{C_4, C_{2t}\}$ -decomposition of $K_{2m,2n}$, when t > 2. Lee and Chu [13, 14] obtained a necessary and sufficient condition for the existence of a full $\{P_k, S_k\}$ -decomposition of $K_{n,n}$ and $K_{m,n}$. Lee and Lin [15] obtained a necessary and sufficient condition for the existence of a full $\{pC_k, qS_{k+1}\}$ -decomposition of $K_{n,n} - I$. Abueida and Lian [7] obtained necessary and sufficient conditions for the existence of a $\{pC_k, qS_{k+1}\}$ -decomposition of K_n for some n. Recently, the authors [12] obtained some necessary and sufficient conditions for the existence of a full $\{P_{k+1}, C_k\}$ -decomposition of K_n and $K_{m,n}$.

In this paper, we study only the existence of a full $\{P_{k+1}, C_k\}$ -decomposition of $K_{m,n}(\lambda)$, we abbreviate the notation for such decomposition as (k; p, q)-decomposition of $K_{m,n}(\lambda)$. The obvious necessary condition for such existence is $k(p + q) = |E(K_{m,n}(\lambda))|$. As we consider only cases where all vertices are of even degree, the case $p \neq 1$ is also obviously necessary, since the presents of a single path in the decomposition would give two vertices of odd degree and the resulting graph is not cycle decomposable. Call the situation with $k(p + q) = |E(K_{m,n}(\lambda))|$, all vertex degrees are even, and $p \neq 1$ the good case.

We prove that in the good case $K_{m,n}(\lambda)$ has a (k; p, q)-decomposition, when $\lambda \equiv 0 \pmod{2}$, $m, n \geq \frac{k}{2}$, m+n > k, and k(p+q) = 2mn for $k \equiv 0 \pmod{2}$. Further, we show that if $K_{m,n}(\lambda)$, $\lambda \geq 3$, $k \equiv 0 \pmod{4}$ (resp., $k \equiv 2 \pmod{4}$) has a (k; p, q)-decomposition in the good case with $k/2 \leq m, n \leq k$, (resp., $k/2 \leq m, n \leq 3k/2$,) then such decomposition also exists in the good case, when $\lambda \geq 3$; $m, n \geq k$ (resp., $m, n \geq 3k/2$).

To prove our results, we use the following:

Theorem 1 [12]. Let p and q be nonnegative integers and k, m, n be positive even integers such that $k \equiv 0 \pmod{4}$. For $m \leq n$, the graph $K_{m,n}$ has a (k; p, q)-decomposition if and only if $m \geq \frac{k}{2}$, $n \geq \lceil \frac{k+1}{2} \rceil$, k(p+q) = mn, and $p \neq 1$.

Theorem 2 [22]. $K_{m,n}^*$ has a $\overrightarrow{C_k}$ -decomposition if and only if $m \geq \frac{k}{2}$, $n \geq \frac{k}{2}$, and k divides 2mn.

By considering the underlying graph of $K_{m,n}^*$, we have the following from Theorem 2.

Theorem 3. The graph $K_{m,n}(2)$ has a C_k -decomposition if and only if $m \geq \frac{k}{2}$, $n \geq \frac{k}{2}$, and k divides 2mn.

2. (k; p, q)-Decomposition of $K_{m,n}(\lambda)$ when $k \equiv 0 \pmod{2}$

In this section, we investigate the existence of (k; p, q)-decomposition of $K_{m,n}(\lambda)$, when $k \equiv 0 \pmod{2}$.

Construction 4. Let \mathbb{C}_{λ} and \mathbb{C}_{μ} be two cycles of length k, where $\mathbb{C}_{\lambda} = (x_1x_2\cdots x_kx_1)$ and $\mathbb{C}_{\mu} = (y_1y_2\cdots y_ky_1)$. If v is a common vertex of \mathbb{C}_{λ} and \mathbb{C}_{μ} such that at least one neighbour of v from each cycle (say, x_i and y_j) does not belongs to the other cycle, then we have two edge-disjoint paths of length k, say \mathbb{P}_{λ} and \mathbb{P}_{μ} from \mathbb{C}_{λ} and \mathbb{C}_{μ} as follows (see Figure 1), where $\mathbb{P}_{\lambda} = (\mathbb{C}_{\lambda} - vx_i) \cup vy_j, \mathbb{P}_{\mu} = (\mathbb{C}_{\mu} - vy_j) \cup vx_i$.



