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SOME TOUGHNESS RESULTS IN INDEPENDENT DOMINATION CRITICAL GRAPHS

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Abstract

A subset S of $V(G)$ is an independent dominating set of G if S is independent and each vertex of G is either in S or adjacent to some vertex of S. Let $i(G)$ denote the minimum cardinality of an independent dominating set of G. A graph G is k-i-critical if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. In this paper, we establish that if G is a connected 3-i-critical graph and S is a vertex cutset of G with $|S| \geq 3$, then $\omega(G-S) \leq \frac{1+\sqrt{8|S|+1}}{2}$, improving a result proved by Ao [3], where $\omega(G-S)$ denotes the number of components of $G-S$. We also provide a characterization of the connected 3-i-critical graphs G attaining the maximum number of $\omega(G-S)$ when S is a minimum cutset of size 2 or 3.

Keywords: domination critical, toughness.

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1. Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complete graph of order n is denoted by K_n. A *star* is a complete bipartite graph $K_{1,n}$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. S is a *clique* if $G[S]$ is complete and S is *independent* if no two vertices of S are adjacent. G is a *split graph* if $V(G)$ is partitioned into a clique and an independent set. The number of components of G is denoted by $\omega(G)$. A vertex subset S of $V(G)$ is a *cutset* of G if $\omega(G-S) > \omega(G)$. For a vertex $v \in V(G)$, the neighborhood of v in G, denoted by $N_G(v)$, is the set of all vertices of $V(G)$ which are adjacent to v. For $S \subseteq V(G)$, $N_G(v) \cap S$ and $S \cup \bigcup_{x \in S} N_G(x)$ are denoted by $N_S(v)$ and $N_G[S]$, respectively.

For subsets S and T of $V(G)$, we say that S *dominates* T, denoted by $S \succ T$, if $T \subseteq N_G[S]$. If $S \succ T$ where $S = \{s\}$, then we write $s \succ T$ instead of $\{s\} \succ T$. Further, if $T = V(H)$ where H is a subgraph of G, then we write $S \succ H$ instead of $S \succ V(H)$ and we say that S is a *dominating set of* H. Thus S is a dominating set of G if each vertex of $V(G)$ is either in S or adjacent to some vertex of S. The minimum cardinality of a dominating set of G is called the *domination number* of G and denoted by $\gamma(G)$.

For a subgraph H of G, if $S \succ H$ and S is independent, then we say that S is an *independent dominating set of* H and denoted by $S \succ_i H$. Thus S is an independent dominating set of G if $S \succ_i G$. The minimum cardinality of an independent dominating set of G is called the *independent domination number* of G and denoted by $i(G)$. Observe that for any graph G, $\gamma(G) \leq i(G)$ and if $\gamma(G) = 1$, then $i(G) = 1$.

The concept of domination critical was first introduced by Sumner and Blitch [6] in 1983. A graph G is *k-γ-critical* if $\gamma(G) = k$, but $\gamma(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. Sumner and Blitch [6] also provided a characterization of 2-γ-critical graphs and some crucial results which have been become useful tools in investigating this topic. Since then k-γ-critical graphs have gained considerable attention. They have been investigated with some other graph parameters such as toughness, matching and Hamiltonicity. Most of these results concern 3-γ-critical graphs. The reader is directed to excellent books by Dehmer [4] and by Haynes *et al.* [5] for more details and for references therein.

In 1994, Ao [3] introduced the concept of "independent domination critical". A graph G is *k-i-critical* if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. In [3], Ao established some properties of 3-i-critical graphs including diameter and a sufficient condition for such graphs to be Hamiltonian. She also showed that if G is a connected 3-i-critical graph and S is a vertex cutset of G, then $\omega(G - S) \leq |S| + 1$. We found that the upper bound of this result can be improved if $|S| \geq 3$. In fact, in this paper, we shall prove the following main theorem in Section 2.

