# CRITICAL GRAPHS FOR $\boldsymbol{R}\left(\boldsymbol{P}_{n}, \boldsymbol{P}_{m}\right)$ AND THE STAR-CRITICAL RAMSEY NUMBER FOR PATHS 

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#### Abstract

The graph Ramsey number $R(G, H)$ is the smallest integer $r$ such that every 2-coloring of the edges of $K_{r}$ contains either a red copy of $G$ or a blue copy of $H$. The star-critical Ramsey number $r_{*}(G, H)$ is the smallest integer $k$ such that every 2-coloring of the edges of $K_{r}-K_{1, r-1-k}$ contains either a red copy of $G$ or a blue copy of $H$. We will classify the critical graphs, 2-colorings of the complete graph on $R(G, H)-1$ vertices with no red $G$ or blue $H$, for the path-path Ramsey number. This classification will be used in the proof of $r_{*}\left(P_{n}, P_{m}\right)$.


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## 1. Introduction

For graphs $G$ and $H$, the graph Ramsey number $R(G, H)$ is the smallest integer $r$ such that every 2-coloring of the edges of $K_{r}$ contains either a red copy of $G$ or a blue copy of $H$. Furthermore, there exists a $(G, H)$-free coloring of $K_{r-1}$, a 2-coloring of the edges that does not contain a red copy of $G$ or a blue copy of $H$. A $(G, H)$-free coloring of $K_{r-1}$ is known as the critical graph for $R(G, H)$. The following section is devoted to classifying the critical graphs for $R\left(P_{n}, P_{m}\right)$.

The star-critical Ramsey number $r_{*}(G, H)$, introduced in [3], is the smallest integer $k$ such that every 2-coloring of the edges of $K_{r}-K_{1, r-1-k}$ contains either a red copy of $G$ or a blue copy of $H$. In other words, this is the largest star that can be removed from $K_{r}$ yet every 2-coloring is still forced to contain either a
red $G$ or a blue $H$. To emphasize the size of the star, we can view the graph $K_{r}-K_{1, r-1-k}$ as $K_{r-1} \sqcup K_{1, k}$, the union of $K_{r-1}$ and $K_{1, k}$ such that the $k$ vertices of the star are vertices of $K_{r-1}$.

The graph $G+H$ is the disjoint union of $G$ and a graph $H$, whereas the graph $G+\{v\}$ is the disjoint union of $G$ and a vertex $v$. The deletion of edges of a subgraph $H$ from $G$ will be denoted as $G-H$ and the deletion of a vertex as $G-\{v\}$. The join of $G$ and $H$, denoted by $G \vee H$, is the graph $G+H$ with the addition of the edges $\{x y: x \in V(G)$ and $y \in V(H)\}$. For a graph $G$ with a 2 -coloring of the edges, the red subgraph will be denoted as $G^{\rho}$ which consists of all the vertices of $G$ along with the red edge set. Similarly, $G^{\beta}$ will denote the blue subgraph. The figures of the critical graphs contain solid red edges and dashed blue edges. The thick edges between cliques denote all edges between the sets of vertices. The following facts can be easily checked.

Observation 1.1. The Ramsey number $R(G, H)=R(H, G)$ and the critical graphs for $R(H, G)$ are the critical graphs for $R(G, H)$ with the colors red and blue interchanged.

Observation 1.2. Let $G^{\prime}$ be a subgraph of $G$ and $H^{\prime}$ be a subgraph of $H$. If $R\left(G^{\prime}, H^{\prime}\right)=R(G, H)$, then the class of critical graphs for $R\left(G^{\prime}, H^{\prime}\right)$ is a subset of the class of critical graphs for $R(G, H)$.

In the following sections, we will consider the path-path graph Ramsey number. A path on $n$ vertices is denoted by $P_{n}$ and the length of the path refers to the number of vertices on the path. The critical graphs for $R\left(P_{n}, P_{m}\right)$ will be classified using a sequence of lemmas that depend on the parity of the length of the path. We will see that the $\left(P_{n}, P_{m}\right)$-free colorings of $K_{R\left(P_{n}, P_{m}\right)-1}$ must belong to an infinite family of graphs. The star-critical Ramsey number $r_{*}\left(P_{n}, P_{m}\right)$ will be determined using this classification.

## 2. The Critical Graphs for $R\left(P_{n}, P_{m}\right)$

The path-path graph Ramsey number was obtained by Gerencsér and Gyárfás [1] in 1967 and is stated as Theorem 2.1. The graphs $G_{1}$ and $G_{2}$ in Definition 2.4 are presented in [1] as examples to establish the lower bound of $R\left(P_{n}, P_{m}\right)$. A complete list of the critical graphs for the path-path Ramsey number is described in Definition 2.4 and classified in Proposition 2.6.

Theorem 2.1 [1]. For all $n \geq m \geq 2, R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$.
Before classifying the critical graphs for $R\left(P_{n}, P_{m}\right)$ with $m \geq 4$, the critical graphs for $m=2$ and $m=3$ are as follows.

Proposition 2.2. For given $n \geq m$, let $r=R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$ and $c$ be a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$.

