

ON THE SIGNED (TOTAL) k -INDEPENDENCE NUMBER IN GRAPHS

ABDOLLAH KHODKAR

Department of Mathematics
University of West Georgia
Carrollton, GA 30118, USA

e-mail: akhodkar@westga.edu

BABAK SAMADI

Department of Mathematics
Arak University
Arak, Iran

e-mail: b-samadi@araku.ac.ir

AND

LUTZ VOLKMANN

Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany

e-mail: volkm@math2.rwth-aachen.de

Abstract

Let G be a graph. A function $f : V(G) \rightarrow \{-1, 1\}$ is a signed k -independence function if the sum of its function values over any closed neighborhood is at most $k - 1$, where $k \geq 2$. The signed k -independence number of G is the maximum weight of a signed k -independence function of G . Similarly, the signed total k -independence number of G is the maximum weight of a signed total k -independence function of G . In this paper, we present new bounds on these two parameters which improve some existing bounds.

Keywords: domination in graphs, signed k -independence, limited packing, tuple domination.

2010 Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Throughout this paper, let G be a finite connected graph with vertex set $V = V(G)$, edge set $E = E(G)$, minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. We use [12] for terminology and notation which are not defined here. For any vertex $v \in V$, $N(v) = \{u \in G \mid uv \in E(G)\}$ denotes the *open neighborhood* of v in G , and $N[v] = N(v) \cup \{v\}$ denotes its *closed neighborhood*. A set $S \subseteq V$ is a *dominating set* (*total dominating set*) in G if each vertex in $V \setminus S$ (in V) is adjacent to at least one vertex in S . The *domination number* $\gamma(G)$ (*total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (total dominating set) in G . A subset $B \subseteq V(G)$ is a *packing set* (an *open packing set*) in G if for every distinct vertices $u, v \in B$, $N[u] \cap N[v] = \emptyset$ ($N(u) \cap N(v) = \emptyset$). The *packing number* (*open packing number*) $\rho(G)$ ($\rho_o(G)$) is the maximum cardinality of a packing set (an open packing set) in G .

Harary and Haynes [4] introduced the concept of tuple domination as a generalization of domination in graphs. Let $1 \leq k \leq \delta(G) + 1$. A set $D \subseteq V$ is a *k-tuple dominating set* in G if $|N[v] \cap D| \geq k$, for all $v \in V(G)$. The *k-tuple domination number*, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality of a *k-tuple dominating set*. In fact, the authors of [4] showed that every graph G with $\delta \geq k - 1$ has a *k-tuple dominating set* and hence a *k-tuple domination number*. It is easy to see that $\gamma_{\times 1}(G) = \gamma(G)$. This concept has been studied by several authors including [1, 2, 6]. A generalization of total domination titled *k-tuple total domination* (or *k-total domination*) was introduced by Kulli [5] as a subset $S \subseteq V(G)$ such that $|N(v) \cap S| \geq k$, for all $v \in V(G)$, where $1 \leq k \leq \delta(G)$. The *k-tuple total domination number*, denoted by $\gamma_{\times k, t}(G)$, is the minimum cardinality of a *k-tuple total dominating set*. We note that $\gamma_{\times 1, t}(G) = \gamma_t(G)$. For more information on various dominations the reader can consult [1].

Gallant *et al.* [2] introduced the concept of limited packing in graphs and exhibited some real-world applications in network security, market saturation and codes. A set of vertices $B \subseteq V$ is called a *k-limited packing set* in G if $|N[v] \cap B| \leq k$ for all $v \in V$, where $k \geq 1$. The *k-limited packing number*, $L_k(G)$, is the maximum number of vertices in a *k-limited packing set*. Replacing $N[v]$ by $N(v)$ in the definition of *k-limited packing*, one can define the *k-total limited packing set*. The *k-total limited packing number*, $L_{k, t}(G)$, is the maximum number of vertices in a *k-total limited packing* in G (see [7]). When $k = 1$ we have $L_1(G) = \rho(G)$ and $L_{1, t}(G) = \rho_o(G)$.

