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ON THE SIGNED (TOTAL) k-INDEPENDENCE NUMBER IN GRAPHS

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Abstract

Let G be a graph. A function $f : V(G) \to \{-1, 1\}$ is a signed k-independence function if the sum of its function values over any closed neighborhood is at most $k-1$, where $k \geq 2$. The signed k-independence number of G is the maximum weight of a signed k-independence function of G. Similarly, the signed total k-independence number of G is the maximum weight of a signed total k-independence function of G. In this paper, we present new bounds on these two parameters which improve some existing bounds.

Keywords: domination in graphs, signed k-independence, limited packing, tuple domination.

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1. Introduction

Throughout this paper, let G be a finite connected graph with vertex set $V = V(G)$, edge set $E = E(G)$, minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. We use [12] for terminology and notation which are not defined here. For any vertex $v \in V$, $N(v) = \{u \in G \mid uv \in E(G)\}$ denotes the *open neighborhood* of v in G, and $N[v] = N(v) \cup \{v\}$ denotes its *closed neighborhood*. A set $S \subseteq V$ is a *dominating set* (*total dominating set*) in G if each vertex in $V \setminus S$ (in V) is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ (*total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (total dominating set) in G. A subset $B \subseteq V(G)$ is a *packing set* (an *open packing set*) in G if for every distinct vertices $u, v \in B$, $N[u] \cap N[v] = \emptyset$ ($N(u) \cap N(v) = \emptyset$). The *packing number* (*open packing number*) $\rho(G)$ ($\rho_o(G)$) is the maximum cardinality of a packing set (an open packing set) in G.

Harary and Haynes [4] introduced the concept of tuple domination as a generalization of domination in graphs. Let $1 \leq k \leq \delta(G)+1$. A set $D \subseteq V$ is a *k-tuple dominating set* in G if $|N[v] \cap D| \geq k$, for all $v \in V(G)$. The *k-tuple domination number*, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality of a k-tuple dominating set. In fact, the authors of [4] showed that every graph G with $\delta \geq k-1$ has a k-tuple dominating set and hence a k-tuple domination number. It is easy to see that $\gamma_{\times 1}(G) = \gamma(G)$. This concept has been studied by several authors including [1, 2, 6]. A generalization of total domination titled *k-tuple total domination* (or *k-total domination*) was introduced by Kulli [5] as a subset $S \subseteq V(G)$ such that $|N(v) \cap S| \geq k$, for all $v \in V(G)$, where $1 \leq k \leq \delta(G)$. The *k-tuple total domination number*, denoted by $\gamma_{\times k,t}(G)$, is the minimum cardinality of a k-tuple total dominating set. We note that $\gamma_{\times 1,t}(G) = \gamma_t(G)$. For more information on various dominations the reader can consult [1].

Gallant *et al.* [2] introduced the concept of limited packing in graphs and exhibited some real-world applications in network security, market saturation and codes. A set of vertices $B \subseteq V$ is called a *k-limited packing set* in G if $|N[v] \cap B| \leq k$ for all $v \in V$, where $k \geq 1$. The *k-limited packing number*, $L_k(G)$, is the maximum number of vertices in a k-limited packing set. Replacing $N[v]$ by $N(v)$ in the definition of k-limited packing, one can define the *k-total limited packing set*. The *k-total limited packing number*, $L_{k,t}(G)$, is the maximum number of vertices in a k-total limited packing in G (see [7]). When $k = 1$ we have $L_1(G) = \rho(G)$ and $L_{1,t}(G) = \rho_o(G)$.

Volkmann [8] introduced the concept of signed k-independence number in graphs. Let $k \geq 2$ be an integer. A function $f : V(G) \to \{-1, 1\}$ is a *signed k-independence function* (SkIF) if the sum of its function values over any closed neighborhood is at most $k-1$. That is, $f(N[v]) \leq k-1$ for all $v \in V(G)$. The weight of a SkIF f is $w(f) = f(V(G)) = \sum_{v \in V(G)} f(v)$. The *signed k-*

independence number (SkIN) of G, denoted $\alpha_s^k(G)$, is the maximum weight of a SkIF of G. If we replace $N[v]$ with $N(v)$ in the definition of SkIF, we will have *a signed total k-independence function* (STkIF). The *signed total k-independence number* (STkIN) of G, denoted $\alpha_{st}^k(G)$, is the maximum weight of a STkIF of G. This concept was introduced and studied in [9].