If $m=2$ and $n>2$, then the resulting graph is

$$
H_{1}:\left\{\begin{array}{l}
H_{1}^{\rho}=K_{n-1}, \\
H_{1}^{\beta}=(n-1) K_{1} .
\end{array}\right.
$$

If $m=3$ and $n>3$, then for any $i \in\left\{0,1,2, \ldots,\left\lfloor\frac{n-1}{2}\right\rfloor\right\}$ the resulting graph is

$$
H_{2}:\left\{\begin{array}{l}
H_{2}^{\rho}=K_{n-1}-i K_{2}, \\
H_{2}^{\beta}=i K_{2} .
\end{array}\right.
$$

If $n=m=2$, then the graph is a single vertex. If $n=m=3$, then the graph is a single edge colored red or blue.
Proof. A $\left(P_{n}, P_{2}\right)$-free coloring of $K_{n-1}$ does not have any blue edges and so the graph must be a red $K_{n-1}$ for $n>2$ and a single vertex if $n=2$. A $\left(P_{n}, P_{3}\right)$-free coloring of $K_{n-1}$ does not have a blue $P_{3}$ which implies that the blue edges must form a matching. Thus, if $n>3$, the graph has red subgraph $K_{n-1}-i K_{2}$ and blue subgraph $i K_{2}$ for any $i \in\left\{0,1,2, \ldots,\left\lfloor\frac{n-1}{2}\right\rfloor\right\}$. If $n=3$, the graph consists of a single edge which may be colored red or blue.

The following fact will be frequently used throughout this paper. It simply states that a path can be created between two sets of vertices by alternating between the sets. The length of the path is twice the size of the smaller set of vertices plus one vertex.

Observation 2.3. Let $G=A \vee B$. If $|V(A)| \geq k$ and $|V(B)| \geq k+1$, then there is a path $P_{2 k+1}$ that alternates between vertices in $A$ and vertices in $B$ starting and ending with a vertex in B. See Figure 2.


Figure 1. A path $P_{2 k+1}$ between $A$ and $B$.
Definition 2.4. For given $n$ and $m$ with $n \geq m \geq 4$, let $r=R\left(P_{n}, P_{m}\right)=$ $n+\left\lfloor\frac{m}{2}\right\rfloor-1$ and $A_{k}$ be any 2 -coloring of the graph $K_{k}$. Define the class of graphs $\mathcal{G}$, shown in Figure 2, to consist of the five families of $K_{r-1}$ listed below.

$$
\begin{aligned}
& \text { If } n \geq m, G_{1}:\left\{\begin{array}{l}
G_{1}^{\rho}=K_{n-1}+A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\rho}, \\
G_{1}^{\beta}=A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\beta} \vee(n-1) K_{1} .
\end{array}\right. \\
& \text { If } n \geq m \text { with } m \text { odd, } G_{2}:\left\{\begin{array}{l}
G_{2}^{\rho}=\left(K_{n-1}-K_{2}\right)+A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\rho}, \\
G_{2}^{\beta}=A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\beta} \vee\left(K_{2}+(n-3) K_{1}\right) .
\end{array}\right. \\
& \text { If } n=m, G_{3}:\left\{\begin{array}{l}
G_{3}^{\rho}=A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\rho} \vee(n-1) K_{1}, \\
G_{3}^{\beta}=K_{n-1}+A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\beta} .
\end{array}\right. \\
& \text { If } n=m \text { with } m \text { odd, } G_{4}:\left\{\begin{array}{l}
G_{4}^{\rho}=A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\rho} \vee\left(K_{2}+(n-3) K_{1}\right), \\
G_{4}^{\beta}=\left(K_{n-1}-K_{2}\right)+A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\beta} .
\end{array}\right. \\
& \text { If } n=m+1 \text { with } m \text { odd, } G_{5}:\left\{\begin{array}{l}
G_{5}^{\rho}=A_{\frac{n}{2}-1}^{\rho} \vee(m-1) K_{1}, \\
G_{5}^{\beta}=K_{m-1}+A_{\frac{n}{2}-1 .}^{\beta} .
\end{array}\right.
\end{aligned}
$$

The graphs in Definition 2.4 (see Figure 2) are $\left(P_{n}, P_{m}\right)$-free colorings. The red subgraphs of $G_{1}$ and $G_{2}$ do not contain a red path on $n$ vertices since each component has size at most $n-1$. A longest path in the blue subgraph of $G_{1}$ is a blue $P_{m-1}$ by Observation 2.3 using blue edges between $K_{n-1}^{\rho}$ and $A_{\left\lfloor\frac{m}{2}\right\rfloor-1}$. A longest path in the blue subgraph of $G_{2}$ is also a blue $P_{m-1}$ using a longest blue $P_{m-2}$ by Observation 2.3 between $\left(K_{n-1}-K_{2}\right)^{\rho}$ and $A_{\left\lfloor\frac{m}{2}\right\rfloor-1}$ and the single blue edge. The graphs $G_{3}$ and $G_{4}$ are $\left(P_{n}, P_{m}\right)$-free colorings since they are the graphs $G_{1}$ and $G_{2}$ with the colors red and blue interchanged. The blue subgraph of $G_{5}$ does not contain a blue path on $m$ vertices since each component has size at most $m-1$. A longest path in the red subgraph of $G_{5}$ is a red $P_{n-1}$ by Observation 2.3 using the red edges between $K_{m-1}^{\beta}$ and $A_{\frac{n}{2}-1}$.