Volkman [8] introduced the concept of signed *k*-independence number in graphs. Let $k \geq 2$ be an integer. A function $f : V(G) \rightarrow \{-1, 1\}$ is a *signed k-independence function* (SkIF) if the sum of its function values over any closed neighborhood is at most $k - 1$. That is, $f(N[v]) \leq k - 1$ for all $v \in V(G)$. The weight of a SkIF f is $w(f) = f(V(G)) = \sum_{v \in V(G)} f(v)$. The *signed k-*

independence number ($SkIN$) of G , denoted $\alpha_s^k(G)$, is the maximum weight of a $SkIF$ of G . If we replace $N[v]$ with $N(v)$ in the definition of $SkIF$, we will have a *signed total k -independence function* ($STkIF$). The *signed total k -independence number* ($STkIN$) of G , denoted $\alpha_{st}^k(G)$, is the maximum weight of a $STkIF$ of G . This concept was introduced and studied in [9].

Throughout this paper, for a graph G of order n we assume that $n \geq k$ ($n \geq k + 1$), otherwise $\alpha_s^k(G) = n$ ($\alpha_{st}^k(G) = n$). Volkmann [8] showed that for every graph G of order n , $\alpha_s^k(G) = n$ if and only if $\Delta(G) \leq k - 2$. It is easy to see that $\alpha_{st}^k(G) = n$ if and only if $\Delta(G) \leq k - 1$ (see [9]). Hence, throughout this paper, we also assume that $\Delta \geq k - 1$ ($\Delta \geq k$) when we deal with the $SkDN$ ($STkDN$) of a graph G .

In this paper, we present some sharp upper and lower bounds for the parameters $\alpha_s^k(G)$ and $\alpha_{st}^k(G)$, which improve and generalize some well-known bounds presented in [3, 8, 9, 10, 11].

2. UPPER BOUNDS

In this section, we present some sharp upper bounds on $\alpha_s^k(G)$ and $\alpha_{st}^k(G)$. First, we introduce some notation. Let G be a graph and $f : V(G) \rightarrow \{-1, 1\}$ be a $SkIF$ ($STkIF$) of G . We define

$$\begin{aligned} V^+ &= \{v \in V \mid f(v) = 1\}, \quad n_+ = |V^+|, \\ V^- &= \{v \in V \mid f(v) = -1\}, \quad n_- = |V^-|, \\ V^o &= \{v \in V \mid \deg(v) - k \equiv 1 \pmod{2}\}, \\ V^e &= \{v \in V \mid \deg(v) - k \equiv 0 \pmod{2}\}, \\ G^+ &= G[V^+] \text{ and } G^- = G[V^-]. \end{aligned}$$

Note that $G[A]$ is the subgraph of G induced by A , for every $A \subseteq V(G)$. For convenience, let $[V^+, V^-]$ be the set of edges having one end point in V^+ and the other in V^- . Finally, $\deg_{G^+}(v) = |N(v) \cap V^+|$ and $\deg_{G^-}(v) = |N(v) \cap V^-|$. We make use of the following observation to show that our bounds are sharp.

Observation 1. *Let $k \geq 2$ be an integer. Then*

- (i) $\alpha_s^k(K_n) = \begin{cases} k - 2 & n \equiv k \pmod{2}, \\ k - 1 & \text{otherwise,} \end{cases}$ (see [8]).
- (ii) $\alpha_{st}^k(K_n) = \begin{cases} k - 2 & n \equiv k \pmod{2}, \\ k - 3 & \text{otherwise.} \end{cases}$
- (iii) $\alpha_{st}^k(K_{p,p}) = \begin{cases} 2k - 4 & p \equiv k \pmod{2}, \\ 2k - 2 & \text{otherwise,} \end{cases}$ (see [9]).

Our next aim is to obtain upper bounds on $\alpha_s^k(G)$ and $\alpha_{st}^k(G)$ in terms of the order, k , minimum and maximum degrees of the graph.

Theorem 2. Let $k \geq 2$ be an integer and let G be a graph of order n .