Theorem 1. *Let G be a connected 3-i-critical graph and S a vertex cutset of G. If $|S| \geq 3$, then $\omega(G-S) \leq \frac{1+\sqrt{8|S|+1}}{2}$. Further, the bound is sharp.*

In studying the concept of critical graphs, we always seek for a characterization. This also applies to connected k-i-critical graphs. It is easy to see that K_n, where n is a positive integer, is the only 1-i-critical graph. Ao [3] showed that G is 2-i-critical if and only if the complement of G is a union of stars. A problem that arises is that of characterizing connected k-i-critical graphs for $k \geq 3$. However, this seems to be a difficult task. So, to investigate a characterization of such graphs, we sometime add some addition hypothesis. In [1], the authors proved that if G is a connected k-i-critical graph, for $k \geq 3$, with a cutvertex u, then $\omega(G-u) \leq k-1$. Further, a characterization of such graphs G with $\omega(G-u) = k-1$ was given. So the characterization of connected 3-i-critical graphs with a cutvertex is known. According to Theorem 1 and Ao's result in the previous paragraph, if G is a connected 3-i-critical graph with a cutset S of size at most five, then $\omega(G-S)$ is at most three. A problem that arises is that of characterizing such graphs. In Section 3, we establish that if G is a connected 3-i-critical graph with a minimum cutset S of size 2 or 3 and $\omega(G-S) = 3$, then G must be in one of six classes. In [2], a characterization of the connected 3-i-critical graphs G with a minimum cutset S of size 2 and $\omega(G-S) = 2$ is given. We then conclude this section by posing the following remaining cases.

1. Characterize the connected 3-i-critical graphs G with a minimum cutset S of size 3 and $\omega(G-S) = 2$.

2. Characterize the connected 3-i-critical graphs G with a minimum cutset S of size 4 or 5 and $\omega(G-S) = k$ where $k \in \{2, 3\}$.

Note that the solution of Problem 1 together with our results in Section 3 provides a complete characterization of the 3-i-critical graphs with a minimum cutset of size 3.

2. The Proof of Theorem 1

We begin this section with some terminology and preliminary results that we will need when establishing our main results. For a pair of non-adjacent vertices u and v of G, I_{uv} denotes a minimum independent dominating set of $G+uv$. Our first two results follow immediately from the definition of k-i-critical graphs.

Lemma 2. *Let G be a connected k-i-critical graph and let u and v be two non-adjacent vertices of G. Then $|I_{uv}| = k-1$ and $|I_{uv} \cap \{u,v\}| = 1$.*

Lemma 3. *Let G be a connected 3-i-critical graph and let u and v be two non-adjacent vertices of G. Then*

(1) *$I_{uv} = \{u, w\}$ or $I_{uv} = \{v, w\}$ for some $w \in V(G) - \{u, v\}$.*

(2) *If $\{w\} = I_{uv} - \{u, v\}$, then $\{u, v, w\}$ is independent.*

Our next result is a useful tool for establishing our main results.

Lemma 4. *Let G be a connected 3-i-critical graph and let u and v be two non-adjacent vertices of G. If $I_{uv} = \{u, w\}$ for some $w \in V(G) - \{u, v\}$, then $I_{uw} - \{u, w\} = \{v\}$, i.e., $I_{uw} = \{u, v\}$ or $I_{uw} = \{w, v\}$.*

Proof. By Lemma 3(2), $\{u, v, w\}$ is independent. Let $x \in V(G) - \{u, v, w\}$. Then $xu \in E(G)$ or $xw \in E(G)$ since $I_{uv} = \{u, w\}$. Then $x \notin I_{uw}$ by Lemma 3(2). Thus $I_{uw} - \{u, w\} = \{v\}$ as required. This proves our lemma. ■

We are now ready to prove Theorem 1.