Throughout this paper, for a graph G of order n we assume that $n \geq k$ $(n \geq k+1)$, otherwise $\alpha_s^k(G) = n$ $(\alpha_{st}^k(G) = n)$. Volkmann [8] showed that for every graph G of order n, $\alpha_s^k(G) = n$ if and only if $\Delta(G) \leq k-2$. It is easy to see that $\alpha_{st}^k(G) = n$ if and only if $\Delta(G) \leq k-1$ (see [9]). Hence, throughout this paper, we also assume that $\Delta \geq k-1$ $(\Delta \geq k)$ when we deal with the SkDN (STkDN) of a graph G.

In this paper, we present some sharp upper and lower bounds for the parameters $\alpha_s^k(G)$ and $\alpha_{st}^k(G)$, which improve and generalize some well-known bounds presented in [3, 8, 9, 10, 11].

2. Upper Bounds

In this section, we present some sharp upper bounds on $\alpha_s^k(G)$ and $\alpha_{st}^k(G)$. First, we introduce some notation. Let G be a graph and $f : V(G) \longrightarrow \{-1, 1\}$ be a SkIF (STkIF) of G. We define

$V^+ = \{v \in V \mid f(v) = 1\}$, $n_+ = |V^+|$,
$V^- = \{v \in V \mid f(v) = -1\}$, $n_- = |V^-|$,
$V^o = \{v \in V \mid \deg(v) - k \equiv 1 \pmod 2\}$,
$V^e = \{v \in V \mid \deg(v) - k \equiv 0 \pmod 2\}$,
$G^+ = G[V^+]$ and $G^- = G[V^-]$.

Note that $G[A]$ is the subgraph of G induced by A, for every $A \subseteq V(G)$. For convenience, let $[V^+, V^-]$ be the set of edges having one end point in V^+ and the other in V^-. Finally, $\deg_{G^+}(v) = |N(v) \cap V^+|$ and $\deg_{G^-}(v) = |N(v) \cap V^-|$. We make use of the following observation to show that our bounds are sharp.

Observation 1. *Let $k \geq 2$ be an integer. Then*

(i) $\alpha_s^k(K_n) = \begin{cases} k-2 & n \equiv k \pmod 2, \\ k-1 & \text{otherwise}, \end{cases}$ (see [8]).

(ii) $\alpha_{st}^k(K_n) = \begin{cases} k-2 & n \equiv k \pmod 2, \\ k-3 & \text{otherwise}. \end{cases}$

(iii) $\alpha_{st}^k(K_{p,p}) = \begin{cases} 2k-4 & p \equiv k \pmod 2, \\ 2k-2 & \text{otherwise}, \end{cases}$ (see [9]).

Our next aim is to obtain upper bounds on $\alpha_s^k(G)$ and $\alpha_{st}^k(G)$ in terms of the order, k, minimum and maximum degrees of the graph.

Theorem 2. *Let $k \geq 2$ be an integer and let G be a graph of order n.*

(i) *If $\delta \geq k-1$, then* $\alpha_s^k(G) \leq \dfrac{\left(\left\lfloor \frac{\Delta+k}{2} \right\rfloor - \left\lceil \frac{\delta-k}{2} \right\rceil - 1\right) n}{\left\lfloor \frac{\Delta+k}{2} \right\rfloor + \left\lceil \frac{\delta-k}{2} \right\rceil + 1}$.

(ii) *If $\delta \geq k$, then* $\alpha_{st}^k(G) \leq \dfrac{\left(\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - \left\lceil \frac{\delta-k+1}{2} \right\rceil\right) n}{\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor + \left\lceil \frac{\delta-k+1}{2} \right\rceil}$.

In addition, these bounds are sharp.