- (i) If $\delta \geq k - 1$, then $\alpha_s^k(G) \leq \frac{\left(\left\lfloor \frac{\Delta + k}{2} \right\rfloor - \left\lceil \frac{\delta - k}{2} \right\rceil - 1\right)n}{\left\lfloor \frac{\Delta + k}{2} \right\rfloor + \left\lceil \frac{\delta - k}{2} \right\rceil + 1}$.
- (ii) If $\delta \geq k$, then $\alpha_{st}^k(G) \leq \frac{\left(\left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor - \left\lceil \frac{\delta - k + 1}{2} \right\rceil\right)n}{\left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor + \left\lceil \frac{\delta - k + 1}{2} \right\rceil}$.

In addition, these bounds are sharp.

Proof. We only prove (i), as (ii) can be proved similarly. Let f be a SkIF of G and $v \in V^+$. Since $f(N[v]) \leq k - 1$, the vertex v has at least $\left\lceil \frac{\delta - k}{2} \right\rceil + 1$ neighbours in V^- . Therefore $|[V^+, V^-]| \geq \left(\left\lceil \frac{\delta - k}{2} \right\rceil + 1\right)|V^+|$. Now let $v \in V^-$. Since f is a SkIF, it follows that the vertex v has at most $\left\lfloor \frac{\Delta + k}{2} \right\rfloor$ neighbours in V^+ . This implies that $|[V^+, V^-]| \leq \left\lfloor \frac{\Delta + k}{2} \right\rfloor |V^-|$. Hence,

$$\left(\left\lceil \frac{\delta - k}{2} \right\rceil + 1\right)|V^+| \leq \left\lfloor \frac{\Delta + k}{2} \right\rfloor |V^-|.$$

Using $|V^+| = \frac{n + w(f)}{2}$ and $|V^-| = \frac{n - w(f)}{2}$, we obtain the desired bound. The equality in part (i) holds for K_n and the equality in part (ii) holds for $K_{n,n}$ by Observation 1. ■

Wang *et al.* [11] proved that if G is a graph of order n with no isolated vertices, then $\alpha_{st}^2(G) \leq \left(\frac{\Delta - 2 \lfloor \frac{\delta}{2} \rfloor}{\Delta}\right)n$. Moreover, Volkmann in [9] generalized this result to $\alpha_{st}^k(G) \leq \frac{n}{\Delta} \left(\Delta - 2 \left\lceil \frac{\delta + 1 - k}{2} \right\rceil\right)$, when $\delta \geq k - 1$.

Since

$$\left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor + \left\lceil \frac{\delta - k + 1}{2} \right\rceil \leq \Delta,$$

we deduce from Theorem 2 part (ii) that

$$\alpha_{st}^k(G) \leq \frac{\left(\left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor - \left\lceil \frac{\delta - k + 1}{2} \right\rceil\right)n}{\left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor + \left\lceil \frac{\delta - k + 1}{2} \right\rceil} \leq \frac{n}{\Delta} \left(\Delta - 2 \left\lceil \frac{\delta + 1 - k}{2} \right\rceil\right).$$

Therefore the upper bound in Theorem 2 part (ii) is an improvement of its corresponding result in [9] (in [11] when $k = 2$).

Corollary 3. *Let $k \geq 2$ be an integer and let G be an r -regular graph of order n . Then*

- (i) $\alpha_s^k(G) \leq \begin{cases} (k-1)n/(r+1) & k \equiv r \pmod{2}, \\ (k-2)n/(r+1) & \text{otherwise.} \end{cases}$
- (ii) $\alpha_{st}^k(G) \leq \begin{cases} (k-2)n/r & k \equiv r \pmod{2}, \\ (k-1)n/r & \text{otherwise.} \end{cases}$

Note that the upper bound given in part (i) of Corollary 3 can also be found in [8].

A relationship between the signed k -independence number and the domination number of a graph G was also established in [8] as follows.