***Proof of Theorem* 1.** Put $t = \omega(G-S)$. If $t \leq 3$, then we are done. So suppose that $t \geq 4$. Let $C_1, C_2, \ldots, C_t$ be the components of $G - S$ and $v_i \in V(C_i)$ for $1 \leq i \leq t$. Consider $G + v_iv_j$ for $1 \leq i < j \leq t$. Let $\{z_{ij}\} = I_{v_iv_j} - \{v_i, v_j\}$. Since $t \geq 4$ and $z_{ij} \succ \bigcup_{k=1}^{t} V(C_k) - (V(C_i) \cup V(C_j))$, it follows that $z_{ij} \in S$. Since by Lemma 3(2), $\{v_i, v_j, z_{ij}\}$ is independent, we have $|\{z_{ij} : 1 \leq i < j \leq t\}| = \binom{t}{2}$. Hence, $|S| \geq \frac{t(t-1)}{2}$, implying that $t \leq \frac{1+\sqrt{8|S|+1}}{2}$ as required.

We next show that our bound is sharp. Let H be a spilt graph consisting of one independent set $U = \{u_1, u_2, \ldots, u_t\}$ of size t and one clique $\overline{U} = \{\overline{u}_{ij} : \{i, j\} \subseteq \{1, 2, \ldots, t\}\}$ of size $\binom{t}{2}$ such that each vertex $u_{ij} \in \overline{U}$ is adjacent to all vertices in U with exception of the pair u_i and u_j. Clearly, $i(H) = 3$ and $\{u_j, \overline{u}_{ij}\} \succ_i H + u_iu_j$. Hence, H is a 3-i-critical graph satisfying the bound of our result. ■

3. A CHARACTERIZATION OF THE CONNECTED 3-i-CRITICAL GRAPHS G HAVING A MINIMUM CUTSET S OF SIZE 2 OR 3 AND $\omega(G - S) = 3$

In this section, we establish a characterization of the connected 3-i-critical graphs G having a minimum cutset S of size 2 or 3 and $\omega(G - S) = 3$. We begin our section by providing six classes of connected 3-i-critical graphs having a minimum cutset of size 2 and 3. The last two theorems in this section show that if G is a connected 3-i-critical graph having a minimum cutset of size 2 or 3 and $\omega(G - S) = 3$, then G must belong to one of these six classes.

I. The class $\mathscr{H}_1$

For positive integers m and $n \geq 2$, define a graph $G \in \mathscr{H}_1$ of order $n + m + 4$ as follows. Set $V(G) = \{a, b, u, v\} \cup X \cup Y$ where $|X| = n$ and $|Y| = m$. The edges of G are defined as follows. $G[X] = K_n$ and $G[Y] = K_m$. Further, join

u and v to every vertex of $\{a, b\} \cup X$. Finally, join each vertex of X to every vertex of Y. This defines the class $\mathscr{H}_1$. Figure 1 illustrates our construction. It is not difficult to show that a graph $G \in \mathscr{H}_1$ is 3-i-critical containing $\{u, v\}$ as a minimum cutset. Note that in our diagram a "double line" denotes the join.

II. The class $\mathscr{H}_2$

For positive integer $n \geq 3$, define a graph $G \in \mathscr{H}_2$ of order $n+6$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3, v\} \cup X$ where $|X| = n$. The edges of G are defined as follows. $G[X] = K_n$; join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Further, join v to every vertex of X. Finally, add the edge u_1u_2. This defines the class $\mathscr{H}_2$. Figure 2 illustrates our construction. It is not difficult to show that a graph $G \in \mathscr{H}_2$ is 3-i-critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

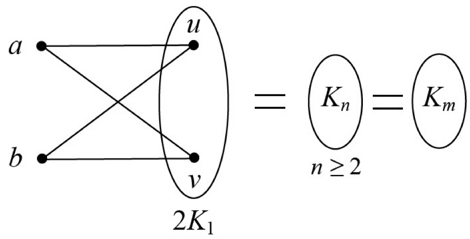

Figure 1. The structure of a graph in the class $\mathscr{H}_1$.

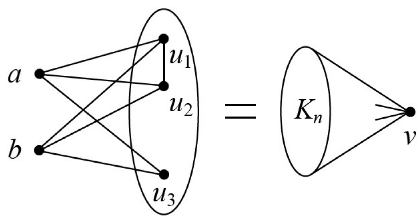

Figure 2. The structure of a graph in the class $\mathscr{H}_2$.

