# UPPER BOUNDS ON THE SIGNED TOTAL $(k, k)$-DOMATIC NUMBER OF GRAPHS 

Lutz Volkmann<br>Lehrstuhl II für Mathematik<br>RWTH Aachen University<br>52056 Aachen, Germany<br>e-mail: volkm@math2.rwth-aachen.de


#### Abstract

Let $G$ be a graph with vertex set $V(G)$, and let $f: V(G) \longrightarrow\{-1,1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the neighborhood of $v$, then $f$ is a signed total $k$-dominating function on $G$. A set $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ of distinct signed total $k$-dominating functions on $G$ with the property that $\sum_{i=1}^{d} f_{i}(x) \leq k$ for each $x \in V(G)$, is called a signed total $(k, k)$-dominating family (of functions) on $G$. The maximum number of functions in a signed total $(k, k)$-dominating family on $G$ is the signed total $(k, k)$-domatic number of $G$.

In this article we mainly present upper bounds on the signed total $(k, k)$ domatic number, in particular for regular graphs.


Keywords: signed total $(k, k)$-domatic number, signed total $k$-dominating function, signed total $k$-domination number, regular graphs.
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## 1. Terminology and Introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants (see, for example the monographs by Haynes, Hedetniemi and Slater [1, 2]). In this paper we continue the investigations of the signed total $(k, k)$-domatic number, introduced by Sheikholeslami and Volkmann [5] in 2010.

We consider finite, undirected and simple graphs $G$ with vertex set $V(G)$. The order $n=n(G)$ of a graph $G$ is the number of its vertices. If $v$ is a vertex of the graph $G$, then $N(v)=N_{G}(v)$ is the open neighborhood of $v$, i.e., the set
of all vertices adjacent to $v$. The number $d_{G}(v)=d(v)=|N(v)|$ is the degree of the vertex $v \in V(G)$, and $\delta(G)$ and $\Delta(G)$ are the minimum degree and maximum degree of $G$, respectively. A graph $G$ is regular of degree $r$ if $\delta(G)=\Delta(G)=r$. Such graphs are called $r$-regular. The complete graph of order $n$ is denoted by $K_{n}$. If $A \subseteq V(G)$ and $f$ is a mapping from $V(G)$ into some set of numbers, then $f(A)=\sum_{x \in A} f(x)$.

If $k \geq 1$ is an integer, then the signed total $k$-dominating function was defined by Wang [6] as a two-valued function $f: V(G) \longrightarrow\{-1,1\}$ such that $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$. The sum $f(V(G))$ is called the weight $w(f)$ of $f$. The minimum of weights $w(f)$, taken over all signed total $k$-dominating functions $f$ on $G$, is called the signed total $k$-domination number of $G$, denoted by $\gamma_{s t}^{k}(G)$. The special case $k=1$ was defined and investigated by Zelinka [7] in 2001. Further information on this parameter can be found in the article [3] by Henning.

A set $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ of distinct signed total $k$-dominating functions on $G$ with the property that $\sum_{i=1}^{d} f_{i}(x) \leq k$ for each vertex $x \in V(G)$, is called in [5] a signed total $(k, k)$-dominating family on $G$. The maximum number of functions in a signed total $(k, k)$-dominating family on $G$ is the signed total $(k, k)$-domatic number of $G$, denoted by $d_{s t}^{k}(G)$. As the assumption $\delta(G) \geq k$ is necessary, we always assume that when we discuss $\gamma_{s t}^{k}(G)$ or $d_{s t}^{k}(G)$, all graphs involved satisfy $\delta(G) \geq k$. The special case $k=1$ of the signed total $(k, k)$-domatic number was defined and investigated by Henning [4] in 2006.

In this paper we continue the studies of the signed total $(k, k)$-domatic number, which is an extension of the classical signed total domatic number. First we present upper bounds on $d_{s t}^{k}(G)$ for regular graphs in terms of order. As an application of some of these upper bounds and some known results, we prove that $d_{s t}^{k}(G) \leq n-3$ for each graph $G$ of order $n \geq 4$. For the complete graph $K_{n}$ we show that $d_{s t}^{n-3}\left(K_{n}\right)=n-3$, and therefore this bound is sharp.

## 2. Regular Graphs

Throughout this section, if $f$ is a signed total $k$-dominating function on a graph $G$, then we let $P$ and $M$ denote the sets of those vertices in $G$ which are assigned under $f$ the values 1 and -1 , respectively. Thus $|P|+|M|=n(G)$.

Theorem 2.1. If $k \geq 2$ is an even integer, and $G$ is a $2 r$-regular graph of odd order $n=2 q+1 \geq 3$, then

$$
d_{s t}^{k}(G) \leq\left\lfloor\frac{k n}{k+1}\right\rfloor .
$$

In addition, if $2 r<(n k) /(k+1)$, then

$$
d_{s t}^{k}(G) \leq\left\lfloor\frac{k n}{k+3}\right\rfloor .
$$

Proof. If $f$ is an arbitrary signed total $k$-dominating function on $G$, then we show that

$$
\begin{equation*}
|P| \geq q+\frac{k+2}{2} \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
|P| \geq q+\frac{k+4}{2} \tag{2}
\end{equation*}
$$

when $2 r<(n k) /(k+1)$. The condition $\sum_{x \in N(y)} f(x) \geq k$ for each vertex $y \in$ $V(G)$ implies that each vertex $u \in P$ is adjacent to at most $(2 r-k) / 2$ vertices in $M$ and each vertex $v \in M$ is adjacent to at least $(2 r+k) / 2$ vertices in $P$. Therefore we obtain

$$
|P| \cdot \frac{2 r-k}{2} \geq(2 q+1-|P|) \frac{2 r+k}{2}
$$

and thus

$$
\begin{equation*}
|P| \geq \frac{(2 r+k)(2 q+1)}{4 r} \tag{3}
\end{equation*}
$$