Theorem 4. *If $k \geq 2$ is an integer and G is a graph of order n with minimum degree $\delta \geq k-1$, then $\alpha_s^k(G) + 2\gamma(G) \leq n$.*

This result can be improved by considering the concept of tuple domination. Moreover, in a similar fashion, we establish a relationship between the signed total k -independence number and the total domination number of a graph as follows.

Theorem 5. *If $k \geq 2$ is an integer and G is a graph of order n with minimum degree δ , then*

- (i) *if $\delta \geq k-1$, then $\alpha_s^k(G) + 2\gamma(G) \leq n - 2 \left\lceil \frac{\delta - k}{2} \right\rceil$,*
- (ii) *if $\delta \geq k$, then $\alpha_{st}^k(G) + 2\gamma_t(G) \leq n - 2 \left\lceil \frac{\delta - k - 1}{2} \right\rceil$,*

and these bounds are sharp.

Proof. We only prove (i), as (ii) can be proved similarly. Let f be a SkIF of G and $v \in V^+$. Since $f(N[v]) \leq k-1$, the vertex v has at least $\left\lceil \frac{\delta - k}{2} \right\rceil + 1$ neighbours in V^- . Hence, $|N[v] \cap V^-| = \deg_{G^-}(v) \geq \left\lceil \frac{\delta - k}{2} \right\rceil + 1$. Now let $v \in V^-$. Since $f(N[v]) \leq k-1$, we deduce that $\deg_{G^-}(v) \geq \left\lceil \frac{\delta - k}{2} \right\rceil$. Thus $|N[v] \cap V^-| \geq \left\lceil \frac{\delta - k}{2} \right\rceil + 1$. This shows that V^- is a $\left(\left\lceil \frac{\delta - k}{2} \right\rceil + 1\right)$ -tuple dominating set in G and hence $\gamma_{\times(\lceil \frac{\delta-k}{2} \rceil + 1)}(G) \leq |V^-|$. Since $|V^-| = \frac{n - w(f)}{2}$, it follows that

$$(1) \quad w(f) + 2\gamma_{\times(\lceil \frac{\delta-k}{2} \rceil + 1)}(G) \leq n.$$

Now let D be a minimum $\left(\left\lceil \frac{\delta - k}{2} \right\rceil + 1\right)$ -tuple dominating set in G and let $u \in D$. It is easy to see that $|N[v] \cap D \setminus \{u\}| \geq \left\lceil \frac{\delta - k}{2} \right\rceil$, for all $v \in V(G)$. Therefore $D \setminus \{u\}$ is a $\left\lceil \frac{\delta - k}{2} \right\rceil$ -tuple dominating set. Hence, $\gamma_{\times(\lceil \frac{\delta - k}{2} \rceil + 1)}(G) - 1 = |D \setminus \{u\}| \geq \gamma_{\times \lceil \frac{\delta - k}{2} \rceil}(G)$. Repeating these inequalities, we obtain

$$\begin{aligned} \gamma_{\times(\lceil \frac{\delta - k}{2} \rceil + 1)}(G) &\geq \gamma_{\times \lceil \frac{\delta - k}{2} \rceil}(G) + 1 \geq \dots \\ (2) \quad &\geq \gamma_{\times 1}(G) + \left\lceil \frac{\delta - k}{2} \right\rceil = \gamma(G) + \left\lceil \frac{\delta - k}{2} \right\rceil. \end{aligned}$$

The result now follows by (1) and (2). The upper bounds are both sharp for the complete graph K_n . $\blacksquare$

Lemma 6. *The following statements hold.*

- (i) *If f is a SkIF of G , then $2|E(G[V^-])| \geq 2|E(G[V^+])| + 2|V^+| - kn + n_o$,*
- (ii) *If f is a STkIF of G , then $2|E(G[V^-])| \geq 2|E(G[V^+])| - (k - 1)n + n_e$,*
where $n_o = |V^o|$ and $n_e = |V^e|$.