III. The class $\mathscr{H}_3$

For positive integers n, m_1 and m_2, define a graph $G \in \mathscr{H}_3$ of order $n + m_1 + m_2 + 5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X \cup Y_1 \cup Y_2$ where $|X| = n$, $|Y_1| = m_1$ and $|Y_2| = m_2$. The edges of G are defined as follows. $G[X \cup Y_1 \cup Y_2] = K_{n+m_1+m_2}$; join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Further, for $1 \leq i \leq 2$, join u_i to every vertex of Y_i. Finally, add the edge u_1u_2. This defines the class $\mathscr{H}_3$. Figure 3 illustrates our construction. It is not difficult to show that a graph $G \in \mathscr{H}_3$ is 3-i-critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

IV. The class $\mathscr{H}_4$

For a positive integer n, define a graph $G \in \mathscr{H}_4$ of order $n + 5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X$ where $|X| = n$. The edges of G are defined as follows. $G[X] = K_n$ and join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. This defines the class $\mathscr{H}_4$. Figure 4 illustrates our construction. It is easy to see that a graph $G \in \mathscr{H}_4$ is 3-i-critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

V. The class $\mathscr{H}_5$

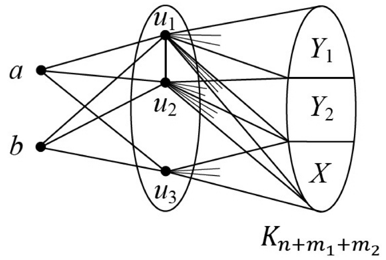

Figure 3. The structure of a graph in the class $\mathscr{H}_3$.

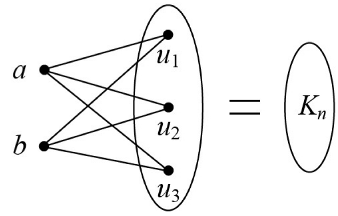

Figure 4. The structure of a graph in the class $\mathscr{H}_4$.

For positive integers $n \geq 2$ and m, define a graph $G \in \mathscr{H}_5$ of order $n+m+5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X \cup Y$ where $|X| = n$ and $|Y| = m$. The edges of G are defined as follows. $G[X \cup Y] = K_{n+m}$; join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Finally, join u_1 to every vertex of Y. This defines the class $\mathscr{H}_5$. Figure 5 illustrates our construction. It is not difficult to see that a graph $G \in \mathscr{H}_5$ is 3-i-critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

VI. The class $\mathscr{H}_6$

For non-negative integer n and positive integers m_1, m_2 and m_3, define a graph $G \in \mathscr{H}_6$ of order $n + m_1 + m_2 + m_3 + 5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X \cup Y_1 \cup Y_2 \cup Y_3$ where $|X| = n$ and $|Y_i| = m_i$ for $1 \leq i \leq 3$. The edges of G are defined as follows. $G[X \cup Y_1 \cup Y_2 \cup Y_3] = K_{n+m_1+m_2+m_3}$, join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Finally, for $1 \leq i \leq 3$; join u_i to every vertex of Y_i. This defines the class $\mathscr{H}_6$. Figure 6 illustrates our construction. It is easy to see that a graph $G \in \mathscr{H}_6$ is 3-i-critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

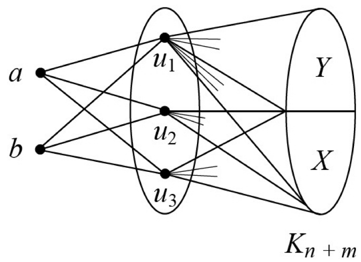

Figure 5. The structure of a graph in the class $\mathscr{H}_5$.

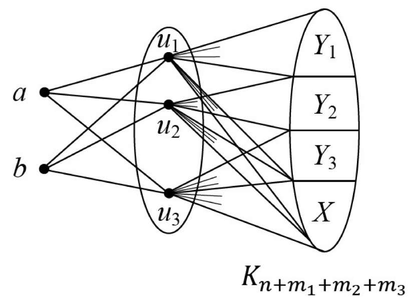

Figure 6. The structure of a graph in the class $\mathscr{H}_6$.