Figure 1. $\mathbb{C}_{\lambda} \cup \mathbb{C}_{\mu} = \mathbb{P}_{\lambda} \cup \mathbb{P}_{\mu}$.

Remark 5. Let $k \in \mathbb{N}$. If G and H have a (k; p, q)-decomposition, then $G \oplus H$ has such a decomposition.

Lemma 6. Let p, q be nonnegative integers and $\{k, m, n\} \in \mathbb{N}$ such that $k \equiv 0 \pmod{2}$ and m + n > k. The graph $K_{m,n}(2)$ has a (k; p, q)-decomposition if and only if $m, n \geq k/2$, k(p+q) = 2mn, and $p \neq 1$.

Proof. Necessity. Conditions $m, n \ge k/2$, k(p+q) = 2mn, and $p \ne 1$ are trivial.

Sufficiency. Let $k \equiv 0 \pmod{2}$. In order to have a C_k -decomposition in $K_{m,n}(2)$, we can always find u, v such that $k = 2uv, m = ru, n = sv, r \geq v$, and $s \geq u$, where r and s are positive integers. We denote the vertices of the partite sets of $K_{ru,sv}$ by $x_i, 0 \leq i \leq ru - 1$ and $y_j, 0 \leq j \leq sv - 1$. By Theorem 3, the

graph $K_{ru,sv}(2)$ has a C_{2uv} -decomposition as follows:

$$\mathbb{C}_{\lambda\mu} = \left(\cdots \left(\cdots x_{(\mu+i)u+j} y_{(\lambda+j)v+i} \cdots \right)_{0 \le i \le v-1} \right)_{0 \le j \le u-1}, \\ 0 \le \lambda \le s-1; 0 \le \mu \le r-1,$$

where the indices of x are to be taken with modulo ru and those of y with modulo sv. Now we construct the required number of P_{k+1} from the C_k -decomposition given above, in two cases.

Case 1: p is even. For a fixed μ and $0 \leq \lambda \leq s - 1$, we can have $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{(\lambda+1)\mu}$ as above. Since $x_{\mu u}y_{\lambda v} \in E(\mathbb{C}_{\lambda\mu})$, $x_{\mu u}y_{(\lambda+u+1)v-1} \in E(\mathbb{C}_{(\lambda+1)\mu})$, $y_{\lambda v} \notin V(\mathbb{C}_{(\lambda+1)\mu})$, and $y_{(\lambda+u+1)v-1} \notin V(\mathbb{C}_{\lambda\mu})$, we have two edge-disjoint paths of length k, say $\mathbb{P}_{\lambda\mu}$ and $\mathbb{P}_{(\lambda+1)\mu}$ from $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{(\lambda+1)\mu}$ as follows (see Figure 2).

$$\mathbb{P}_{\lambda\mu} = (\mathbb{C}_{\lambda\mu} - x_{\mu u} y_{\lambda v}) \cup x_{\mu u} y_{(\lambda+u+1)v-1},$$
$$\mathbb{P}_{(\lambda+1)\mu} = (\mathbb{C}_{(\lambda+1)\mu} - x_{\mu u} y_{(\lambda+u+1)v-1}) \cup x_{\mu u} y_{\lambda v}.$$



Figure 2. $\mathbb{C}_{\lambda\mu} \cup \mathbb{C}_{(\lambda+1)\mu} = \mathbb{P}_{\lambda\mu} \cup \mathbb{P}_{(\lambda+1)\mu}$.

