# ON SUPER EDGE-ANTIMAGICNESS OF SUBDIVIDED STARS 

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#### Abstract

Enomoto, Llado, Nakamigawa and Ringel (1998) defined the concept of a super $(a, 0)$-edge-antimagic total labeling and proposed the conjecture that every tree is a super $(a, 0)$-edge-antimagic total graph. In the support of this conjecture, the present paper deals with different results on super ( $a, d$ )-edge-antimagic total labeling of subdivided stars for $d \in\{0,1,2,3\}$.


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## 1. Introduction

All graphs in this paper are finite, undirected and simple. For a graph $G, V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. A $(v, e)$-graph $G$ is a graph such that $|V(G)|=v$ and $|E(G)|=e$. A general reference for graph-theoretic ideas can be seen in [30]. A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called a total labeling. Some labelings use the vertex-set only or the edge-set only and we shall call them vertex-labelings or edge-labelings, respectively.

Definition 1.1. An $(s, d)$-edge-antimagic vertex $((s, d)$-EAV) labeling of a $(v, e)$ graph $G$ is a bijective function $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ such that the set of edgesums of all edges in $G, w(x y)=\{\lambda(x)+\lambda(y): x y \in E(G)\}$, forms an arithmetic progression $\{s, s+d, s+2 d, \ldots, s+(e-1) d\}$, where $s>0$ and $d \geq 0$ are two fixed integers.

Definition 1.2. A bijection $\lambda: V(G) \cup E(G) \rightarrow\{1,2, \ldots, v+e\}$ is called an (a,d)-edge-antimagic total $((a, d)$-EAT) labeling of a $(v, e)$-graph $G$ if the set of edge-weights $\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in V(G)\}$ forms an arithmetic progression starting from $a$ and having common difference $d$, where $a>0$ and $d \geq 0$ are two fixed integers. A graph that admits an $(a, d)$-EAT labeling is called an $(a, d)$-EAT graph.
Definition 1.3. If $\lambda$ is an $(a, d)$-EAT labeling such that $\lambda(V(G))=\{1,2, \ldots, v\}$, then $\lambda$ is called a super $(a, d)$-EAT labeling and $G$ is known as a super $(a, d)$-EAT graph.
In Definitions 1.2 and 1.3, if $d=0$ then an ( $a, 0$ )-EAT labeling is called an edgemagic total (EMT) labeling and a super ( $a, 0$ )-EAT labeling is called a super edge magic total (SEMT) labeling. Moreover, in general $a$ is called a minimum edge-weight but particularly a magic constant when $d=0$. The definition of an ( $a, d$ )-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [27] as a natural extension of magic valuation defined by Kotzig and Rosa [19, 20]. A super $(a, d)$-EAT labeling is a natural extension of the notion of super edgemagic labeling defined by Enomoto, Llado, Nakamigawa and Ringel. Moreover, Enomoto et al. [7] proposed the following conjecture.

Conjecture 1.1. Every tree admits a super ( $a, 0$ )-EAT labeling.
In the support of this conjecture, many authors have considered a super (a,0)-EAT labeling for different particular classes of trees. Lee and Shah [21] verified this conjecture by a computer search for trees with at most 17 vertices. For different values of $d$, the results related to a super ( $a, d$ )-EAT labeling can
be found for w-trees [12], extended w-trees [13, 14], stars [22], subdivided stars [15, 16, 17, 18, 25, 26, 23, 24], path-like trees [3], caterpillars [19, 20, 29], disjoint union of stars and books [9], and wheels, fans and friendship graphs [28], paths and cycles [27] and complete bipartite graphs [1]. For detail studies of a super ( $a, d$ )-EAT labeling reader can see $[2,4,5,6,8,9,10,11]$.