Remark 2.5. In Proposition 2.2, the $\left(P_{n}, P_{m}\right)$-free coloring when $m=2$, the graph $H_{1}$, belongs to the family of graphs $G_{1}$ in $\mathcal{G}$ of Definition 2.4, but when $m=3$ the graph $H_{2}$ does not belong to $\mathcal{G}$.

Proposition 2.6. For given $n$ and $m$ with $n \geq m \geq 4$, let $r=R\left(P_{n}, P_{m}\right)=$ $n+\left\lfloor\frac{m}{2}\right\rfloor-1$. If $c$ is a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$, then the resulting graph must belong to the class of graphs $\mathcal{G}$ in Definition 2.4.

The following sequence of lemmas will prove Proposition 2.6. We will first classify the critical graphs for odd $n \geq m+1$ in Lemma 2.9 and for odd $n=m$ in Lemma 2.11. Then we will proceed with even $n \geq m+1$ with $m$ odd in Lemma 2.14 and $m$ even in Lemma 2.15. Finally, Lemma 2.16 will classify the critical graphs for even $n=m$. Note that the order of the lemmas is necessary as Lemma 2.16 invokes Lemma 2.15 which invokes Lemma 2.14. Also, Lemma 2.9 is used in the proof of Lemma 2.14. Within the proofs of the lemmas, the cycle-path and


Figure 2. Critical graphs for $R\left(P_{n}, P_{m}\right)$. The graph $G_{1}$ is for all $n \geq m$. The graphs $G_{2}$ and $G_{4}$ are when $m$ is odd. The graphs $G_{3}$ and $G_{4}$ are when $n=m$. The graph $G_{5}$ is for $n=m+1$ with $m$ odd.
cycle-cycle Ramsey numbers (Theorems 2.7 and 2.8) will be used in addition to Observation 2.3 and the observation that the graph of a clique $K_{n-1}$ and an edge adjacent to any vertex of the clique contains a $P_{n}$.

Theorem 2.7 [4]. For $n \geq m \geq 2$,

$$
R\left(C_{n}, P_{m}\right)= \begin{cases}\max \left\{n+\left\lfloor\frac{m}{2}\right\rfloor-1,2 m-1\right\} & \text { for } n \text { odd } \\ n+\left\lfloor\frac{m}{2}\right\rfloor-1 & \text { for } n \text { even } .\end{cases}
$$

Theorem 2.8 [5]. For $n \geq m \geq 3$,

$$
R\left(C_{n}, C_{m}\right)= \begin{cases}2 n-1 & \text { for } m \text { odd, }(n, m) \neq(3,3) \\ n+\frac{m}{2}-1 & \text { for } m, n \text { both even },(n, m) \neq(4,4) \\ \max \left\{n+\frac{m}{2}-1,2 m-1\right\} & \text { for } m \text { even and } n \text { odd }\end{cases}
$$

Lemma 2.9. For given $n$ and $m$ with $n$ odd and $n \geq m+1 \geq 5$, let $r=$ $R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$. If $c$ is a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$, then the resulting graph is either $G_{1}$ or $G_{2}$ as in Definition 2.4.

Proof. Let $r=R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$ and $c$ be a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$. Since $R\left(C_{n-1}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-2=r-1$ and the graph does not have a blue $P_{m}$, the graph must have a red $C_{n-1}=\left(v_{1} v_{2} \ldots v_{n-1}\right)$. A red edge from any $v_{i}$ to a vertex not on the cycle creates a red $P_{n}$. Thus, there cannot be any such red edges and the red $C_{n-1}$ has all blue edges to the remaining $\left\lfloor\frac{m}{2}\right\rfloor-1$ vertices. Using these blue edges and Observation 2.3, there is a blue

$$
P_{2\left(\left\lfloor\frac{m}{2}\right\rfloor-1\right)+1}=P_{2\left\lfloor\frac{m}{2}\right\rfloor-1}= \begin{cases}P_{m-1} & \text { if } m \text { is even } \\ P_{m-2} & \text { if } m \text { is odd }\end{cases}
$$

If $m$ is even, then there is a blue $P_{m-1}$ which can begin and end at any vertex on the red $C_{n-1}$. Thus, a single blue edge between two vertices on the red cycle would create a blue $P_{m}$. This implies that there cannot be any blue edges within the red cycle and the resulting graph is $G_{1}$. If $m$ is odd, then there is a blue $P_{m-2}$ that can begin and end at any vertex on the red $C_{n-1}$. Thus, a single blue edge between two vertices on the red cycle would extend this path to a blue $P_{m-1}$. Two blue edges between vertices on the red cycle may be either disjoint or share a vertex. In either case, the blue $P_{m-2}$ could be extended to a blue $P_{m}$ using these blue edges at the beginning and end of the blue $P_{m-2}$ if they are disjoint, or at the end of the blue $P_{m-2}$ if they share a vertex. This implies that there can be at most one blue edge between two vertices on the cycle and the resulting graph is either $G_{1}$ or $G_{2}$.

In Lemma 2.11, we classify the critical graphs for odd $n=m$. Within the proof, we use the Ramsey number $R\left(C_{n-1}, C_{m-1}\right)$ for $n=m \geq 7$. The following result will be used in the case when $n=m=5$.