Proof. We only prove (ii). Let $v \in V^-$. Since $f(N(v)) \leq k - 1$, we observe that $\deg_{G^-}(v) \geq \deg_{G^+}(v) - k + 1$ and $\deg_{G^-}(v) \geq \deg_{G^+}(v) - k + 2$ when $v \in V^- \cap V^e$. We infer that

$$\begin{aligned} 2|E(G[V^-])| &= \sum_{v \in V^-} \deg_{G^-}(v) \\ &= \sum_{v \in V^- \cap V^o} \deg_{G^-}(v) + \sum_{v \in V^- \cap V^e} \deg_{G^-}(v) \\ &\geq \sum_{v \in V^- \cap V^o} (\deg_{G^+}(v) - k + 1) \\ &\quad + \sum_{v \in V^- \cap V^e} (\deg_{G^+}(v) - k + 2) \\ &= |[V^+, V^-]| - (k - 1)|V^-| + |V^- \cap V^e|. \end{aligned}$$

This implies

$$(3) \quad |[V^+, V^-]| \leq 2|E(G[V^-])| + (k - 1)|V^-| - |V^- \cap V^e|.$$

Now let $v \in V^+$. Since $f(N(v)) \leq k - 1$, we have $\deg_{G^+}(v) \leq \deg_{G^-}(v) + k - 1$

and $\deg_{G^+}(v) \leq \deg_{G^-}(v) + k - 2$ when $v \in V^+ \cap V^e$. It follows that

$$\begin{aligned}
 2|E(G[V^+])| &= \sum_{v \in V^+} \deg_{G^+}(v) \\
 &= \sum_{v \in V^+ \cap V^o} \deg_{G^+}(v) + \sum_{v \in V^+ \cap V^e} \deg_{G^+}(v) \\
 &\leq \sum_{v \in V^+ \cap V^o} (\deg_{G^-}(v) + k - 1) \\
 &\quad + \sum_{v \in V^+ \cap V^e} (\deg_{G^-}(v) + k - 2) \\
 &= |[V^+, V^-]| + (k - 1)|V^+| - |V^+ \cap V^e|.
 \end{aligned}$$

This implies

$$(4) \quad |[V^+, V^-]| \geq 2|E(G[V^+])| - (k - 1)|V^+| + |V^+ \cap V^e|.$$

Combining (3) and (4), we obtain (ii). ■

Theorem 7. Let $k \geq 2$ be an integer, and let G be a graph of order n and minimum degree δ . Then

- (i) $\alpha_s^k(G) \leq n - \left\lceil \frac{1}{2} \left(-\delta - k + \sqrt{(\delta + k)^2 + 8n(\delta - k + 2) + 8n_o} \right) \right\rceil$,
- (ii) $\alpha_{st}^k(G) \leq n - \left\lceil \frac{1}{2} \left(3 - \delta - k + \sqrt{(\delta + k - 3)^2 + 8n(\delta - k + 1) + 8n_e} \right) \right\rceil$.

Proof. We only proof (ii). Let $v \in V^-$. Then $2\deg_{G^-}(v) \geq \deg(v) - k + 1$. Since $\deg_{G^-}(v) \leq |V^-| - 1$, it follows that

$$(5) \quad \sum_{v \in V^-} (\deg(v) - k + 1) \leq 2 \sum_{v \in V^-} \deg_{G^-}(v) \leq 2|V^-|(|V^-| - 1).$$

Furthermore, we have

$$\begin{aligned}
 2|E(G[V^+])| - 2|E(G[V^-])| &= \sum_{v \in V^+} \deg_{G^+}(v) - \sum_{v \in V^-} \deg_{G^-}(v) \\
 &= \sum_{v \in V^+} (\deg(v) - \deg_{G^-}(v)) \\
 &\quad - \sum_{v \in V^-} (\deg(v) - \deg_{G^+}(v)) \\
 &= \sum_{v \in V^+} \deg(v) - |[V^+, V^-]|
 \end{aligned}$$

$$\begin{aligned}
& - \sum_{v \in V^-} \deg(v) + |[V^+, V^-]| \\
& = \sum_{v \in V^+} \deg(v) - \sum_{v \in V^-} \deg(v).
\end{aligned}$$