Proof. We only prove (i), as (ii) can be proved similarly. Let f be a SkIF of G and $v \in V^+$. Since $f(N[v]) \leq k-1$, the vertex v has at least $\left\lceil \frac{\delta-k}{2} \right\rceil + 1$ neighbours in V^-. Therefore $|[V^+, V^-]| \geq \left(\left\lceil \frac{\delta-k}{2} \right\rceil + 1\right) |V^+|$. Now let $v \in V^-$. Since f is a SkIF, it follows that the vertex v has at most $\left\lfloor \frac{\Delta+k}{2} \right\rfloor$ neighbours in V^+. This implies that $|[V^+, V^-]| \leq \left\lfloor \frac{\Delta+k}{2} \right\rfloor |V^-|$. Hence,

$$\left(\left\lceil \frac{\delta-k}{2} \right\rceil + 1\right) |V^+| \leq \left\lfloor \frac{\Delta+k}{2} \right\rfloor |V^-|.$$

Using $|V^+| = \dfrac{n+w(f)}{2}$ and $|V^-| = \dfrac{n-w(f)}{2}$, we obtain the desired bound. The equality in part (i) holds for K_n and the equality in part (ii) holds for $K_{n,n}$ by Observation 1. ∎

Wang *et al.* [11] proved that if G is a graph of order n with no isolated vertices, then $\alpha_{st}^2(G) \leq \left(\dfrac{\Delta - 2\left\lfloor \frac{\delta}{2} \right\rfloor}{\Delta}\right) n$. Moreover, Volkmann in [9] generalized this result to $\alpha_{st}^k(G) \leq \dfrac{n}{\Delta}\left(\Delta - 2\left\lceil \dfrac{\delta+1-k}{2} \right\rceil\right)$, when $\delta \geq k-1$.

Since

$$\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor + \left\lceil \frac{\delta-k+1}{2} \right\rceil \leq \Delta,$$

we deduce from Theorem 2 part (ii) that

$$\alpha_{st}^k(G) \leq \frac{\left(\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - \left\lceil \frac{\delta-k+1}{2} \right\rceil\right) n}{\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor + \left\lceil \frac{\delta-k+1}{2} \right\rceil} \leq \frac{n}{\Delta}\left(\Delta - 2\left\lceil \frac{\delta+1-k}{2} \right\rceil\right).$$

Therefore the upper bound in Theorem 2 part (ii) is an improvement of its corresponding result in [9] (in [11] when $k = 2$).

Corollary 3. *Let $k \geq 2$ be an integer and let G be an r-regular graph of order n. Then*

(i) $\alpha_s^k(G) \leq \begin{cases} (k-1)n/(r+1) & k \equiv r\,(\text{mod } 2), \\ (k-2)n/(r+1) & \text{otherwise.} \end{cases}$

(ii) $\alpha_{st}^k(G) \leq \begin{cases} (k-2)n/r & k \equiv r\,(\text{mod } 2), \\ (k-1)n/r & \text{otherwise.} \end{cases}$

Note that the upper bound given in part (i) of Corollary 3 can also be found in [8].

A relationship between the signed k-independence number and the domination number of a graph G was also established in [8] as follows.

Theorem 4. *If $k \geq 2$ is an integer and G is a graph of order n with minimum degree $\delta \geq k-1$, then $\alpha_s^k(G) + 2\gamma(G) \leq n$.*

This result can be improved by considering the concept of tuple domination. Moreover, in a similar fashion, we establish a relationship between the signed total k-independence number and the total domination number of a graph as follows.

Theorem 5. *If $k \geq 2$ is an integer and G is a graph of order n with minimum degree δ, then*

(i) *if $\delta \geq k-1$, then $\alpha_s^k(G) + 2\gamma(G) \leq n - 2\left\lceil \frac{\delta-k}{2} \right\rceil$,*

(ii) *if $\delta \geq k$, then $\alpha_{st}^k(G) + 2\gamma_t(G) \leq n - 2\left\lceil \frac{\delta-k-1}{2} \right\rceil$,*

and these bounds are sharp.