Similarly, we can find pairs of paths of length k from the pairs of cycles $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{(\lambda+1)\mu}$, where $\lambda = 0, 2, \ldots, s-2$ or s-1 and $0 \leq \mu \leq r-1$. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

Now for a fixed λ and $0 \leq \mu \leq r-1$, we can have $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{\lambda(\mu+1)}$ as above. Since $x_{\mu p} y_{(\lambda+p)q-1} \in E(\mathbb{C}_{\lambda\mu})$, $x_{(\mu+q+1)p-1} y_{(\lambda+p)q-1} \in E(\mathbb{C}_{\lambda(\mu+1)})$, $x_{\mu p} \notin V(\mathbb{C}_{\lambda(\mu+1)})$, and $x_{(\mu+q+1)p-1} \notin V(\mathbb{C}_{\lambda\mu})$, we have two edge-disjoint paths of length k, say $\mathbb{P}_{\lambda\mu}$ and $\mathbb{P}_{\lambda(\mu+1)}$ from $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{\lambda(\mu+1)}$ as follows (see Figure 3).

$$\mathbb{P}_{\lambda\mu} = \left(\mathbb{C}_{\lambda\mu} - x_{\mu p} y_{(\lambda+p)q-1}\right) \cup x_{(\mu+q+1)p-1} y_{(\lambda+p)q-1},\\ \mathbb{P}_{\lambda(\mu+1)} = \left(\mathbb{C}_{\lambda(\mu+1)} - x_{(\mu+q+1)p-1} y_{(\lambda+p)q-1}\right) \cup x_{\mu p} y_{(\lambda+p)q-1}$$



Figure 3. $\mathbb{C}_{\lambda\mu} \cup \mathbb{C}_{\lambda(\mu+1)} = \mathbb{P}_{\lambda\mu} \cup \mathbb{P}_{\lambda(\mu+1)}.$

Similarly, we can find pairs of paths of length k from the pairs of cycles $\mathbb{C}_{\lambda\mu}$ and $\mathbb{C}_{\lambda(\mu+1)}$, where $\mu = 0, 2, \ldots, r-2$ or r-1. Hence we have the desired paths.

Case 2: p is odd. Fixing $v = \gcd(n, k/2)$, we have u = k/2v, s = n/v. Since k divides 2mn, i.e. 2uv divides 2mn and v divides n, we have r = m/u.



Figure 4. $\mathbb{C}_{00} \cup \mathbb{C}_{01} \cup \mathbb{C}_{02}$.

Subcase 2a: $(v+2)u - 1 \leq m$ and $v+2 \leq r$. Since $r \geq 3$ and $s \geq 1$, we can have \mathbb{C}_{00} , \mathbb{C}_{01} , and \mathbb{C}_{02} (see Figure 4). By applying a procedure similar to Construction 4, we have three edge-disjoint paths of length k, say \mathbb{P}_{00} , \mathbb{P}_{01} , and \mathbb{P}_{02} from \mathbb{C}_{00} , \mathbb{C}_{01} , and \mathbb{C}_{02} as follows (see Figure 5).

Decomposition of Complete Bipartite Multigraphs into Paths ... 721



Figure 5. $\mathbb{P}_{00} \cup \mathbb{P}_{01} \cup \mathbb{P}_{02}$.

By applying a procedure similar to Case 1, the remaining pairs of cycles $\mathbb{C}_{\lambda\mu} \oplus \mathbb{C}_{\lambda(\mu+1)}$, (λ, μ) , $(\lambda, \mu+1) \neq (0,0), (0,1), (0,2)$ decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.



Figure 6. $\mathbb{C}_{00} \cup \mathbb{C}_{10} \cup \mathbb{C}_{20}$.

Subcase 2b: $(u+2)v - 1 \leq n$ and $u+2 \leq s$. Since $r \geq 1$ and $s \geq 3$, we can have \mathbb{C}_{00} , \mathbb{C}_{10} , and \mathbb{C}_{20} (see Figure 6). By applying a procedure similar to

Construction 4, we have three edge-disjoint paths of length k, say \mathbb{P}_{00} , \mathbb{P}_{10} , and \mathbb{P}_{20} from \mathbb{C}_{00} , \mathbb{C}_{10} , and \mathbb{C}_{20} as follows (see Figure 7).



Figure 7. $\mathbb{P}_{00} \cup \mathbb{P}_{10} \cup \mathbb{P}_{20}$.