Proposition 2.10 [3]. For a given $n \geq 3$, let $r=R\left(P_{n}, C_{4}\right)=n+1$. If $c$ is a $\left(P_{n}, C_{4}\right)$-free coloring of $K_{r-1}$, then the resulting graph must belong to the class of graphs $H_{i}$ for $i=0, \ldots,\left\lfloor\frac{n-1}{2}\right\rfloor$ such that

$$
\begin{aligned}
H_{i}^{\rho} & =\left(K_{n-1}-i K_{2}\right)+K_{1} \\
H_{i}^{\beta} & =K_{1, n-1} \cup i K_{2}
\end{aligned}
$$

Note that the blue subgraph $H_{i}^{\beta}=K_{1, n-1} \cup i K_{2}$ is obtained by adding $i$ disjoint edges to the complete bipartite graph. Thus, the vertices of $i K_{2}$ are vertices of $K_{1, n-1}$. This is illustrated in Figure 3.

Lemma 2.11. For given $n$ and $m$ with $n=m \geq 5$ odd, let $r=R\left(P_{n}, P_{m}\right)=$ $n+\left\lfloor\frac{m}{2}\right\rfloor-1$. If c is a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$, then the resulting graph is $G_{i}$ with $i=1,2,3$ or 4 as in Definition 2.4.

Proof. For given $n$ and $m$ with $n=m$ odd, let $r=R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$ and $c$ be a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$. For $n, m \geq 7, R\left(C_{n-1}, C_{m-1}\right)=$


Figure 3. Critical graph $H_{3}$ for $R\left(P_{n}, C_{4}\right)$.
$(n-1)+\frac{m-1}{2}-1=n+\left\lfloor\frac{m}{2}\right\rfloor-2=r-1$. Thus, the graph $K_{r-1}$ has either a red $C_{n-1}$ or a blue $C_{m-1}$. If there is a red $C_{n-1}$, then the same proof as in Lemma 2.9 holds and the resulting graph is either $G_{1}$ or $G_{2}$. If there is a blue $C_{m-1}$, then the proof in Lemma 2.9 holds with the colors red and blue interchanged and the resulting graph is either $G_{3}$ or $G_{4}$.

If $n=m=5$, then the cycle-cycle Ramsey number $R\left(C_{4}, C_{4}\right)=6 \neq 5=r-1$. In this case, let $c$ be a ( $P_{5}, P_{5}$ )-free coloring of $K_{5}$. Since $R\left(P_{5}, C_{4}\right)=6$ and there is no red $P_{5}$, either the graph has a blue $C_{4}$ or the graph has a $\left(P_{5}, C_{4}\right)$-free coloring. If there is a blue $C_{4}$, then the same proof as in Lemma 2.9 holds and the resulting graph is either $G_{3}$ or $G_{4}$. Otherwise, the graph has a $\left(P_{5}, C_{4}\right)$-free coloring. Then, by Proposition 2.10, the graph must be $H_{0}$ or $H_{1}$. Note that $H_{2}$ contains a blue $P_{5}$. The graphs $H_{0}$ and $H_{1}$ are the graphs $G_{1}$ and $G_{2}$, respectively.

Next, we will consider the cases when $n$ is even. In the proofs of the lemmas with even $n$, Proposition 2.12 will be used, which is stated as Exercise 7.2 .38 in West [6], and Lemma 2.13.

Proposition 2.12 [6]. If $G$ is connected with minimum degree $k \geq 2$ and $G$ has more than $2 k$ vertices, then $G$ has a path on $2 k+1$ vertices.

Lemma 2.13. Let $G=A \vee B$ with $|V(A)| \geq k$ and $|V(B)| \geq k+i$. If there is a path on $i$ vertices in $B$, then $G$ has a path $P_{2 k+i}$.

Proof. By Observation 2.3, there is a path $P_{2 k+1}$ that alternates between vertices in $A$ and vertices in $B$ starting and ending with a vertex in $B$. Since $G=A \vee B$, this path can begin at any vertex in $B$. Thus, we may begin the path of length $2 k+1$ at an endpoint of the path on $i$ vertices in $B$ and the graph has a path of length $2 k+i$. Note that there are at least $k$ vertices in $B$ not on the path of length $i$ and so an endpoint of this $P_{i}$ and $k$ vertices not on the path $P_{i}$ are sufficient to create the path $P_{2 k+i}$.

Lemma 2.14. For given $n$ and $m$ with even $n$ and odd $m=2 k+1$ such that $n \geq m+1 \geq 6$, let $r=R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$. If $c$ is a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$, then the resulting graph is either $G_{1}, G_{2}$ or $G_{5}$ as in Definition 2.4.

Proof. For given $n$ and $m$ with even $n$ and odd $m=2 k+1$ such that $n \geq$ $m+1 \geq 6$, let $r=R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1=n+\left\lfloor\frac{2 k+1}{2}\right\rfloor-1=n+k-1$ and $c$ be a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$. If $K_{r-1}^{\beta}$ is connected with the degree of each vertex at least $k$, then there is a blue path on at least $2 k+1$ vertices by Proposition 2.12 and hence a blue $P_{m}$. Therefore, either there is a vertex in $K_{r-1}^{\beta}$ of degree at most $k-1$ or $K_{r-1}^{\beta}$ is disconnected.