Definition 1.4. Let $n_{i} \geq 1,1 \leq i \leq r$, and $r \geq 3$. A subdivided star $T\left(n_{1}, n_{2}, \ldots, n_{r}\right)$ is a tree obtained by inserting $n_{i}-1$ vertices to each of the $i$ th edge of the star $K_{1, r}$, where for all $n_{i}=1, T \underbrace{(1,1, \ldots, 1)}_{r-\text { times }} \cong K_{1, r}$. Moreover suppose that $V(G)=\{c\} \cup\left\{x_{i}^{l_{i}}: 1 \leq i \leq r ; 1 \leq l_{i} \leq n_{i}\right\}$ is the vertex-set and $E(G)=\left\{c x_{i}^{1}: 1 \leq i \leq r\right\} \cup\left\{x_{i}^{l_{i}} x_{i}^{l_{i}+1}: 1 \leq i \leq r ; 1 \leq l_{i} \leq n_{i}-1\right\}$ is the edgeset of the subdivided star $G \cong T\left(n_{1}, n_{2}, \ldots, n_{r}\right)$, thus $v=|V(G)|=\sum_{i=1}^{r} n_{i}+1$ and $e=|E(G)|=\sum_{i=1}^{r} n_{i}$.
$\mathrm{Lu}[23,24]$ called the subdivided star $T\left(n_{1}, n_{2}, n_{3}\right)$ as a three-path tree and proved that it is a super $(a, 0)$-EAT graph if $n_{1}$ and $n_{2}$ are odd with $n_{3}=n_{2}+1$ or $n_{3}=n_{2}+2$. Ngurah et al. [25] proved that the subdivided star $T\left(n_{1}, n_{2}, n_{3}\right)$ is also a super ( $a, 0$ )-EAT graph if $n_{3}=n_{2}+3$ or $n_{3}=n_{2}+4$. Salman et al. [26] found a super $(a, 0)$-EAT labeling of subdivided stars $T \underbrace{(n, n, n, \ldots, n)}_{r-\text { times }}$, where $n \in\{2,3\}$. Moreover, Javaid et al. [15, 16, 17] found the super $(a, d)$-EAT labelings on different subclasses of subdivided stars for $d \in\{0,1,2\}$. However, the investigation of the different results related to a super ( $a, d$ )-EAT labeling of the subdivided star $T\left(n_{1}, n_{2}, n_{3}, \ldots, n_{r}\right)$ with unequal $n_{i}$ for $1 \leq i \leq r$ is still open. In this paper, we investigate a super $(a, d)$-EAT labeling on the subdivided stars for all possible values of $d$.

## 2. Basic Results

In this section, we present some basic results which will be used frequently to prove the main results.

Ngurah et al. [25] found lower and upper bounds of the minimum edge-weight $a$ for a subclass of the subdivided stars, which is stated as follows.

Lemma 2.1. If $T\left(n_{1}, n_{2}, n_{3}\right)$ is a super $(a, 0)$-EAT graph, then $\frac{1}{2 l}\left(5 l^{2}+3 l+6\right) \leq$ $a \leq \frac{1}{2 l}\left(5 l^{2}+11 l-6\right)$, where $l=\sum_{i=1}^{3} n_{i}$.

The lower and upper bounds of the minimum edge-weight $a$ for another subclass of subdivided stars established by Salman et al. [26] are given below.

Lemma 2.2. If $T \underbrace{(n, n, \ldots, n)}_{n-\text { times }}$ is a super (a,0)-EAT graph, then $\frac{1}{2 l}\left(5 l^{2}+(9-\right.$ $\left.2 n) l+n^{2}-n\right) \leq a \leq \frac{1}{2 l}\left(5 l^{2}+(2 n+5) l+n-n^{2}\right)$, where $l=n^{2}$.

Moreover, the following lemma presents the lower and upper bound of the minimum edge-weight $a$ for the most generalized subclass of subdivided stars proved by Javaid and Bhatti [17, 18].

Lemma 2.3. If $T\left(n_{1}, n_{2}, n_{3}, \ldots, n_{r}\right)$ has a super ( $\left.a, d\right)$-EAT labeling, then $\frac{1}{2 l}\left(5 l^{2}+\right.$ $\left.r^{2}-2 l r+9 l-r-(l-1) l d\right) \leq a \leq \frac{1}{2 l}\left(5 l^{2}-r^{2}+2 l r+5 l+r-(l-1) l d\right)$, where $l=\sum_{i=1}^{r} n_{i}$ and $d \in\{0,1,2,3\}$.

Bača and Miller [4] state a necessary condition for a graph to be super $(a, d)$ EAT, which provides an upper bound on the parameter $d$. Let a $(v, e)$-graph $G$ be a super $(a, d)$-EAT. The minimum possible edge-weight is at least $v+4$. The maximum possible edge-weight is no more than $3 v+e-1$. Thus $a+(e-1) d \leq$ $3 v+e-1$ or $d \leq \frac{2 v+e-5}{e-1}$. For any subdivided star, where $v=e+1$, it follows that $d \leq 3$.

Let us recall the following proposition which we will use frequently in the proofs of the main results.