By applying a procedure similar to Case 1, the remaining pairs of cycles $\mathbb{C}_{\lambda\mu} \oplus \mathbb{C}_{(\lambda+1)\mu}$ (λ,μ) , $(\lambda+1,\mu) \neq (0,0), (1,0), (2,0)$ decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

Subcase 2c: $(v+1)u-1 \leq m$, $(u+1)v-1 \leq n$, $u+1 \leq s$, and $v+1 \leq r$, m or $n \neq k/2$. Since $r, s \geq 2$ we can have \mathbb{C}_{00} , \mathbb{C}_{10} , and \mathbb{C}_{11} . By applying a procedure similar to Case 1, we have two edge-disjoint paths of length k, say \mathbb{P}_{10} and \mathbb{P}_{11} from \mathbb{C}_{10} and \mathbb{C}_{11} as follows:

$$\mathbb{P}_{10} = \left(\mathbb{C}_{10} - x_0 y_{(u+1)v-1}\right) \cup x_{(v+1)u-1} y_{(u+1)v-1},\\ \mathbb{P}_{11} = \left(\mathbb{C}_{11} - x_{(v+1)u-1} y_{(u+1)v-1}\right) \cup x_0 y_{(u+1)v-1}.$$

Now consider \mathbb{C}_{00} and \mathbb{P}_{11} (see Figure 8); since $x_0y_{uv-1} \in E(\mathbb{C}_{00})$, $x_{(v+1)u-2}y_{uv-1} \in E(\mathbb{P}_{11})$, $x_{(v+1)u-2} \notin V(\mathbb{C}_{00})$, and $x_0 \in V(\mathbb{P}_{11})$, we have two edge-disjoint paths of length k, say \mathbb{P}_{00} and $\hat{\mathbb{P}}_{11}$ from \mathbb{C}_{00} and \mathbb{P}_{11} as follows (see Figure 9).

$$\mathbb{P}_{00} = (\mathbb{C}_{00} - x_0 y_{uv-1}) \cup x_{(v+1)u-2} y_{uv-1}, \\ \hat{\mathbb{P}}_{11} = (\mathbb{P}_{11} - x_{(v+1)u-2} y_{uv-1}) \cup x_0 y_{uv-1}.$$

By applying a procedure similar to Case 1, the remaining pairs of cycles both $\mathbb{C}_{\lambda\mu} \oplus \mathbb{C}_{(\lambda+1)\mu}$ and $\mathbb{C}_{\lambda\mu} \oplus \mathbb{C}_{\lambda(\mu+1)}$, (λ, μ) , $(\lambda+1, \mu)$ $(\lambda, \mu+1) \neq (0, 0)$, (0, 1), (1, 1)

Decomposition of Complete Bipartite Multigraphs into Paths ... 723



Figure 9. \mathbb{P}_{00} and $\hat{\mathbb{P}}_{11}$.

decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

Subcase 2d: m = k/2 + 1 and n = k/2. When m = k/2 + 1 and n = k/2, we have s = p = 1 and r = q + 1. Since $\lambda = 2$ and $0 \le \mu \le r - 1$, we can have \mathbb{C}_{00} and \mathbb{C}_{01} (see Figure 10). By applying a procedure similar to Case 1, we have two edge-disjoint paths of length k, say \mathbb{P}_{00} and \mathbb{P}_{01} from \mathbb{C}_{00} and \mathbb{C}_{01} as follows (see Figure 11).

$$\mathbb{P}_{00} = (\mathbb{C}_{00} - x_0 y_{2a-3}) \cup x_{2a-2} y_{2a-3},$$

$$\mathbb{P}_{01} = (\mathbb{C}_{01} - x_{2a-2} y_{2a-3}) \cup x_0 y_{2a-3}.$$



Figure 10. $\mathbb{C}_{00} \cup \mathbb{C}_{01}$.



Figure 11. $\mathbb{P}_{00} \cup \mathbb{P}_{01}$.

Let a = r + 1/2. Now we consider \mathbb{P}_{00} and \mathbb{C}_{0a} (see Figure 12). Since $x_{2a-1}y_{a-2} \in E(\mathbb{C}_{a0}), x_{a-1}y_{a-2} \in E(\mathbb{P}_{00})$, and $x_{a-1} \notin V(\mathbb{C}_{a0})$ we have two edgedisjoint paths of length k, say \mathbb{P}_{0a} and $\hat{\mathbb{P}}_{00}$ from \mathbb{C}_{0a} and \mathbb{P}_{00} as follows (see Figure 13).