Case I: There is a vertex in $K_{r-1}^{\beta}$ of degree at most $k-1$. This implies that there is a vertex $v$ in $K_{r-1}^{\rho}$ of degree at least $(r-2)-(k-1)=(n+k-1-2)-$ $(k-1)=n-2$. Note that $\left|V\left(K_{r-1}-\{v\}\right)\right|=n+k-3$ and the cycle-cycle Ramsey number, $R\left(C_{n-2}, C_{m-1}\right)=n-2+\frac{m-1}{2}-1=n-2+\frac{2 k+1-1}{2}-1=n+k-3$ (except when $n=6$ and $m=5$ ) by Theorem 2.8. Thus, $K_{r-1}-\{v\}$ has either a red $C_{n-2}$ or a blue $C_{m-1}$.

Suppose that $K_{r-1}-\{v\}$ has a red $C_{n-2}$. If $v$ has a red edge to a vertex on the red $C_{n-2}$ and to a vertex not on the red $C_{n-2}$, then the graph contains a red $P_{n}$. Since $K_{r-1}$ does not have a red $P_{n}$, the vertex $v$ must have exactly $n-2$ red edges to the red $C_{n-2}$ and $k-1$ blue edges to the remaining $k-1$ vertices. A red edge from a vertex of the red $C_{n-2}$ to any of the $k-1$ vertices creates a red $P_{n}$. Hence, all edges from the red $C_{n-2}$ must be blue (see Figure 4).


Figure 4. A red $C_{n-2}$ with blue edges to remaining vertices.
By Observation 2.3, there is a blue $P_{2(k-1)+1}=P_{m-2}$ that can begin and end at any vertex on the red $C_{n-1}$. Note that this red cycle on $n-1$ vertices includes the vertex $v$. Thus, a single blue edge between two vertices on the red cycle would extend this path to a blue $P_{m-1}$ and two blue edges between vertices on the red cycle would create a blue $P_{m}$. Thus, there can be at most one blue edge between two vertices on the red cycle and the resulting graph is either $G_{1}$ or $G_{2}$.

If $n=6$ and $m=5$, then the cycle-cycle Ramsey number $R\left(C_{4}, C_{4}\right)=$ $6 \neq 5=n+k-3$. In this case, let $c$ be a ( $P_{6}, P_{5}$ )-free coloring of $K_{6}$. Since $R\left(P_{6}, C_{4}\right)=7$ and the graph does not contain a red $P_{6}$, it has either a blue $C_{4}$ or a $\left(P_{6}, C_{4}\right)$-free coloring. If there is a blue $C_{4}$, then the same proof as below holds and the resulting graph is $G_{5}$. Otherwise, the graph has a $\left(P_{6}, C_{4}\right)$-free coloring. Then, by Proposition 2.10, the graph must be $H_{0}$ or $H_{1}$. Note that $H_{2}$ contains a blue $P_{5}$. The graphs $H_{0}$ and $H_{1}$ are the graphs $G_{1}$ and $G_{2}$, respectively.

Suppose that $K_{r-1}-\{v\}$ has a blue $C_{m-1}$. Since the graph does not have a blue $P_{m}$, there cannot be any blue edges between a vertex on the blue $C_{m-1}$ and a vertex not on the blue $C_{m-1}$. Therefore, all edges between the vertices of the blue $C_{m-1}$ and the remaining $n-k-2$ vertices are red. We will now show that the only $\left(P_{n}, P_{m}\right)$-free coloring occurs when $n=m+1$. Moreover, if $n \geq m+3$ and the graph has a blue $C_{m-1}$, then every graph contains a red $P_{n}$.

If $n=m+1$, then the graph has a red path $P_{2(n-k-2)+1}=P_{2(n-1)-m}=$ $P_{m}=P_{n-1}$ with endpoints on the cycle and so there cannot be any red edges between vertices on the blue $C_{m-1}$. Thus, there is a blue $K_{m-1}$ with all red edges to the remaining $n-k-2$ vertices. Note that $n-k-2=n-\frac{m-1}{2}-2=$ $\frac{2 n-m-3}{2}=\frac{2 n-(m+1)-2}{2}=\frac{n-2}{2}=\frac{n}{2}-1$ and the resulting graph is $G_{5}$.

If $m+3 \leq n \leq m+(k+1)$, then $m-1 \geq n-k-2$. By Observation 2.3, the graph has a red $P_{2(n-k-2)}=P_{2 n-2 k-4}=P_{2 n-2-(2 k+1)-1}=P_{2(n-1)-(m+1)}$ and hence a red $P_{n}$.

If $m+(k+2) \leq n \leq m+2 k$, then $m-1<n-k-2$. By Observation 2.3, the graph has a red $P_{2(m-1)+1}=P_{m+2 k}$ and hence a red $P_{n}$.

If $n \geq m+(2 k+1)$, then $m-1<n-k-2$. By Observation 2.3 , the graph has a red $P_{2(m-1)+1}=P_{m+2 k}$. Let $A$ denote the graph consisting of the $n-k-2$ vertices not on the blue $C_{m-1}$.