Proposition 2.4 [3]. If $a(v, e)$-graph $G$ has an ( $s, d)$-EAV labeling, then
(i) $G$ has a super $(s+v+1, d+1)$-EAT labeling,
(ii) $G$ has a super $(s+v+e, d-1)$-EAT labeling.

## 3. Super $(a, d)$-EAT Labeling of Subdivided Stars

This section deals with the main results related to super ( $a, d$ )-EAT labelings on more generalized families of subdivided stars for all possible values of $d$.

### 3.1. When $n \equiv 0(\bmod 2)$

The results of super $(a, d)$-EAT labelings for different values of $d$ on the class of subdivided stars $T\left(n, n+1, n_{3}, \ldots, n_{r}\right)$ when $n \equiv 0(\bmod 2)$ are as follows.

Theorem 3.1. For $n \equiv 0(\bmod 2)$ and $r \geq 3, G \cong T\left(n, n+1, n_{3}, \ldots, n_{r}\right)$ admits a super ( $a, 0$ )-EAT labeling with $a=2 v+s-1$ and a super ( $a, 2$ )-EAT labeling with $a=v+s+1$, where $v=|V(G)|$ and $s=(n+3)+\sum_{m=3}^{r}\left[2^{m-3}(n+1)\right]$ and $n_{m}=2^{m-2}(n+1)$ for $3 \leq m \leq r$.

Proof. According to the definition of graph $G$, we have that $v=(2 n+2)+$ $\sum_{m=3}^{r}\left[2^{m-2}(n+1)\right]$ and $e=v-1$. Define $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as follows.

$$
\lambda(c)=1 .
$$

For even $1 \leq l_{i} \leq n_{i}$, where $i=1,2$ and $3 \leq i \leq r$, let

$$
\begin{gathered}
\lambda(u)= \begin{cases}1+\frac{l_{1}}{2}, & \text { for } u=x_{1}^{l_{1}}, \\
(n+2)-\frac{l_{2}}{2}, & \text { for } u=x_{2}^{l_{2}} .\end{cases} \\
\lambda\left(x_{i}^{l_{i}}\right)=(n+2)+\sum_{m=3}^{i}\left[2^{m-3}(n+1)\right]-\frac{l_{i}}{2} .
\end{gathered}
$$

For odd $1 \leq l_{i} \leq n_{i}$ and $\alpha=(n+1)+\sum_{m=3}^{r}\left[2^{m-3}(n+1)\right]$, where $i=1,2$ and $3 \leq i \leq r$, let

$$
\lambda(u)= \begin{cases}\alpha+\frac{l_{1}+1}{2}, & \text { for } u=x_{1}^{l_{1}} \\ (\alpha+n+2)-\frac{l_{2}+1}{2} & \text { for } u=x_{2}^{l_{2}}\end{cases}
$$

and $\lambda\left(x_{i}^{l_{i}}\right)=(\alpha+n+2)+\sum_{m=3}^{i}\left[2^{m-3}(n+1)\right]-\frac{l_{i}+1}{2}$.
The set of all edge-sums $\{\lambda(x)+\lambda(y): x y \in E(G)\}$ generated by the above formulas forms a consecutive integer sequence $(\alpha+1)+1,(\alpha+1)+2, \ldots,(\alpha+1)+e$, where $s=\alpha+2$. Therefore, by Proposition 2.1, $\lambda$ can be extended to a super ( $a, 0$ )-EAT labeling with $a=2 v-1+s=2 v+(n+2)+\sum_{m=3}^{r}\left[2^{m-3}(n+1)\right]$ and to a super ( $a, 2$ )-EAT labeling with $a=v+1+s=v+(n+4)+\sum_{m=3}^{r}\left[2^{m-3}(n+1)\right]$.

Theorem 3.2. For $n \equiv 0(\bmod 2)$ and $r \geq 3, G \cong T\left(n, n+1, n_{3}, \ldots, n_{r}\right)$ admits a super $(a, 1)$-EAT labeling with $a=2 v+2$, where $v=|V(G)|$ and $n_{m}=2^{m-2}(n+1)$ for $3 \leq m \leq r$.