Figure 13. $\mathbb{P}_{00} \cup \mathbb{P}_{01}$.

By applying a procedure similar to Case 1, the remaining pairs of cycles $\mathbb{C}_{0\mu}$ and $\mathbb{C}_{0(\mu+1)}$, $2 \leq \mu \neq a \leq r-1$ decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

Theorem 7. Let p, q be nonnegative integers and $\{k, m, n, \lambda\} \in \mathbb{N}$ such that $k \equiv \lambda \equiv 0 \pmod{2}$, $m+n > k \geq 4$, and k divides 2mn. If $m, n \geq k/2$, $k(p+q) = \lambda mn$, and $p \neq 1$, then the graph $K_{m,n}(\lambda)$ has a (k; p, q)-decomposition.

Proof. When $\lambda \geq 2$, we can write $K_{m,n}(\lambda) = (\lambda/2) K_{m,n}(2)$. By Lemma 6 and Remark 5, the graph $(\lambda/2) K_{m,n}(2)$ has a (k; p, q)-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Remark 8.

- 1. Let k, m, n be positive even integers such that $k \ge 4$. If the graph $K_{m,n}(\lambda)$ has a (k; p, q)-decomposition, then for every positive integer x, the graph $K_{m,n}(x\lambda)$ has a (k; p, q)-decomposition.
- 2. Let k, m, n be positive even integers such that $k \ge 4$. If the graph $K_{m,n}(\lambda)$ has a (k; p, q)-decomposition, then for all positive integers r and s, the graph $K_{rm,sn}(\lambda)$ has a (k; p, q)-decomposition.
- 3. Let k, n_1, n_2, \ldots, n_m be positive even integers such that $k \ge 4$. If the graph $K_{n_i,n_j}(\lambda)$, for $1 \le i \ne j \le m$ has a (k; p, q)-decomposition, then the graph $K_{n_1,n_2,\ldots,n_m}(\lambda)$ has a (k; p, q)-decomposition.

3. (k; p, q)-Decomposition of $K_{m,n}(\lambda)$, when $\lambda \geq 3$

In this section, we investigate the existence of a (k; p, q)-decomposition of $K_{m,n}(\lambda)$, when $\lambda \geq 3$ and $\lambda m \equiv \lambda n \equiv k \equiv 0 \pmod{2}$.

Theorem 9. Let $\{k, m, n, \lambda\} \in \mathbb{N}$ and i, j be nonnegative integers such that $\lambda \geq 3$, $\lambda m \equiv \lambda n \equiv 0 \pmod{2}$, and $k \equiv 0 \pmod{4}$. If $K_{\frac{k}{2}+i,\frac{k}{2}+j}(\lambda)$, $0 \leq i, j \leq k/2$ has a (k; p, q)-decomposition, then the graph $K_{m,n}(\lambda)$, where $m, n \geq k$, has a (k; p, q)-decomposition.

Proof. By the hypothesis, let m = tk + x and n = sk + y, where t and s are positive integers, x and y are nonnegative integers such that $0 \le x, y < k$.

When x = y = 0, we can write $K_{m,n}(\lambda) = K_{tk,sk}(\lambda) = \lambda t s K_{k,k}$. When x = y = k/2, we can write

$$K_{m,n}(\lambda) = K_{(t-1)k+\frac{3k}{2},(s-1)k+\frac{3k}{2}}(\lambda)$$

= $K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,\frac{3k}{2}}(\lambda) \oplus K_{\frac{3k}{2},(s-1)k}(\lambda) \oplus K_{\frac{3k}{2},\frac{3k}{2}}(\lambda)$
= $((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)\lambda K_{k,\frac{3k}{2}} \oplus (s-1)\lambda K_{\frac{3k}{2},k} \oplus \lambda K_{\frac{3k}{2},\frac{3k}{2}}.$