Before establishing our main results in this section, we need the following lemma.

Lemma 5. *Let G be a connected 3-i-critical graph and S a minimum cutset where $2 \leq |S| \leq 5$. Suppose $\omega(G - S) = 3$. Then*

(1) *There is a vertex of $G - S$, say x, such that $x \succ S$.*

(2) *At least two components of $G - S$ are singleton.*

(3) *Each component of $G - S$ is complete.*

Proof. Let C_1, C_2 and C_3 be the components of $G - S$.

(1) Suppose to the contrary that no vertex of $G - S$ dominates S. Then, since S is a minimum cutset, $|V(C_i)| \geq 2$ for $1 \leq i \leq 3$. For $1 \leq i \leq 3$, let u_i and v_i be two distinct vertices of $V(C_i)$. Put $T = \{\{x, y\} : x \in \{u_i, v_i\}, y \in \{u_j, v_j\}$ for $1 \leq i < j \leq 3\}$. It is easy to see that $|T| = 12$. We now consider $G + xy$ for $\{x, y\} \in T$. For each $\{x, y\} \in T$, let $\{z_{xy}\} = I_{xy} - \{x, y\}$. Since $\omega(G - S) = 3$ and $|V(C_i)| \geq 2$ for $1 \leq i \leq 3$, it follows that $z_{xy} \in S$. Consequently, there are at least 3 distinct elements of T, say $\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}$ such that $z_{x_1y_1} = z_{x_2y_2} = z_{x_3y_3}$ because $|S| \leq 5$ and $|T| = 12$.

We may now assume without loss of generality that $\{x_1, y_1\} = \{u_1, u_2\}$ and that $I_{x_1y_1} = I_{u_1u_2} = \{u_1, z_{u_1u_2}\}$. Then $z_{u_1u_2} \succ (V(C_2) \cup V(C_3)) - \{u_2\}$. By Lemma 3(2), $\{x_2, y_2, x_3, y_3\} \cap \{v_2, u_3, v_3\} = \emptyset$. Thus $\{x_2, y_2, x_3, y_3\} \subseteq \{u_1, v_1, u_2\}$. But this contradicts the fact that $\{x_i, y_i\} \neq \{x_j, y_j\}$ for $1 \leq i \neq j \leq 3$. This proves (1).

(2) Let $x \in V(G - S)$ such that $x \succ S$. Without loss of generality, we may assume that $x \in V(C_1)$. Choose $y \in V(C_2)$. Consider $G + xy$. Let $\{z\} = I_{xy} - \{x, y\}$. By Lemma 3(2), $z \notin S$. Then $z \in V(C_3)$. We first suppose that $I_{xy} = \{x, z\}$. Then $V(C_2) = \{y\}$. Now consider $G + xz$. Then, by Lemma 4, $I_{xz} = \{x, y\}$ or $I_{xz} = \{z, y\}$. If $I_{xz} = \{x, y\}$, then $|V(C_3)| = 1$. If $I_{xz} = \{z, y\}$, then $|V(C_1)| = 1$. Hence, G contains at least two singleton components as required.

We now suppose that $I_{xy} = \{y, z\}$. By similar arguments, G contains at least two singleton components. This proves (2).