Proof. We define the vertex labeling $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as follows.

$$
\lambda(c)=1 .
$$

For $1 \leq l_{i} \leq n_{i}$, where $i=1,2$ and $3 \leq i \leq r$,

$$
\lambda(u)= \begin{cases}l_{1}+1, & \text { for } u=x_{1}^{l_{1}}, \\ (2 n+3)-l_{2}, & \text { for } u=x_{2}^{l_{2}},\end{cases}
$$

and $\lambda\left(x_{i}^{l_{i}}\right)=(2 n+3)+\sum_{m=3}^{i}\left[2^{m-2}(n+1)\right]-l_{i}$.
Suppose $\alpha=(2 n+2)+\sum_{m=3}^{r}\left[2^{m-2}(n+1)\right]$ and define $\lambda: E(G) \rightarrow\{v+$ $1, v+2, \ldots, v+e\}$ as follows. For $l_{i}=1$, where $i=1,2$ and $3 \leq i \leq r$, let

$$
\lambda(c u)= \begin{cases}(2 \alpha-1), & \text { for } u=x_{1}^{1}, \\ 2 \alpha-(n+1), & \text { for } u=x_{2}^{1},\end{cases}
$$

and $\lambda\left(c x_{i}^{1}\right)=2 \alpha-(n+1)-\sum_{m=3}^{i}\left[2^{m-3}(n+1)\right]$.
For $1 \leq l_{i} \leq n_{i}-1$, where $i=1,2$ and $3 \leq i \leq r$,

$$
\lambda\left(x_{i}^{1_{i}} x_{i}^{1_{i}+1}\right)= \begin{cases}(2 \alpha-1)-l_{1}, & \text { for } i=1, \\ 2 \alpha-2(n+1)+l_{2}, & \text { for } i=2\end{cases}
$$

and $\lambda\left(x_{i}^{1_{i}} x_{i}^{1_{i}+1}\right)=2 \alpha-2(n+1)-\sum_{m=3}^{i}\left[2^{m-2}(n+1)\right]+l_{i}$, for $3 \leq i \leq r$.
The set of edge-weights $\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in V(G)\}$ generated by the above formulas forms an integer sequence $2 v+2,2 v+3, \ldots, 2 v+1+e$ with difference 1 . Consequently, $\lambda$ is a super ( $a, 1$ )-EAT labeling with $a=2 v+2$.

Theorem 3.3. For $n \equiv 0(\bmod 2)$ and $r \geq 3$, the graph $G \cong T\left(n, n+1, n_{3}, \ldots, n_{r}\right)$ admits a super ( $a, 3$ )-EAT labeling with $a=v+4$, where $v=|V(G)|$ and $n_{m}=2^{m-2}(n+1)$ for $3 \leq m \leq r$.

Proof. Consider the vertex labeling as in the proof of Theorem 3.2. Now, we define the edge labeling $\lambda: E(G) \rightarrow\{v+1, v+2, \ldots, v+e\}$ as follows.

For $l_{i}=1$, where $i=1,2$ and, $3 \leq i \leq r$, let

$$
\lambda(c u)= \begin{cases}(\alpha+1), & \text { for } u=x_{1}^{1} \\ (\alpha+n+1), & \text { for } u=x_{2}^{1}\end{cases}
$$

and $\lambda\left(c x_{i}^{1}\right)=(\alpha+n+1)+\sum_{m=3}^{i}\left[2^{m-3}(n+1)\right]$.
For $1 \leq l_{i} \leq n_{i}-1$,

$$
\lambda\left(x_{i}^{1_{i}} x_{i}^{1_{i}+1}\right)= \begin{cases}(\alpha+1)+l_{1}, & \text { for } i=1 \\ (\alpha+2 n+2)-l_{2}, & \text { for } i=2\end{cases}
$$

and $\lambda\left(x_{i}^{1_{i}} x_{i}^{1_{i}+1}\right)=(\alpha+2 n+2)+\sum_{m=3}^{i}\left[2^{m-2}(n+1)\right]-l_{i}$, for $3 \leq i \leq r$.
The set of edge-weights $\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in V(G)\}$ generated by the above formulas forms an integer sequence $(v+1)+3(1),(v+1)+3(2),(v+1)+$ $3(3), \ldots,(v+1)+3(e)$ with difference 3 . Consequently, $\lambda$ is a super ( $a, 3$ )-EAT labeling with $a=v+4$.

### 3.2. When $n \equiv 1(\bmod 2)$

The results of super $(a, d)$-EAT labelings for different values of $d$ on the class of subdivided stars $T\left(n, n, n+1, n_{4}, \ldots, n_{r}\right)$ when $n \equiv 1(\bmod 2)$ are as follows.