Since $k \equiv 0 \pmod{4}$, by Theorem 1 the graphs $K_{k,k}$, $K_{k,\frac{3k}{2}}$, and $K_{\frac{3k}{2},\frac{3k}{2}}$ have a (k; p, q)-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 1: x = 0 and 0 < y < k. When 0 < y < k/2, we can write

$$\begin{split} K_{m,n}(\lambda) &= K_{tk,(s-1)k+\frac{k}{2}+\frac{k}{2}+y}(\lambda) = K_{tk,(s-1)k+\frac{k}{2}}(\lambda) \oplus K_{tk,y+\frac{k}{2}}(\lambda) \\ &= (t\lambda)K_{k,(s-1)k+\frac{k}{2}} \oplus tK_{k,y+\frac{k}{2}}(\lambda) \\ &= (t(s-1)\lambda)K_{k,k} \oplus (t\lambda)K_{k,\frac{k}{2}} \oplus tK_{k,y+\frac{k}{2}}(\lambda). \end{split}$$

By Theorem 1, the graphs $K_{k,k}$, $K_{k,\frac{k}{2}}$ both have a (k; p, q)-decomposition and by the hypothesis, the graph $K_{k,y+\frac{k}{2}}(\lambda)$ has a (k; p, q)-decomposition.

When $k/2 \le y < k$, we can write

$$K_{m,n}(\lambda) = K_{tk,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda)$$
$$= (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda).$$

By Theorem 1, the graph $K_{k,k}$ has a (k; p, q)-decomposition and by the hypothesis, the graph $K_{k,y}(\lambda)$ has a (k; p, q)-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 2: k/2 < x < k and $k/2 \le y < k$. We can write

$$K_{m,n}(\lambda) = K_{tk+x,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) \oplus K_{x,sk}(\lambda) \oplus K_{x,y}(\lambda)$$
$$= (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda) \oplus sK_{x,k}(\lambda) \oplus K_{x,y}(\lambda),$$

and $\lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda(tsk + sx + ty) + \lambda xy/k$. By Theorem 1, the graph $K_{k,k}$ has a (k; p, q)-decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{x,k}(\lambda)$ both have a (k; p, q)-decomposition. Since k divides λmn , we have k divides λxy and also $k/2 \leq x, y < k$, then by the hypothesis, $K_{x,y}(\lambda)$ has a (k; p, q)-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 3: $0 < x, y \le k/2$. We can write

$$\begin{split} K_{m,n}(\lambda) &= K_{(t-1)k+(k+x),(s-1)k+(k+y)}(\lambda) \\ &= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,k+y}(\lambda) \oplus K_{k+x,(s-1)k}(\lambda) \oplus K_{k+x,k+y}(\lambda) \\ &= (t-1)(s-1)K_{k,k}(\lambda) \oplus (t-1)K_{k,k+y}(\lambda) \oplus (s-1)K_{k+x,k}(\lambda) \\ &\oplus K_{k/2,k+y}(\lambda) \oplus K_{k/2+x,k+y}(\lambda) \\ &= \lambda(t-1)(s-1)K_{k,k} \oplus (t-1)K_{k,k/2}(\lambda) \oplus (t-1)K_{k,k/2+y}(\lambda) \\ &\oplus (s-1)K_{k/2,k}(\lambda) \oplus (s-1)K_{k/2+x,k}(\lambda) \oplus K_{k/2,k+y}(\lambda) \\ &\oplus K_{k/2+x,k/2}(\lambda) \oplus K_{k/2+x,k/2+y}(\lambda), \end{split}$$

and $\lambda mn/k = \lambda (tk+x)(sk+y)/k = \lambda k(t-1)(s-1) + \lambda (t-1)(k+y) + \lambda (k+x)(s-1) + \lambda (k+x+y) + (\lambda xy)/k$. By Theorem 1, the graphs $K_{k,k}$ and $K_{k/2,k}$ both