(3) By (2), we may assume without loss of generality that $|V(C_1)| = |V(C_2)| = 1$. Put $\{x\} = V(C_1)$ and $\{y\} = V(C_2)$. Since S is a minimum cutset, $x \succ S$ and $y \succ S$. Suppose to the contrary that C_3 is not complete. Then there exist $a, b \in V(C_3)$ where $ab \notin E(G)$. Consider $G + xa$. Let $\{z\} = I_{xa} - \{x, a\}$. Because no vertex of $\{x, a\}$ is adjacent to a vertex of $\{y, b\}$, $z \succ \{y, b\}$. Then $z \in S$ since $y \in V(C_2)$ and $b \in V(C_3)$. But this contradicts Lemma 3(2) since $x \succ S$. This proves (3) and completes the proof of our lemma. ∎

Theorem 6. *Let G be a connected 3-i-critical graph and S a minimum cutset of size 2. If $G - S$ contains exactly three components, then $G \in \mathscr{H}_1$.*

Proof. Let C_1, C_2 and C_3 be the components of $G - S$. By Lemma 5(2) we may assume that $|V(C_1)| = |V(C_2)| = 1$. Put $\{a\} = V(C_1)$, $\{b\} = V(C_2)$ and $S = \{u, v\}$. Since S is a minimum cutset, $N_G(a) = N_G(b) = \{u, v\}$. Further, if $|V(C_3)| = 1$, then it is easy to see that $i(G) \leq 2$, a contradiction. Hence, $|V(C_3)| \geq 2$. By Lemma 5(3), C_3 is complete. Note that $N_{V(C_3)}(u) \neq \emptyset$ and $N_{V(C_3)}(v) \neq \emptyset$ since S is a minimum cutset. If there is a vertex $y \in N_{V(C_3)}(u) - N_{V(C_3)}(v)$, then $\{y, v\} \succ_i G$, a contradiction. Similarly, if there is a vertex $y \in N_{V(C_3)}(v) - N_{V(C_3)}(u)$, then $\{y, u\} \succ_i G$, again a contradiction. Note that $uv \notin E(G)$. Hence, $N_{V(C_3)}(u) \cup N_{V(C_3)}(v) = N_{V(C_3)}(u) \cap N_{V(C_3)}(v)$. It is easy to see that $|N_{V(C_3)}(u) \cup N_{V(C_3)}(v)| = |N_{V(C_3)}(u) \cap N_{V(C_3)}(v)| \geq 2$ since S is a minimum cutset of size 2. Because $i(G) = 3$, $|V(C_3) - (N_{V(C_3)}(u) \cup N_{V(C_3)}(v))| \geq 1$. Therefore, $G \in \mathscr{H}_1$ as required. This completes the proof of our theorem. ■

Theorem 7. *Let G be a connected 3-i-critical graph and S a minimum cutset of size 3. If $G - S$ contains exactly three components, then $G \in \mathscr{H}_i$ for $2 \leq i \leq 6$.*

Proof. Let C_1, C_2 and C_3 be the components of $G - S$. By Lemma 5(2) we may assume that $|V(C_1)| = |V(C_2)| = 1$. Put $\{a\} = V(C_1)$, $\{b\} = V(C_2)$ and $S = \{u_1, u_2, u_3\}$. Since S is a minimum cutset, $N_G(a) = N_G(b) = \{u_1, u_2, u_3\}$. Further, if $|V(C_3)| = 1$, then S must be independent since $i(G) = 3$ and thus $G \in \mathscr{H}_4$. So we may suppose that $|V(C_3)| \geq 2$. Note that C_3 is complete by Lemma 5(3). Further, $N_{V(C_3)}(u_i) \neq \emptyset$ for $1 \leq i \leq 3$ since S is a minimum cutset.

Claim 1. $G[S]$ contains at most one edge.

Proof. Assume, without loss of generality, that $u_1u_2, u_2u_3 \in E(G)$. Then $u_2 \succ S \cup \{a, b\}$. It is easy to see that $i(G) \leq 2$ since C_3 is complete. But this contradicts the fact that $i(G) = 3$. This settles our claim. □

We now distinguish two cases according to the number of edges in $G[S]$.

Case 1: $G[S]$ contains exactly one edge, say u_1u_2.

Claim 2. $N_{V(C_3)}(u_3) = N_{V(C_3)}(u_1) \cap N_{V(C_3)}(u_2)$.