Proof. We only prove (i), as (ii) can be proved similarly. Let f be a SkIF of G and $v \in V^+$. Since $f(N[v]) \leq k-1$, the vertex v has at least $\left\lceil \frac{\delta-k}{2} \right\rceil + 1$ neighbours in V^-. Hence, $|N[v] \cap V^-| = \deg_{G^-}(v) \geq \left\lceil \frac{\delta-k}{2} \right\rceil + 1$. Now let $v \in V^-$. Since $f(N[v]) \leq k-1$, we deduce that $\deg_{G^-}(v) \geq \left\lceil \frac{\delta-k}{2} \right\rceil$. Thus $|N[v] \cap V^-| \geq \left\lceil \frac{\delta-k}{2} \right\rceil + 1$. This shows that V^- is a $\left(\left\lceil \frac{\delta-k}{2} \right\rceil + 1\right)$-tuple dominating set in G and hence $\gamma_{\times(\lceil \frac{\delta-k}{2} \rceil+1)}(G) \leq |V^-|$. Since $|V^-| = \frac{n-w(f)}{2}$, it follows that

$$w(f) + 2\gamma_{\times(\lceil \frac{\delta-k}{2} \rceil+1)}(G) \leq n. \tag{1}$$

Now let D be a minimum $\left(\left\lceil\frac{\delta-k}{2}\right\rceil+1\right)$-tuple dominating set in G and let $u \in D$. It is easy to see that $|N[v] \cap D \setminus \{u\}| \geq \left\lceil\frac{\delta-k}{2}\right\rceil$, for all $v \in V(G)$. Therefore $D\setminus\{u\}$ is a $\left\lceil\frac{\delta-k}{2}\right\rceil$-tuple dominating set. Hence, $\gamma_{\times(\lceil\frac{\delta-k}{2}\rceil+1)}(G)-1 = |D \setminus \{u\}| \geq \gamma_{\times\lceil\frac{\delta-k}{2}\rceil}(G)$. Repeating these inequalities, we obtain

$$\begin{aligned}\gamma_{\times(\lceil\frac{\delta-k}{2}\rceil+1)}(G) &\geq \gamma_{\times\lceil\frac{\delta-k}{2}\rceil}(G)+1 \geq \cdots \\ &\geq \gamma_{\times 1}(G)+\left\lceil\frac{\delta-k}{2}\right\rceil = \gamma(G)+\left\lceil\frac{\delta-k}{2}\right\rceil.\end{aligned} \tag{2}$$

The result now follows by (1) and (2). The upper bounds are both sharp for the complete graph K_n. ■

Lemma 6. *The following statements hold.*
(i) *If f is a SkIF of G, then* $2|E(G[V^-])| \geq 2|E(G[V^+])| + 2|V^+| - kn + n_o$,
(ii) *If f is a STkIF of G, then* $2|E(G[V^-])| \geq 2|E(G[V^+])| - (k-1)n + n_e$,
where $n_o = |V^o|$ and $n_e = |V^e|$.

Proof. We only prove (ii). Let $v \in V^-$. Since $f(N(v)) \leq k-1$, we observe that $\deg_{G^-}(v) \geq \deg_{G^+}(v)-k+1$ and $\deg_{G^-}(v) \geq \deg_{G^+}(v)-k+2$ when $v \in V^- \cap V^e$. We infer that

$$\begin{aligned}2|E(G[V^-])| &= \sum_{v\in V^-}\deg_{G^-}(v) \\ &= \sum_{v\in V^-\cap V^o}\deg_{G^-}(v) + \sum_{v\in V^-\cap V^e}\deg_{G^-}(v) \\ &\geq \sum_{v\in V^-\cap V^o}(\deg_{G^+}(v)-k+1) \\ &+ \sum_{v\in V^-\cap V^e}(\deg_{G^+}(v)-k+2) \\ &= |[V^+,V^-]| - (k-1)|V^-| + |V^- \cap V^e|.\end{aligned}$$