726

have a (k; p, q)-decomposition and by the hypothesis, the graphs $K_{k,k/2+y}(\lambda)$, $K_{k/2+x,k}(\lambda)$, both have a (k; p, q)-decomposition. Since k divides λmn and $k \equiv 0$ (mod 4), we have k divides $\lambda(k/2 + x)(k/2 + y)$, 2 divides λx , and 2 divides λy and $k/2 \leq (k/2+x), (k/2+y) \leq k$. Then by the hypothesis, the graphs $K_{k/2+x,k/2+y}(\lambda), K_{k/2+x,k/2}(\lambda), \text{ and } K_{k/2,k/2+y}(\lambda) \text{ have a } (k; p, q) \text{-decomposition.}$ The graph $K_{k/2,k+y}(\lambda)$ can be viewed as $K_{k/2,k/2}(\lambda) \oplus K_{k/2,k/2+y}(\lambda) = \lambda K_{k/2,k/2} \oplus K_{k/2,k/2}(\lambda)$ $K_{k/2,k/2+y}(\lambda)$. By Theorem 2, the graph $K_{k/2,k/2}$ has a C_k -decomposition and by the hypothesis, the graph $K_{k/2,k/2+y}(\lambda)$ has a (k; p, q)-decomposition. Now for any pair of cycles of length k, one from the graph $\lambda K_{k/2,k/2}$, say \mathbb{C}_{α} and the other from the graph $K_{k/2,k/2+y}(\lambda)$, say \mathbb{C}_{β} , we have a common vertex in $\mathbb{C}_{\alpha} \oplus \mathbb{C}_{\beta}$, say v, such that at least one neighbor of v from each cycle does not belongs to the other cycle. Then by the Construction 4 we have two edge-disjoint paths of length k from \mathbb{C}_{α} and \mathbb{C}_{β} . By applying a similar procedure to the remaining pairs of cycles, we have edge-disjoint pairs of paths. Hence the graph $K_{k/2,k+y}(\lambda)$ has a (k; p, q)-decomposition. Therefore, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 4: $0 < x \le k/2$ and k/2 < y < k. We can write

$$\begin{split} K_{m,n}(\lambda) &= K_{(t-1)k+(k+x),sk+y}(\lambda) \\ &= K_{(t-1)k,sk}(\lambda) \oplus K_{(t-1)k,y}(\lambda) \oplus K_{k+x,sk}(\lambda) \oplus K_{k+x,y}(\lambda) \\ &= ((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k+x,k}(\lambda) \oplus K_{k+x,y}(\lambda) \\ &= ((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k/2,k}(\lambda) \oplus sK_{k/2+x,k}(\lambda) \\ &\oplus K_{k/2,y}(\lambda) \oplus K_{k/2+x,y}(\lambda), \end{split}$$

and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda((t-1)sk + (t-1)y + sk/2 + s(k/2+x)) + \lambda(k+x)y/k$. By Theorem 1, the graphs $K_{k,k}$ and $K_{k/2,k}$ both have a (k; p, q)-decomposition. Since k divides λmn , we have 2 divides λy , k divides $xy\lambda$ and also $k/2 \leq (k/2+x), y \leq k$, then by the hypothesis, the graphs $K_{k,y}(\lambda), K_{k/2+x,k}(\lambda)$, and $K_{k/2+x,y}(\lambda)$ have a (k; p, q)-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Theorem 10. Let $\{k, m, n, \lambda\} \in \mathbb{N}$ and i, j be nonnegative integers such that $\lambda \geq 3$, $\lambda m \equiv \lambda n \equiv 0 \pmod{2}$, and $k \equiv 2 \pmod{4}$. If $K_{\frac{k}{2}+i,\frac{k}{2}+j}(\lambda)$, $0 \leq i, j \leq k$ has a (k; p, q)-decomposition, then the graph $K_{m,n}(\lambda)$, where $m, n \geq 3k/2$, has a (k; p, q)-decomposition.

Proof. By the hypothesis, let m = tk + x and n = sk + y, where t and s are positive integers, x and y are nonnegative integers such that $0 \le x, y < k$.