Proof. It is easy to see that if there exists $x \in N_{V(C_3)}(u_3)$ but $x \notin N_{V(C_3)}(u_1) \cap N_{V(C_3)}(u_2)$, then $\{x, u_1\} \succ_i G$ or $\{x, u_2\} \succ_i G$, a contradiction. Hence, $N_{V(C_3)}(u_3) \subseteq N_{V(C_3)}(u_1) \cap N_{V(C_3)}(u_2)$. On the other hand, if $x \in N_{V(C_3)}(u_1) \cap N_{V(C_3)}(u_2)$ but $x \notin N_{V(C_3)}(u_3)$, then $\{x, u_3\} \succ_i G$, again a contradiction. Hence, $N_{V(C_3)}(u_1) \cap N_{V(C_3)}(u_2) \subseteq N_{V(C_3)}(u_3)$. This settles our claim. □

Now put $Y = V(C_3) - N_{V(C_3)}(u_3)$. It is easy to see that $Y \neq \emptyset$ since $i(G) = 3$.

Claim 3. Each vertex of Y is adjacent to at most one vertex of $\{u_1, u_2\}$.

Proof. Clearly, if there exists $y \in Y$ such that $yu_1 \in E(G)$ and $yu_2 \in E(G)$, then $\{y, u_3\} \succ_i G$, a contradiction. This settles our claim. □

Subcase 1.1: $|Y| = 1$. Put $\{v\} = Y$. If $vu_j \in E(G)$ for some $1 \leq j \leq 2$, then $\{u_j, u_3\} \succ_i G$, a contradiction. Hence, $vu_j \notin E(G)$ for $1 \leq j \leq 2$. Consequently, $N_G(v) = V(C_3) - \{v\}$. Because S is a minimum cutset, $|V(C_3) - \{v\}| \geq 3$. Therefore, $G \in \mathscr{H}_2$.

Subcase 1.2: $|Y| \geq 2$.

Claim 4. Each vertex of Y is adjacent to exactly one vertex of $\{u_1, u_2\}$. Further, $|N_Y(u_1)| \geq 1$ and $|N_Y(u_2)| \geq 1$.

Proof. We first suppose to the contrary that there is a vertex $y \in Y$ such that $yu_1 \notin E(G)$ and $yu_2 \notin E(G)$. Thus $N_S(y) = \emptyset$. Since $|Y| \geq 2$, there is $y_1 \in Y - \{y\}$. Consider $G + u_3y_1$. Let $\{z_1\} = I_{u_3y_1} - \{u_3, y_1\}$. By Lemma 3(2), $z_1 \notin V(C_3) \cup \{a, b\}$ since C_3 is complete and $\{a, b\} \subseteq N_G(u_3)$. Then $z_1 \in \{u_1, u_2\}$. If $I_{u_3y_1} = \{u_3, z_1\}$, then no vertex of $I_{u_3y_1}$ is adjacent to y, a contradiction. Hence, $I_{u_3y_1} = \{y_1, z_1\}$. We may assume that $z_1 = u_1$. Then $u_1y_1 \notin E(G)$. Now consider $G + u_1y$. Let $\{z_2\} = I_{u_1y} - \{u_1, y\}$. By Lemma 3(2), $z_2 \notin V(C_3) \cup \{a, b, u_2\}$ since C_3 is complete and $\{a, b, u_2\} \subseteq N_G(u_1)$. Thus $z_2 = u_3$. If $I_{u_1y} = \{u_1, u_3\}$, then no vertex of I_{u_1y} is adjacent to y_1, a contradiction. Hence, $I_{u_1y} = \{y, u_3\}$. But then no vertex of I_{u_1y} is adjacent to u_2, again a contradiction. This proves that each vertex of Y is adjacent to at least one vertex of $\{u_1, u_2\}$. It then follows by Claim 3 that each vertex of Y is adjacent to exactly one vertex of $\{u_1, u_2\}$.