Theorem 3.4. For $n \equiv 1(\bmod 2)$ and $r \geq 4, G \cong T\left(n, n, n+1, n_{4}, \ldots, n_{r}\right)$ admits a super $(a, 0)$-EAT labeling with $a=2 v+s-1$ and a super $(a, 2)$-EAT labeling with $a=v+s+1$, where $v=|V(G)|$ and $s=\frac{3(n+1)}{2}+\sum_{m=4}^{r}\left[2^{m-4}(n+1)\right]$, and $n_{m}=2^{m-3}(n+1)$ for $4 \leq m \leq r$.

Proof. Let $G \cong T\left(n, n, n+1, n_{4}, \ldots, n_{r}\right)$. Then $v=(3 n+2)+\sum_{m=4}^{r}\left[2^{m-3}(n+1)\right]$ and $e=v-1$. Define $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as follows.

$$
\lambda(c)=\frac{n+1}{2}
$$

For even $1 \leq l_{i} \leq n_{i}$, where $i=1,2,3$ and $4 \leq i \leq r$,

$$
\begin{gathered}
\lambda(u)= \begin{cases}\frac{n+1}{2}-\frac{l_{1}}{2}, & \text { for } u=x_{2}^{l_{1}}, \\
\frac{n+1}{2}+\frac{l_{2}}{2}, & \text { for } u=x_{2}^{l_{2},} \\
\frac{3(n+1)}{2}-\frac{l_{3}}{2}, & \text { for } u=x_{3}^{l_{3} .}\end{cases} \\
\lambda\left(x_{i}^{l_{i}}\right)=\frac{3(n+1)}{2}+\sum_{m=4}^{i}\left[2^{m-4}(n+1)\right]-\frac{l_{i}}{2} .
\end{gathered}
$$

For odd $1 \leq l_{i} \leq n_{i}$ and $\alpha=\frac{3(n+1)}{2}+\sum_{m=4}^{r}\left[2^{m-4}(n+1)\right]$, where $i=1,2,3$ and $4 \leq i \leq r$,

$$
\lambda(u)= \begin{cases}\alpha+\frac{n+3}{2}-\frac{l_{1}+1}{2}, & \text { for } u=x_{1}^{l_{1}} \\ \alpha+\frac{n+1}{2}+\frac{l_{2}+1}{2}, & \text { for } u=x_{2}^{l_{2}} \\ \alpha+\frac{3 n+5}{2}-\frac{l_{3}+1}{2}, & \text { for } u=x_{3}^{l_{3}}\end{cases}
$$

and

$$
\lambda\left(x_{i}^{l_{i}}\right)=\alpha+\frac{3 n+5}{2}+\sum_{m=4}^{i}\left[2^{m-4}(n+1)\right]-\frac{l_{i}+1}{2}
$$

The set of all edge-sums $\{\lambda(x)+\lambda(y): x y \in E(G)\}$ generated by the above formulas forms a consecutive integer sequence $(\alpha+1)+1,(\alpha+1)+2, \ldots,(\alpha+1)+e$, where $s=\alpha+2$. Therefore, by Proposition $2.1, \lambda$ can be extended to a super $(a, 0)$-EAT labeling with $a=2 v+s-1=2 v+\frac{3(n+1)}{2}+\sum_{m=4}^{r}\left[2^{m-4}(n+1)\right]$ and to a super $(a, 2)$-EAT labeling with $a=v+1+s=v+\frac{3 n+7}{2}+\sum_{m=4}^{r}\left[2^{m-4}(n+1)\right]$.

Theorem 3.5. For $n \equiv 1(\bmod 2)$ and $r \geq 4, G \cong T\left(n, n, n+1, n_{4}, \ldots, n_{r}\right)$ admits a super $(a, 1)$-EAT labeling with $a=2 v+2$, where $v=|V(G)|$ and $n_{m}=2^{m-3}(n+1)$ for $4 \leq m \leq r$.

Proof. We define the vertex labeling $\lambda: V(G) \rightarrow\{1,2, \ldots, v\}$ as follows.

$$
\lambda(c)=n+1
$$

For $1 \leq l_{i} \leq n_{i}$, where $i=1,2,3$ and $4 \leq i \leq r$, let

$$
\lambda(u)= \begin{cases}(n+1)-l_{1}, & \text { for } u=x_{1}^{l_{1}} \\ (n+1)+l_{2}, & \text { for } u=x_{2}^{l_{2}} \\ 3(n+1)-l_{3}, & \text { for } u=x_{3}^{l_{3}}\end{cases}
$$

and $\lambda\left(x_{i}^{l_{i}}\right)=3(n+1)+\sum_{m=4}^{i}\left[2^{m-3}(n+1)\right]-l_{i}$.
Suppose that $\alpha=(2 n+1)+\sum_{m=3}^{r}\left[2^{m-3}(n+1)\right]$ and define $\lambda: E(G) \rightarrow$ $\{v+1, v+2, \ldots, v+e\}$ as follows.