When x = y = k/2, we can write

$$K_{m,n}(\lambda) = K_{(t-1)k+\frac{3k}{2},(s-1)k+\frac{3k}{2}}(\lambda)$$

= $K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,\frac{3k}{2}}(\lambda) \oplus K_{\frac{3k}{2},(s-1)k}(\lambda) \oplus K_{\frac{3k}{2},\frac{3k}{2}}(\lambda)$
= $((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)\lambda K_{k,\frac{3k}{2}} \oplus (s-1)\lambda K_{\frac{3k}{2},k} \oplus \lambda K_{\frac{3k}{2},\frac{3k}{2}}.$

By Theorem 1, the graph $K_{k,k}$, has a (k; p, q)-decomposition and by the hypothesis, the graphs $K_{k,\frac{3k}{2}}$, and $K_{\frac{3k}{2},\frac{3k}{2}}$ both have a (k; p, q)-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 1: $0 \le x, y < k/2$. When $0 \le x, y < k/2$, we have $t, s \ge 2$. We can write

$$K_{m,n}(\lambda) = K_{(t-1)k+(k+x),(s-1)k+(k+y)}(\lambda)$$

= $K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,k+y}(\lambda) \oplus K_{k+x,(s-1)k}(\lambda) \oplus K_{k+x,k+y}(\lambda)$
= $((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)K_{k,k+y}(\lambda) \oplus (s-1)K_{k+x,k}(\lambda)$
 $\oplus K_{k+x,k+y}(\lambda),$

and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda((t-1)(s-1)k + (s-1)(k+x) + (t-1)(k+y)) + \lambda(k+x)(k+y)/k.$

By Theorem 1, the graph $K_{k,k}$ has a (k; p, q)-decomposition and by the hypothesis, the graphs $K_{k,k+y}(\lambda)$ and $K_{k+x,k}(\lambda)$ both have a (k; p, q)-decomposition. Since k divides λmn , we have k divides $\lambda(k+x)(k+y)$ and also $k/2 \leq (k+x), (k+y) \leq 3k/2$, then by the hypothesis, the graph $K_{k+x,k+y}(\lambda)$ has a (k; p, q)-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 2: $k/2 \leq x < k$ and k/2 < y < k. We can write $K_{m,n}(\lambda) = K_{tk+x,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) \oplus K_{x,sk}(\lambda) \oplus K_{x,y}(\lambda) = (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda) \oplus sK_{x,k}(\lambda) \oplus K_{x,y}(\lambda)$, and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda(tsk+sx+ty) + \lambda xy/k$. By Theorem 1, the graph $K_{k,k}$ has a (k; p, q)-decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{x,k}(\lambda)$ both have a (k; p, q)-decomposition. Since k divides λmn , we have k divides λxy and also $k/2 \leq x, y < k$, then by the hypothesis, the graph $K_{x,y}(\lambda)$ has a (k; p, q)-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Case 3: $0 \le x < k/2$ and $k/2 \le y < k$. When $0 \le x < k/2$ and $k/2 \le y < k$, we have $t \ge 2$ and $s \ge 1$. We can write

$$K_{m,n}(\lambda) = K_{(t-1)k+(k+x),sk+y}(\lambda)$$

= $K_{(t-1)k,sk}(\lambda) \oplus K_{(t-1)k,y}(\lambda) \oplus K_{k+x,sk}(\lambda) \oplus K_{k+x,y}(\lambda)$
= $((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k+x,k}(\lambda) \oplus K_{k+x,y}(\lambda)$

728

and $\lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda((t-1)sk+s(k+x)+(t-1)y) + \lambda(k+x)y/k$. By Theorem 1, the graph $K_{k,k}$ has a (k; p, q)-decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{k+x,k}(\lambda)$ both have a (k; p, q)-decomposition. Since k divides λmn , we have k divides $\lambda(k+x)y$ and also $k/2 \leq (k+x), y \leq 3k/2$, then by the hypothesis, the graph $K_{k+x,y}(\lambda)$ has a (k; p, q)-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Acknowledgments

The authors are grateful to the anonymous referees for their valuable suggestions and comments, which improved the presentation of the paper. The first author thank the University Grants Commission for its financial support through the Grant No.F.4-7/2008(BSR)/11-105/2008(BSR)/ December 2012. The second author thank the Department of Science and Technology, Government of India, New Delhi for its financial support through the Grant No. DST/SR/S4/MS:282/13.

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Decomposition of Complete Bipartite Multigraphs into Paths ... 731

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> Received 9 August 2014 Revised 23 February 2015 Accepted 23 February 2015