This implies

$$|[V^+,V^-]| \leq 2|E(G[V^-])| + (k-1)|V^-| - |V^- \cap V^e|. \tag{3}$$

Now let $v \in V^+$. Since $f(N(v)) \leq k-1$, we have $\deg_{G^+}(v) \leq \deg_{G^-}(v) + k - 1$

and $\deg_{G^+}(v) \leq \deg_{G^-}(v) + k - 2$ when $v \in V^+ \cap V^e$. It follows that

$$\begin{aligned}
2|E(G[V^+])| &= \sum_{v\in V^+} \deg_{G^+}(v) \\
&= \sum_{v\in V^+\cap V^o} \deg_{G^+}(v) + \sum_{v\in V^+\cap V^e} \deg_{G^+}(v) \\
&\leq \sum_{v\in V^+\cap V^o} (\deg_{G^-}(v) + k - 1) \\
&+ \sum_{v\in V^+\cap V^e} (\deg_{G^-}(v) + k - 2) \\
&= |[V^+, V^-]| + (k-1)|V^+| - |V^+ \cap V^e|.
\end{aligned}$$

The implies

$$|[V^+, V^-]| \geq 2|E(G[V^+])| - (k-1)|V^+| + |V^+ \cap V^e|. \tag{4}$$

Combining (3) and (4), we obtain (ii). ∎

Theorem 7. *Let $k \geq 2$ be an integer, and let G be a graph of order n and minimum degree δ. Then*

(i) $\alpha_s^k(G) \leq n - \left\lceil \frac{1}{2}\left(-\delta - k + \sqrt{(\delta+k)^2 + 8n(\delta - k + 2) + 8n_o}\right)\right\rceil$,

(ii) $\alpha_{st}^k(G) \leq n - \left\lceil \frac{1}{2}\left(3 - \delta - k + \sqrt{(\delta+k-3)^2 + 8n(\delta - k + 1) + 8n_e}\right)\right\rceil$.

Proof. We only proof (ii). Let $v \in V^-$. Then $2\deg_{G^-}(v) \geq \deg(v) - k + 1$. Since $\deg_{G^-}(v) \leq |V^-| - 1$, it follows that

$$\sum_{v\in V^-} (\deg(v) - k + 1) \leq 2 \sum_{v\in V^-} \deg_{G^-}(v) \leq 2|V^-|(|V^-| - 1). \tag{5}$$

Furthermore, we have

$$\begin{aligned}
2|E(G[V^+])| - 2|E(G[V^-])| &= \sum_{v\in V^+} \deg_{G^+}(v) - \sum_{v\in V^-} \deg_{G^-}(v) \\
&= \sum_{v\in V^+} (\deg(v) - \deg_{G^-}(v)) \\
&- \sum_{v\in V^-} (\deg(v) - \deg_{G^+}(v)) \\
&= \sum_{v\in V^+} \deg(v) - |[V^+, V^-]|
\end{aligned}$$

$$\begin{aligned} &- \sum_{v\in V^-} \deg(v) + |[V^+, V^-]| \\ &= \sum_{v\in V^+} \deg(v) - \sum_{v\in V^-} \deg(v). \end{aligned}$$

Applying part (ii) of Lemma 6, we deduce that

$$\sum_{v\in V^+} \deg(v) - (k-1)n + n_e \leq \sum_{v\in V^-} \deg(v). \tag{6}$$

Combining (5) and (6), we obtain

$$\begin{aligned} 2|V^-|^2 - 2|V^-| &\geq \sum_{v\in V^+} \deg(v) + (1-k)n + n_e + (1-k)|V^-| \\ &\geq \delta|V^+| + (1-k)n + n_e + (1-k)|V^-|. \end{aligned}$$

Using $|V^+| = n - |V^-|$, we infer that

$$2|V^-|^2 + (\delta + k - 3)|V^-| - (\delta - k + 1)n - n_e \geq 0.$$

Solving the above inequality for $|V^-|$ we obtain

$$|V^-| \geq \frac{-(\delta+k-3) + \sqrt{(\delta+k-3)^2 + 8n(\delta-k+1) + 8n_e}}{4}.$$

Using $|V^-| = (n - \alpha^k_{st}(G))/2$, we arrive at the desired bound. ■

The special case $k = 2$ of parts (i) and (ii) of Theorem 7 can be found in [3] and [10], respectively.