We next show that $N_Y(u_1) \neq \emptyset$. Suppose this is not the case. Then u_1 is not adjacent to any vertex of Y. Then each vertex of Y is adjacent to u_2 by the above argument. Consequently, $\{u_2, u_3\} \succ_i G$, a contradiction. Hence, $N_Y(u_1) \neq \emptyset$. Similarly, $N_Y(u_2) \neq \emptyset$. This settles our claim. □

Put $Y_1 = N_Y(u_1)$ and $Y_2 = N_Y(u_2)$. It follows by Claim 4 that $Y_1 \cap Y_2 = \emptyset$ and hence $G \in \mathscr{H}_3$.

Case 2: $G[S]$ is independent.

Claim 5. For each $x \in V(C_3)$, either x is adjacent to every vertex of $\{u_1, u_2, u_3\}$ or x is adjacent to exactly one vertex of $\{u_1, u_2, u_3\}$.

Proof. We first suppose to the contrary that there is a vertex $x_1 \in V(C_3)$ such that x_1 is not adjacent to any vertex of $\{u_1, u_2, u_3\}$. Consider $G + x_1u_1$. Let $\{z\} = I_{x_1u_1} - \{x_1, u_1\}$. By Lemma 3(2) and the fact that C_3 is complete and $\{a, b\} \subseteq N_G(u_1)$, it follows that $z \in \{u_2, u_3\}$. Consequently, $u_2u_3 \in E(G)$, a contradiction. Hence, each vertex of $V(C_3)$ is adjacent to at least one vertex of $\{u_1, u_2, u_3\}$.

We now suppose that there is a vertex $x_2 \in V(C_3)$ such that x_2 is adjacent to, say, u_1 and u_2 but not to u_3. Then $\{x_2, u_3\} \succ_i G$, a contradiction. This settles our claim. □

For $1 \leq i \leq 3$, let Y_i be the set of vertices in C_3 which are adjacent only to u_i in S and let $Y = Y_1 \cup Y_2 \cup Y_3$ and $X = V(C_3) - Y$. Then, by Claim 5, $x \succ S$ for $x \in X$.

Claim 6. If $Y \neq \emptyset$, then either $Y_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$ or $Y = Y_i$ for exactly one $i \in \{1, 2, 3\}$.

Proof. Assume that one $Y_i = \emptyset$ and the other two are not empty, say $Y_1, Y_2 \neq \emptyset$ and $Y_3 = \emptyset$. Then $V(C_3) = Y_1 \cup Y_2 \cup X$. Consider $G + u_1u_2$. Let $\{z\} = I_{u_1u_2} - \{u_1, u_2\}$. By Lemma 3(2), $z \notin V(C_3) \cup \{a, b\}$. Then $z = u_3$. If $I_{u_1u_2} = \{u_1, u_3\}$, then no vertex of $I_{u_1u_2}$ is adjacent to a vertex of Y_2, a contradiction. Hence, $I_{u_1u_2} = \{u_2, u_3\}$. But then no vertex of $I_{u_1u_2}$ is adjacent to a vertex of Y_1, again a contradiction. This settles our claim. □

We now distinguish two subcases.

Subcase 2.1: $Y = \emptyset$. It is easy to see that $V(C_3) = X$ and thus $G \in \mathscr{H}_4$.

Subcase 2.2: $Y \neq \emptyset$. Suppose first that two $Y_i's$ are empty, say $Y_1 \neq \emptyset$ and $Y_2 = Y_3 = \emptyset$. If $X = \emptyset$, then u_1 becomes a cutvertex of G, contradicting the fact that $S = \{u_1, u_2, u_3\}$ is a minimum cutset. Hence, $X \neq \emptyset$. Note that $\{u_1\} \cup X$ is a vertex cutset of G. Since the minimum cardinality of a vertex cutset in G is 3, $|X| \geq 2$. It is easy to see that $G \in \mathscr{H}_5$.

We now suppose that all $Y_i's$ are not empty. Then $G \in \mathscr{H}_6$. This completes the proof of our theorem. ■

Our last result follows immediately from Theorems 6 and 7.

Corollary 8. *Let G be a connected 3-i-critical graph and S a minimum cutset where $2 \leq |S| \leq 3$. If $\omega(G - S) = 3$, then the minimum degree of G is $|S|$.*

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