Applying part (ii) of Lemma 6, we deduce that

$$(6) \quad \sum_{v \in V^+} \deg(v) - (k-1)n + n_e \leq \sum_{v \in V^-} \deg(v).$$

Combining (5) and (6), we obtain

$$\begin{aligned}
2|V^+|^2 - 2|V^-| & \geq \sum_{v \in V^+} \deg(v) + (1-k)n + n_e + (1-k)|V^-| \\
& \geq \delta|V^+| + (1-k)n + n_e + (1-k)|V^-|.
\end{aligned}$$

Using $|V^+| = n - |V^-|$, we infer that

$$2|V^-|^2 + (\delta + k - 3)|V^-| - (\delta - k + 1)n - n_e \geq 0.$$

Solving the above inequality for $|V^-|$ we obtain

$$|V^-| \geq \frac{-(\delta + k - 3) + \sqrt{(\delta + k - 3)^2 + 8n(\delta - k + 1) + 8n_e}}{4}.$$

Using $|V^-| = (n - \alpha_{st}^k(G))/2$, we arrive at the desired bound. $\blacksquare$

The special case $k = 2$ of parts (i) and (ii) of Theorem 7 can be found in [3] and [10], respectively.

Theorem 8. *Let $k \geq 2$ be an integer, and let G be a graph of order n , size m , maximum degree Δ and minimum degree δ . Then*

$$(7) \quad \alpha_{st}^k(G) \leq \left\lfloor \frac{(3\Delta + 2 \lfloor \frac{\Delta+k-1}{2} \rfloor + 3k - 3)n - 8m - 2n_e}{3\Delta + 2 \lfloor \frac{\Delta+k-1}{2} \rfloor - k + 1} \right\rfloor,$$

$$(8) \quad \alpha_{st}^k(G) \leq \left\lfloor \frac{(2 \lfloor \frac{\Delta+k-1}{2} \rfloor - 3\delta + 3k - 3)n + 4m - 2n_e}{3\delta + 2 \lfloor \frac{\Delta+k-1}{2} \rfloor - k + 1} \right\rfloor.$$

Proof. (i) It follows from (4) and Lemma 6 (ii) that

$$\begin{aligned}
2|E(G[V^-])| + |[V^+, V^-]| & \geq 4|E(G[V^+])| - (k-1)n_+ \\
& - (k-1)n + n_e \\
& = 4m - 4|E(G[V^-])| - 4|[V^+, V^-]| \\
& - (k-1)n_+ - (k-1)n + n_e
\end{aligned}$$

and thus

$$6|E(G[V^-])| + 5|[V^+, V^-]| \geq 4m - (k-1)n_+ - (k-1)n + n_e.$$

Using this inequality and the bound

$$2|E(G[V^-])| = \sum_{v \in V^-} (\deg(v) - |N(v) \cap V^+|) \leq \Delta n_- - |[V^+, V^-]|,$$

we arrive at

$$(9) \quad 3\Delta n_- + 2|[V^+, V^-]| \geq 4m - (k-1)n_+ - (k-1)n + n_e.$$

If $v \in V^-$, then $f(N(v)) \leq k-1$ implies that $2|N(v) \cap V^+| \leq \deg(v) + k-1 \leq \Delta + k-1$ and therefore $|N(v) \cap V^+| \leq \lfloor \frac{\Delta+k-1}{2} \rfloor$. This yields

$$(10) \quad |[V^+, V^-]| \leq \left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor n_- = \left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor (n - n_+).$$

We deduce from (9) and (10) that

$$\left(3\Delta + 2 \left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor \right) n_- \geq 4m - (k-1)(n - n_-) - (k-1)n + n_e$$

and so

$$n_- \geq \frac{4m - 2(k-1)n + n_e}{3\Delta + 2 \left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - k + 1}.$$

This yields to

$$\begin{aligned} \alpha_{st}^k(G) &= n - 2n_- \\ &\leq \frac{(3\Delta + 2 \left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - k + 1 + 4(k-1))n - 2n_e - 8m}{3\Delta + 2 \left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - k + 1} \\ &= \frac{(3\Delta + 2 \left\lfloor \frac{\Delta+k-1}{2} \right\rfloor + 3k - 3)n - 2n_e - 8m}{3\Delta + 2 \left\lfloor \frac{\Delta+k-1}{2} \right\rfloor - k + 1}, \end{aligned}$$

and (7) is proved.