For $m+(2 k+1) \leq n<m+4 k$, the Ramsey number

$$
\begin{aligned}
R\left(P_{n-(m+2 k)+1}, P_{m}\right) & =R\left(P_{m}, P_{n-(m+2 k)+1}\right) \\
& =m+\left\lfloor\frac{n-(m+2 k)+1}{2}\right\rfloor-1=\frac{n}{2} .
\end{aligned}
$$

For $n \geq m+4 k$, the Ramsey number

$$
\begin{aligned}
R\left(P_{n-(m+2 k)+1}, P_{m}\right) & =n-(m+2 k)+1+\left\lfloor\frac{m}{2}\right\rfloor-1 \\
& =n-3 k-1
\end{aligned}
$$

Since $n-k-2>R\left(P_{n-(m+2 k)+1}, P_{m}\right)$ and the graph does not have a blue $P_{m}$, there must be a red $P_{n-(m+2 k)+1}$ in $A$. By Lemma 2.13, we can extend this path to a red $P_{n}$ since $P_{2(m-1)+n-(m+2 k)+1}=P_{m+n-2 k-1}=P_{2 k+1+n-2 k-1}=P_{n}$. Note that since $n-k-2-(n-(m+2 k)+1)=m+k-3$, there are at least $3 k-2$ vertices in $A$ not on the path that may be used to form the red $P_{m+2 k}$.

Case II: The graph $K_{r-1}^{\beta}$ is disconnected. Let $X$ and $Y$ be any partition of the graph $K_{r-1}$ such that all edges between $X$ and $Y$ are red. If both $X$ and $Y$ have at least $\frac{n}{2}$ vertices, then there is a red $P_{n}$ by Observation 2.3. Therefore, one of $X$ or $Y$ must have at most $\frac{n}{2}-1$ vertices. Without loss of generality, assume that $|V(Y)| \leq \frac{n}{2}-1$. For some $i=0,1,2, \ldots, \frac{n}{2}-2$, we have that $|V(X)|=\left(\frac{n}{2}+k-1\right)+i$ and $|V(Y)|=\left(\frac{n}{2}-1\right)-i$. Note that if there is a red $P_{2(i+1)}$ in $X$, then the graph contains a red $P_{n}$ by Lemma 2.13.

If $n=m+1$, then $|V(X)|=2 k+i$. Since $R\left(P_{2(i+1)}, P_{m}\right)=R\left(P_{m}, P_{2(i+1)}\right)=$ $m+i, X$ either has a red $P_{2(i+1)}$, a blue $P_{m}$ or $X$ is the critical graph for $R\left(P_{m}, P_{2(i+1)}\right)$ with $m$ odd and the colors interchanged. Since the first two possibilities contradict the graph having a $\left(P_{n}, P_{m}\right)$-free coloring, $X$ must be such a critical graph. By Lemma $2.9, X$ is the graph $G_{3}$, a blue $K_{2 k}$ with all edges to the remaining $i$ vertices colored red. If we move the $i$ vertices to $Y$, then the graph is a blue $K_{2 k}=K_{m-1}$ with all red edges to the $\frac{n}{2}-1$ remaining vertices. Thus, the resulting graph is $G_{5}$.

If $n \geq m+3$, then $|V(X)| \geq \frac{m+3}{2}+k-1+i=m+i$. For $i<k$, the Ramsey number $R\left(P_{2(i+1)}, P_{m}\right)=R\left(P_{m}, P_{2(i+1)}\right)=m+\left\lfloor\frac{2(i+1)}{2}\right\rfloor-1=m+i$. For $i \geq k$, the Ramsey number $R\left(P_{2(i+1)}, P_{m}\right)=2(i+1)+\left\lfloor\frac{m}{2}\right\rfloor-1=2 i+k+1 \geq m+k$. Thus, for all $i$, there must be a red $P_{2(i+1)}$ in $X$ and the graph does not have a ( $P_{n}, P_{m}$ )-free coloring.

Lemma 2.15. For given $n$ and $m$ both even such that $n \geq m+1 \geq 5$, let $r=R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$. If $c$ is a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$, then the resulting graph is $G_{1}$ as in Definition 2.4.

Proof. For given $n$ and $m$ both even such that $n \geq m+1 \geq 5$, let $r=$ $R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$ and $c$ be a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$. For $m$ even, $R\left(P_{n}, P_{m}\right)=R\left(P_{n}, P_{m+1}\right)$. By Lemma 2.14, the critical graphs for $R\left(P_{n}, P_{m+1}\right)$ are $G_{1}, G_{2}$ or $G_{5}$. The critical graphs for $R\left(P_{n}, P_{m}\right)$ must be a subset of these graphs by Observation 1.2. Since both $G_{2}$ and $G_{5}$ contain a blue $P_{m}$, the only critical graph for $R\left(P_{n}, P_{m}\right)$ is $G_{1}$.

Lemma 2.16. For given $n$ and $m$ with $n=m$ even, let $r=R\left(P_{n}, P_{m}\right)=$ $n+\left\lfloor\frac{m}{2}\right\rfloor-1$. If $c$ is a $\left(P_{n}, P_{m}\right)$-free coloring of $K_{r-1}$, then the resulting graph is either $G_{1}$ or $G_{3}$ as in Definition 2.4.