Theorem 8. *Let $k \geq 2$ be an integer, and let G be a graph of order n, size m, maximum degree Δ and minimum degree δ. Then*

$$\alpha^k_{st}(G) \leq \left\lfloor \frac{\left(3\Delta + 2\left\lfloor \frac{\Delta+k-1}{2}\right\rfloor + 3k - 3\right)n - 8m - 2n_e}{3\Delta + 2\lfloor \frac{\Delta+k-1}{2}\rfloor - k + 1} \right\rfloor, \tag{7}$$

$$\alpha^k_{st}(G) \leq \left\lfloor \frac{\left(2\left\lfloor \frac{\Delta+k-1}{2}\right\rfloor - 3\delta + 3k - 3\right)n + 4m - 2n_e}{3\delta + 2\lfloor \frac{\Delta+k-1}{2}\rfloor - k + 1} \right\rfloor. \tag{8}$$

Proof. (i) It follows from (4) and Lemma 6 (ii) that

$$\begin{aligned} 2|E(G[V^-])| + |[V^+, V^-]| &\geq 4|E(G[V^+])| - (k-1)n_+ \\ &- (k-1)n + n_e \\ &= 4m - 4|E(G[V^-])| - 4|[V^+, V^-]| \\ &- (k-1)n_+ - (k-1)n + n_e \end{aligned}$$

and thus

$$6|E(G[V^-])| + 5|[V^+, V^-]| \geq 4m - (k-1)n_+ - (k-1)n + n_e.$$

Using this inequality and the bound

$$2|E(G[V^-])| = \sum_{v \in V^-} (\deg(v) - |N(v) \cap V^+|) \leq \Delta n_- - |[V^+, V^-]|,$$

we arrive at

$$3\Delta n_- + 2|[V^+, V^-]| \geq 4m - (k-1)n_+ - (k-1)n + n_e. \tag{9}$$

If $v \in V^-$, then $f(N(v)) \leq k-1$ implies that $2|N(v) \cap V^+| \leq \deg(v) + k - 1 \leq \Delta + k - 1$ and therefore $|N(v) \cap V^+| \leq \lfloor \frac{\Delta+k-1}{2} \rfloor$. This yields

$$|[V^+, V^-]| \leq \left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor n_- = \left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor (n - n_+). \tag{10}$$

We deduce from (9) and (10) that

$$\left(3\Delta + 2\left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor\right) n_- \geq 4m - (k-1)(n - n_-) - (k-1)n + n_e$$

and so

$$n_- \geq \frac{4m - 2(k-1)n + n_e}{3\Delta + 2\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - k + 1}.$$

This yields to

$$\begin{aligned} \alpha_{st}^k(G) &= n - 2n_- \\ &\leq \frac{\left(3\Delta + 2\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - k + 1 + 4(k-1)\right) n - 2n_e - 8m}{3\Delta + 2\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - k + 1} \\ &= \frac{\left(3\Delta + 2\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor + 3k - 3\right) n - 2n_e - 8m}{3\Delta + 2\left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - k + 1}, \end{aligned}$$

and (7) is proved.

(ii) It follows from (4) and Lemma 6 (ii) that

$$\begin{aligned} 2m - 2|E(G[V^+])| - |[V^+, V^-]| &= 2|E(G[V^-])| + |[V^+, V^-]| \\ &\geq 4|E(G[V^+])| - (k-1)n_+ \\ &- (k-1)n + n_e \end{aligned}$$

and thus

$$2m \geq 6|E(G[V^+])| + |[V^+, V^-]| - (k-1)n_+ - (k-1)n + n_e.$$

Using this inequality and the bound

$$2|E(G[V^+])| = \sum_{v \in V^+} (\deg(v) - |N(v) \cap V^-|) \geq \delta n_+ - |[V^+, V^-]|,$$

we arrive at

$$2m \geq 3\delta n_+ - 2|[V^+, V^-]| - (k-1)n_+ - (k-1)n + n_e.$$

Applying (10), we conclude that

$$2m \geq \left(3\delta + 2\left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor - k + 1\right) n_+ - 2\left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor n - (k-1)n + n_e.$$

Using this inequality and $n_+ = \frac{n + \alpha_{st}^k(G)}{2}$, we obtain the bound (8), and the proof is complete. ■

If $K_{p,p}$ is the complete bipartite graph, then Observation 1 (iii) demonstrates that the inequalities (7) and (8) are sharp, when $k \leq p + 1$.

Using Lemma 6 (i) instead of Lemma 6 (ii), we obtain analogously to the proof of Theorem 8 the following two upper bounds on the signed k-independence number.