(ii) It follows from (4) and Lemma 6 (ii) that

$$\begin{aligned} 2m - 2|E(G[V^+])| - |[V^+, V^-]| &= 2|E(G[V^-])| + |[V^+, V^-]| \\ &\geq 4|E(G[V^+])| - (k-1)n_+ \\ &\quad - (k-1)n + n_e \end{aligned}$$

and thus

$$2m \geq 6|E(G[V^+])| + |[V^+, V^-]| - (k-1)n_+ - (k-1)n + n_e.$$

Using this inequality and the bound

$$2|E(G[V^+])| = \sum_{v \in V^+} (\deg(v) - |N(v) \cap V^-|) \geq \delta n_+ - |[V^+, V^-]|,$$

we arrive at

$$2m \geq 3\delta n_+ - 2|[V^+, V^-]| - (k-1)n_+ - (k-1)n + n_e.$$

Applying (10), we conclude that

$$2m \geq \left(3\delta + 2 \left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor - k + 1\right) n_+ - 2 \left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor n - (k-1)n + n_e.$$

Using this inequality and $n_+ = \frac{n + \alpha_{st}^k(G)}{2}$, we obtain the bound (8), and the proof is complete. ■

If $K_{p,p}$ is the complete bipartite graph, then Observation 1 (iii) demonstrates that the inequalities (7) and (8) are sharp, when $k \leq p + 1$.

Using Lemma 6 (i) instead of Lemma 6 (ii), we obtain analogously to the proof of Theorem 8 the following two upper bounds on the signed k -independence number.

Theorem 9. *Let $k \geq 2$ be an integer, and let G be a graph of order n , size m , maximum degree Δ and minimum degree δ . Then*

$$(11) \quad \alpha_s^k(G) \leq \left\lfloor \frac{(3\Delta + 2 \lfloor \frac{\Delta+k}{2} \rfloor + 3k - 4) n - 8m - 2n_o}{3\Delta + 2 \lfloor \frac{\Delta+k}{2} \rfloor - k + 4} \right\rfloor,$$

$$(12) \quad \alpha_{st}^k(G) \leq \left\lfloor \frac{(2 \lfloor \frac{\Delta+k}{2} \rfloor - 3\delta + 3k - 4) n + 4m - 2n_o}{3\delta + 2 \lfloor \frac{\Delta+k}{2} \rfloor - k + 4} \right\rfloor.$$

The complete graph K_n , when $n + 1 \geq k$, shows that the inequalities (11) and (12) are sharp.

3. LOWER BOUNDS

As an application of the concepts of (total) limited packing we establish some lower bounds on the parameters $\alpha_s^k(G)$ and $\alpha_{st}^k(G)$ of a graph G .

Theorem 10. *Let G be graph of order n and $2 \leq k \leq \Delta(G)$. Then*

$$(i) \quad \alpha_s^k(G) \geq -n + 2 \left\lfloor \frac{\delta + 2\rho(G) + k - 2}{2} \right\rfloor,$$

(ii) $\alpha_{st}^k(G) \geq -n + 2 \left\lfloor \frac{\delta + 2\rho_0(G) + k - 3}{2} \right\rfloor$,
and these bounds are sharp.