Proof. For given $n$ and $m$ with $n=m=2 k$ even, let $r=R\left(P_{2 k}, P_{2 k}\right)=$ $2 k+\left\lfloor\frac{2 k}{2}\right\rfloor-1=3 k-1$ and $c$ be a $\left(P_{2 k}, P_{2 k}\right)$-free coloring of $K_{r-1}$. If $K_{r-1}^{\rho}$ is connected with the degree of each vertex at least $k$, then there is a red $P_{2 k}$ by Proposition 2.12. Similarly, if $K_{r-1}^{\beta}$ is connected with the degree of each vertex at least $k$, then there is a blue $P_{2 k}$ by Proposition 2.12. Therefore, either there is
a vertex in both $K_{r-1}^{\rho}$ and $K_{r-1}^{\beta}$ of degree at most $k-1$ or at least one of $K_{r-1}^{\rho}$ or $K_{r-1}^{\beta}$ is disconnected.

Case I: There is a vertex in both $K_{r-1}^{\rho}$ and $K_{r-1}^{\beta}$ of degree at most $k-1$ and both subgraphs are connected. This implies that there is a vertex $v$ in $K_{r-1}^{\rho}$ of degree at least $2 k-2$ and a vertex $w$ in $K_{r-1}^{\beta}$ of degree at least $2 k-2$. Note that $\left|V\left(K_{r-1}-\{v, w\}\right)\right|=3 k-4$ and the cycle-cycle Ramsey number $R\left(C_{2 k-2}, C_{2 k-2}\right)=3 k-4$ by Theorem 2.8 (except when $k=3$ ). Thus, $K_{r-1}-$ $\{v, w\}$ has either a red $C_{2 k-2}$ or a blue $C_{2 k-2}$.

Suppose that $K_{r-1}-\{v, w\}$ has a red $C_{2 k-2}$. If $v$ has a red edge to a vertex on the red $C_{2 k-2}$ and to a vertex not on the red $C_{2 k-2}$, then the graph contains a red $P_{2 k}$. Since $K_{r-1}$ does not have a red $P_{2 k}$, the vertex $v$ must have exactly $2 k-2$ red edges to the red $C_{2 k-2}$ and $k-1$ blue edges to the remaining $k-1$ vertices which include $w$. A red edge from a vertex of the red $C_{2 k-2}$ to any of the $k-1$ vertices creates a red $P_{2 k}$. Hence, all edges from the red $C_{2 k-2}$ must be blue. Note the graph contains a blue $P_{2 k-1}$ which implies that the red $C_{2 k-2}$ along with the vertex $v$ must be a red $K_{2 k-1}$. Since the red graph is connected, the remaining vertices must form a blue clique and the graph belongs to the class of graphs $G_{1}$. A similar proof holds if we suppose that $K_{r-1}-\{v, w\}$ has a blue $C_{2 k-2}$ resulting with the graph $G_{3}$.

If $k=3$, then the cycle-cycle Ramsey number $R\left(C_{4}, C_{4}\right)=6 \neq 5=3 k-4$. In this case, let $c$ be a $\left(P_{6}, P_{6}\right)$-free coloring of $K_{7}$. Since $R\left(P_{6}, C_{4}\right)=7$ and the graph does not contain a red $P_{6}$, there must be a blue $C_{4}$. The same proof as above holds for $K_{7}-\{v, w\}$ with a blue $C_{4}$ and the resulting graph is $G_{3}$.

Case II: Either $K_{r-1}^{\rho}$ or $K_{r-1}^{\beta}$ is disconnected. Suppose that $K_{r-1}^{\rho}$ is disconnected. Let $X$ be a red component and $Y=K_{r-1}-X$. If both $X$ and $Y$ have at least $\frac{n}{2}=k$ vertices, then the graph contains a blue path $P_{n}$ by Observation 2.3. Therefore, one of $X$ or $Y$ has at most $\frac{n}{2}-1=k-1$ vertices. This implies that $|V(X)|=(2 k-1)+i$ and $|V(Y)|=(k-1)-i$ for some $i \in\left\{0,1,2, \ldots, \frac{n}{2}-2=k-2\right\}$. Note that if there is a blue $P_{2(i+1)}$ in $X$, then the graph contains a blue $P_{2 k}$ by Lemma 2.13. The Ramsey number $R\left(P_{2 k}, P_{2(i+1)}\right)=2 k+i$ and so $X$ either has a red $P_{2 k}$, a blue $P_{2(i+1)}$ or $X$ is the critical graph for $R\left(P_{2 k}, P_{2(i+1)}\right)$. Since the first two possibilities contradict the graph having a ( $P_{2 k}, P_{2 k}$ )-free coloring, $X$ must be such a critical graph. By Lemma 2.15, $X$ must be $G_{1}$, a red $K_{2 k-1}$ with all edges to the remaining $i$ vertices colored blue. Thus, the graph is $G_{1}$, a red $K_{2 k-1}$ with all blue edges to the remaining $i+|V(Y)|=i+(k-1)-i=k-1$ vertices. A similar proof holds if we suppose that $K_{r-1}^{\beta}$ is disconnected resulting with the graph $G_{3}$.