Theorem 9. *Let $k \geq 2$ be an integer, and let G be a graph of order n, size m, maximum degree Δ and minimum degree δ. Then*

$$\alpha_s^k(G) \leq \left\lfloor \frac{\left(3\Delta + 2\left\lfloor \frac{\Delta+k}{2} \right\rfloor + 3k - 4\right) n - 8m - 2n_o}{3\Delta + 2\lfloor \frac{\Delta+k}{2} \rfloor - k + 4} \right\rfloor, \tag{11}$$

$$\alpha_s^k(G) \leq \left\lfloor \frac{\left(2\left\lfloor \frac{\Delta+k}{2} \right\rfloor - 3\delta + 3k - 4\right) n + 4m - 2n_o}{3\delta + 2\lfloor \frac{\Delta+k}{2} \rfloor - k + 4} \right\rfloor. \tag{12}$$

The complete graph K_n, when $n + 1 \geq k$, shows that the inequalities (11) and (12) are sharp.

3. Lower Bounds

As an application of the concepts of (total) limited packing we establish some lower bounds on the parameters $\alpha_s^k(G)$ and $\alpha_{st}^k(G)$ of a graph G.

Theorem 10. *Let G be graph of order n and $2 \leq k \leq \Delta(G)$. Then*

(i) $\alpha_s^k(G) \geq -n + 2\left\lfloor \dfrac{\delta + 2\rho(G) + k - 2}{2} \right\rfloor$,

(ii) $\alpha_{st}^k(G) \geq -n + 2\left\lfloor \dfrac{\delta + 2\rho_0(G) + k - 3}{2} \right\rfloor$,

and these bounds are sharp.

Proof. We only prove part (i), and part (ii) can be proved in a similar fashion. Let B be a $\left\lfloor \dfrac{\delta+k}{2} \right\rfloor$-limited packing set in G. We define $f : V(G) \to \{-1, 1\}$ by

$$f(v) = \begin{cases} +1 & v \in B, \\ -1 & v \in V \setminus B. \end{cases}$$

For all vertices v in $V(G)$,

$$\begin{aligned} f(N[v]) &= 2|N[v] \cap B| - |N[v]| \\ &\leq 2\left\lfloor \frac{\delta+k}{2} \right\rfloor - \delta - 1 \leq k - 1. \end{aligned}$$

Hence, f is a signed k-independence function of G and therefore

$$\alpha_s^k(G) \geq f(V(G)) = 2|B| - n = 2L_{\lfloor \frac{\delta+k}{2} \rfloor}(G) - n. \tag{13}$$

Assume that B' is a maximum $\left\lfloor \dfrac{\delta+k}{2} \right\rfloor$-limited packing set in G. Suppose to the contrary that $V = B'$. If v is a vertex in $V(G)$ with maximum degree, then $\left\lfloor \dfrac{\delta+k}{2} \right\rfloor > |N[v] \cap B'| = \Delta + 1$, a contradiction. Now let $u \in V \setminus B'$. It is easy to check that $B' \cup \{u\}$ is a $\left(\left\lfloor \dfrac{\delta+k}{2} \right\rfloor + 1\right)$-limited packing in G. Thus

$$L_{\lfloor \frac{\delta+k}{2} \rfloor}(G) + 1 = |B' \cup \{u\}| \leq L_{\lfloor \frac{\delta+k}{2} \rfloor + 1}(G).$$

Indeed, we have

$$\begin{aligned} L_{\lfloor \frac{\delta+k}{2} \rfloor}(G) &\geq L_{\lfloor \frac{\delta+k}{2} \rfloor - 1}(G) + 1 \geq \cdots \\ &\geq L_1(G) + \left\lfloor \frac{\delta+k}{2} \right\rfloor - 1 = \rho(G) + \left\lfloor \frac{\delta+k}{2} \right\rfloor - 1. \end{aligned}$$

By (13), we deduce that $\alpha_s^k(G) \geq -n + 2\rho(G) + 2\left\lfloor \dfrac{\delta+k-2}{2} \right\rfloor$, as desired. The equalities hold for the graph K_n. ■

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