Proof. We only prove part (i), and part (ii) can be proved in a similar fashion. Let B be a $\left\lfloor \frac{\delta + k}{2} \right\rfloor$ -limited packing set in G . We define $f : V(G) \rightarrow \{-1, 1\}$ by

$$f(v) = \begin{cases} +1 & v \in B, \\ -1 & v \in V \setminus B. \end{cases}$$

For all vertices v in $V(G)$,

$$\begin{aligned} f(N[v]) &= 2|N[v] \cap B| - |N[v]| \\ &\leq 2 \left\lfloor \frac{\delta + k}{2} \right\rfloor - \delta - 1 \leq k - 1. \end{aligned}$$

Hence, f is a signed k -independence function of G and therefore

$$(13) \quad \alpha_s^k(G) \geq f(V(G)) = 2|B| - n = 2L_{\lfloor \frac{\delta+k}{2} \rfloor}(G) - n.$$

Assume that B' is a maximum $\left\lfloor \frac{\delta + k}{2} \right\rfloor$ -limited packing set in G . Suppose to the contrary that $V = B'$. If v is a vertex in $V(G)$ with maximum degree, then $\left\lfloor \frac{\delta + k}{2} \right\rfloor > |N[v] \cap B'| = \Delta + 1$, a contradiction. Now let $u \in V \setminus B'$. It is easy to check that $B' \cup \{u\}$ is a $\left(\left\lfloor \frac{\delta + k}{2} \right\rfloor + 1 \right)$ -limited packing in G . Thus

$$L_{\lfloor \frac{\delta+k}{2} \rfloor}(G) + 1 = |B' \cup \{u\}| \leq L_{\lfloor \frac{\delta+k}{2} \rfloor + 1}(G).$$

Indeed, we have

$$\begin{aligned} L_{\lfloor \frac{\delta+k}{2} \rfloor}(G) &\geq L_{\lfloor \frac{\delta+k}{2} \rfloor - 1}(G) + 1 \geq \dots \\ &\geq L_1(G) + \left\lfloor \frac{\delta + k}{2} \right\rfloor - 1 = \rho(G) + \left\lfloor \frac{\delta + k}{2} \right\rfloor - 1. \end{aligned}$$

By (13), we deduce that $\alpha_s^k(G) \geq -n + 2\rho(G) + 2 \left\lfloor \frac{\delta + k - 2}{2} \right\rfloor$, as desired. The equalities hold for the graph K_n . ■

REFERENCES

- [1] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann, *k*-domination and *k*-independence in graphs: A survey, *Graphs Combin.* **28** (2012) 1–55.
doi:10.1007/s00373-011-1040-3
- [2] R. Gallant, G. Gunther, B.L. Hartnell and D.F. Rall, *Limited packing in graphs*, *Discrete Appl. Math.* **158** (2010) 1357–1364.
doi:10.1016/j.dam.2009.04.014
- [3] A.N. Ghameshlou, A. Khodkar and S.M. Sheikholeslami, *On the signed bad numbers of graphs*, *Bulletin of the ICA* **67** (2013) 81–93.
- [4] F. Harary and T.W. Haynes, *Double domination in graphs*, *Ars Combin.* **55** (2000) 201–213.
- [5] V. Kulli, *On n -total domination number in graphs*, in: *Graph Theory, Combinatorics, Algorithms and Applications*, SIAM (Philadelphia, 1991) 319–324.
- [6] D.A. Mojdeh, B. Samadi and S.M. Hosseini Moghaddam, *Limited packing vs tuple domination in graphs*, *Ars Combin.*, to appear.
- [7] D.A. Mojdeh, B. Samadi and S.M. Hosseini Moghaddam, *Total limited packing in graphs*, submitted.
- [8] L. Volkmann, *Signed k -independence in graphs*, *Cent. Eur. J. Math.* **12** (2014) 517–528.
doi:10.2478/s11533-013-0357-y
- [9] L. Volkmann, *Signed total k -independence number in graphs*, *Util. Math.*, to appear.
- [10] H.C. Wang and E.F. Shan, *Signed total 2-independence in graphs*, *Util. Math.* **74** (2007) 199–206.
- [11] H.C. Wang, J. Tong and L. Volkmann, *A note on signed total 2-independence in graphs*, *Util. Math.* **85** (2011) 213–223.
- [12] D.B. West, *Introduction to Graph Theory* (Second Edition) (Prentice Hall, USA, 2001).

Received 23 September 2014

Revised 11 February 2015

Accepted 11 February 2015