Proof of Proposition 2.6. Let $r=R\left(P_{n}, P_{m}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor-1$ and $c$ be a ( $P_{n}, P_{m}$ )-free coloring of $K_{r-1}$. For odd $n \geq m+1$, the critical graphs are $G_{1}$
and $G_{2}$ by Lemma 2.9. For odd $n=m$, the critical graphs are $G_{1}, G_{2}, G_{3}$ and $G_{4}$ by Lemma 2.11. For even $n \geq m+1$, the critical graphs are $G_{1}, G_{2}$ and $G_{5}$ if $m$ is odd by Lemma 2.14 or $G_{1}$ if $m$ is even by Lemma 2.15. For even $n=m$, the critical graphs are $G_{1}$ and $G_{3}$ by Lemma 2.16.

## 3. Determination of $r_{*}\left(P_{n}, P_{m}\right)$

Theorem 3.1. For all $n \geq m \geq 4, r_{*}\left(P_{n}, P_{m}\right)=\left\lceil\frac{m}{2}\right\rceil$.
Proof. A $\left(P_{n}, P_{m}\right)$-free coloring of $K_{n+\left\lfloor\frac{m}{2}\right\rfloor-2} \sqcup K_{1,\left\lceil\frac{m}{2}\right\rceil-1}$ is the graph $G_{1}$ as in Definition 2.4 and a vertex $v$ with all red edges to $A_{\left\lfloor\frac{m}{2}\right\rfloor-1}$. Hence, $r_{*}\left(P_{n}, P_{m}\right) \geq$ $\left\lceil\frac{m}{2}\right\rceil$.

Consider a 2 -coloring of $K_{n+\left\lfloor\frac{m}{2}\right\rfloor-1}$ and remove a vertex $v$. Then the underlying complete graph on $n+\left\lfloor\frac{m}{2}\right\rfloor-2$ vertices must have the structure of the critical graphs as in Proposition 2.6.

Case I: The underlying graph is $G_{1}$ in Definition 2.4. The red subgraph of $G_{1}$ is $K_{n-1}+A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\rho}$. If $v$ has a red edge to the red $K_{n-1}$, then the graph contains a red $P_{n}$. If $m$ is even, then the graph contains a blue $P_{m-1}$ and a blue edge from $v$ to the red $K_{n-1}$ creates a blue $P_{m}$. Therefore, $\left\lceil\frac{m}{2}\right\rceil$ edges adjacent to $v$ force $v$ to have an edge to the red clique yielding either a red $P_{n}$ or a blue $P_{m}$. If $m$ is odd, then the graph contains a blue $P_{m-2}$ and $v$ can have one blue edge to the red $K_{n-1}$ without creating a blue $P_{m}$. This implies that $\left\lfloor\frac{m}{2}\right\rfloor+1=\left\lceil\frac{m}{2}\right\rceil$ edges force the graph to have either a red $P_{n}$ or a blue $P_{m}$.

Case II: The underlying graph is $G_{2}$ with $m$ odd in Definition 2.4. The red subgraph of $G_{2}$ is $\left(K_{n-1}-K_{2}\right)+A_{\left\lfloor\frac{m}{2}\right\rfloor-1}^{\rho}$. If $v$ has a red edge to the red $K_{n-1}$, then the graph contains a red $P_{n}$. The graph also contains a blue $P_{m-1}$ which uses the blue edges between the red $\left(K_{n-1}-K_{2}\right)$ and $A_{\left\lfloor\frac{m}{2}\right\rfloor-1}$ and the single blue edge within the red clique. A blue edge from $v$ to the red $\left(K_{n-1}-K_{2}\right)$ creates a blue $P_{m}$. Therefore, $\left\lceil\frac{m}{2}\right\rceil$ edges adjacent to $v$ force $v$ to have an edge to the red clique yielding either a red $P_{n}$ or a blue $P_{m}$.

Case III: The underlying graph is $G_{3}$ with $n=m$ in Definition 2.4. A similar proof holds as in Case I by interchanging the colors red and blue.

Case IV: The underlying graph is $G_{4}$ with $n=m$ odd in Definition 2.4. A similar proof holds as in Case II by interchanging the colors red and blue.

Case V: The underlying graph is $G_{5}$ with $n=m+1$ and $m$ odd in Definition 2.4. The blue subgraph of $G_{5}$ is $K_{m-1}+A_{\frac{n}{2}-1}^{\beta}$. If $v$ has a blue edge to the blue $K_{m-1}$, then the graph contains a blue $P_{m}$. The graph contains a red $P_{n-1}$ and a
red edge from $v$ to the blue $K_{m-1}$ creates a red $P_{n}$. Therefore, $\frac{n}{2}=\frac{m+1}{2}=\left\lceil\frac{m}{2}\right\rceil$ edges adjacent to $v$ force $v$ to have an edge to the red clique yielding either a red $P_{n}$ or a blue $P_{m}$. Thus, it follows that $r_{*}\left(P_{n}, P_{m}\right)=\left\lceil\frac{m}{2}\right\rceil$.

Other classifications of critical graphs and star-critical Ramsey numbers including trees versus complete graphs, multiple copies of $K_{2}$ and $K_{3}$, and paths versus $C_{4}$ can be found in [2] and [3]. The critical graphs for $R\left(P_{n}, C_{m}\right)$ and cycles versus $K_{3}$ and $K_{4}$ have been found as well as their star-critical Ramsey numbers.

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