For $l_{i}=1$, where $i=1,2,3$ and $4 \leq i \leq r$,

$$
\lambda(c u)= \begin{cases}2 \alpha-n, & \text { for } u=x_{1}^{1} \\ 2 \alpha-(n+1), & \text { for } u=x_{2}^{1} \\ 2 \alpha-(2 n+1), & \text { for } u=x_{3}^{1}\end{cases}
$$

and $\lambda\left(c x_{i}^{1}\right)=2 \alpha-(2 n+1)-\sum_{m=4}^{i}\left[2^{m-4}(n+1)\right]$ respectively.
For $1 \leq l_{i} \leq n_{i}-1, i=1,2,3$ and $4 \leq i \leq r$,

$$
\lambda\left(x_{i}^{1_{i}} x_{i}^{1_{i}+1}\right)= \begin{cases}2 \alpha-n+l_{1}, & \text { for } i=1 \\ 2 \alpha-(n+1)-l_{2}, & \text { for } i=2 \\ 2 \alpha-(3 n+2)+l_{3}, & \text { for } i=3\end{cases}
$$

and $\lambda\left(x_{i}^{1_{i}} x_{i}^{1_{i}+1}\right)=2 \alpha-(3 n+2)-\sum_{m=4}^{i}\left[2^{m-3}(n+1)\right]+l_{i}$.
The set of edge-weights $\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in V(G)\}$ generated by the above formulas forms an integer sequence $2 v+2,2 v+3, \ldots, 2 v+1+e$ with difference 1. Consequently, $\lambda$ is a super ( $a, 1$ )-EAT labeling with $a=2 v+2$.

Theorem 3.6. For $n \equiv 1(\bmod 2)$ and $r \geq 4, G \cong T\left(n, n, n+1, n_{4}, \ldots, n_{r}\right)$ admits a super $(a, 3)$-EAT labeling with $a=v+4$, where $v=|V(G)|$ and $n_{m}=$ $2^{m-3}(n+1)$ for $4 \leq m \leq r$.

Proof. Consider the vertex labeling as in the proof of Theorem 3.2.2. Now, we define the edge labeling $\lambda: E(G) \rightarrow\{v+1, v+2, \ldots, v+e\}$ as follows.

For $l_{i}=1$, where $i=1,2,3$ and $4 \leq i \leq r$,

$$
\lambda(c u)= \begin{cases}\alpha+n, & \text { for } u=x_{1}^{1} \\ \alpha+(n+1), & \text { for } u=x_{2}^{1} \\ \alpha+(2 n+1), & \text { for } u=x_{3}^{1}\end{cases}
$$

and $\lambda\left(c x_{i}^{1}\right)=\alpha+(2 n+1)+\sum_{m=4}^{i}\left[2^{m-4}(n+1)\right]$.
For $1 \leq l_{i} \leq n_{i}-1$, where $i=1,2,3$ and $4 \leq i \leq r$,

$$
\lambda\left(x_{i}^{1_{i}} x_{i}^{1_{i}+1}\right)= \begin{cases}\alpha+n-l_{1}, & \text { for } i=1 \\ \alpha+(n+1)+l_{2}, & \text { for } i=2 \\ \alpha+(3 n+1)-l_{3}, & \text { for } i=3\end{cases}
$$

and $\lambda\left(x_{i}^{1_{i}} x_{i}^{1_{i}+1}\right)=\alpha+(3 n+2)+\sum_{m=4}^{i}\left[2^{m-3}(n+1)\right]-l_{i}$.
The set of edge-weights $\{\lambda(x)+\lambda(x y)+\lambda(y): x y \in V(G)\}$ generated by the above formulas forms an integer sequence $(v+1)+3(1),(v+1)+3(2),(v+$ $1)+3(3), \ldots,(v+1)+3(e)$ with difference 3 . Consequently, $\lambda$ admits a super ( $a, 3$ )-EAT labeling with $a=v+4$.

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