If we suppose that $|P| \leq q+\frac{k}{2}$, then the last inequality leads to

$$
q+\frac{k}{2} \geq|P| \geq \frac{(2 r+k)(2 q+1)}{4 r}
$$

It follows that $r>q$. This is a contradiction to the hypothesis $r \leq q$, and thus (1) is proved. If we suppose in the case $2 r<(n k) /(k+1)$ that $|P| \leq q+\frac{k+2}{2}$, then (3) leads to the contradiction $2 r(k+1) \geq k(2 q+1)=k n$. Hence (2) is proved too.

Now let $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ be a signed total $(k, k)$-dominating family on $G$ such that $d=d_{s t}^{k}(G)$. Since $\sum_{i=1}^{d} f_{i}(u) \leq k$ for every $u \in V(G)$, each of these sums contains at least $\lceil(d-k) / 2\rceil$ summands of value -1 . Applying this and inequality (1), we see that the sum

$$
\begin{equation*}
\sum_{x \in V(G)} \sum_{i=1}^{d} f_{i}(x)=\sum_{i=1}^{d} \sum_{x \in V(G)} f_{i}(x) \tag{4}
\end{equation*}
$$

contains at least $(2 q+1)\lceil(d-k) / 2\rceil$ summands of value -1 and at least $d(q+$ $(k+2) / 2)$ summands of value 1 . As the sum (4) consists of exactly $d(2 q+1)$ summands, it follows that
$(2 q+1) \frac{d-k}{2}+d\left(q+\frac{k+2}{2}\right) \leq(2 q+1)\left\lceil\frac{d-k}{2}\right\rceil+d\left(q+\frac{k+2}{2}\right) \leq d(2 q+1)$.
We deduce that

$$
(2 q+1)(d-k)+d(2 q+k+2) \leq 2 d(2 q+1)
$$

and thus $d(k+1) \leq k(2 q+1)$. This yields to the first bound immediately. Using (2) and (4) instead of (1) and (4), we obtain the second bound analogously.

Example 3.10 will demonstrate that the first bound in Theorem 2.1 is sharp. If $k$ is odd in Theorem 2.1, then we can improve the upper bound on the signed total $(k, k)$-domatic number.

Theorem 2.2. If $k \geq 1$ is an odd integer, and $G$ is a $2 r$-regular graph of odd order $n=2 q+1 \geq 3$, then

$$
d_{s t}^{k}(G) \leq\left\lfloor\frac{k n}{k+2}\right\rfloor .
$$

In addition, if $2 r<(n(k+1)) /(k+2)$, then

$$
d_{s t}^{k}(G) \leq\left\lfloor\frac{k n}{k+4}\right\rfloor .
$$

Proof. If $f$ is an arbitrary signed total $k$-dominating function on $G$, then we show that

$$
\begin{equation*}
|P| \geq q+\frac{k+3}{2} \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
|P| \geq q+\frac{k+5}{2} \tag{6}
\end{equation*}
$$

when $2 r<(n(k+1)) /(k+2)$. As $G$ is $2 r$-regular and $k$ is odd, the condition $\sum_{x \in N(y)} f(x) \geq k$ leads to $\sum_{x \in N(y)} f(x) \geq k+1$ for each vertex $y \in V(G)$. This implies that each vertex $u \in P$ is adjacent to at most $(2 r-1-k) / 2$ vertices in $M$ and each vertex $v \in M$ is adjacent to at least $(2 r+k+1) / 2$ vertices in $P$. Therefore we obtain

$$
|P| \cdot \frac{2 r-1-k}{2} \geq(2 q+1-|P|) \frac{2 r+1+k}{2}
$$

and thus

$$
\begin{equation*}
|P| \geq \frac{(2 r+k+1)(2 q+1)}{4 r} \tag{7}
\end{equation*}
$$

If we suppose that $|P| \leq q+\frac{k+1}{2}$, then the last inequality leads to

$$
q+\frac{k+1}{2} \geq|P| \geq \frac{(2 r+k+1)(2 q+1)}{4 r}
$$

It follows that $r>q$. This is a contradiction to the hypothesis $r \leq q$, and thus (5) is proved. If we suppose in the case $2 r<(n(k+1)) /(k+2)$ that $|P| \leq q+\frac{k+3}{2}$, then (7) leads to the contradiction $2 r(k+2) \geq(k+1)(2 q+1)=n(k+1)$. Hence (6) is proved too.

If $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ is a signed total $(k, k)$-dominating family on $G$ such that $d=d_{s t}^{k}(G)$, then the proof of the desired bounds is similar to that of the proof of Theorem 2.1.

The proofs of the next upper bounds for regular graphs are analogous to that of Theorems 2.1 and 2.2.

Theorem 2.3. If $k \geq 2$ is an even integer, and $G$ is a $2 r$-regular graph of even order $n$, then

$$
d_{s t}^{k}(G) \leq\left\lfloor\frac{k n}{k+2}\right\rfloor .
$$

Theorem 2.4. If $k \geq 1$ is an odd integer, and $G$ is a $(2 r+1)$-regular graph of even order $n$, then

$$
d_{s t}^{k}(G) \leq\left\lfloor\frac{k n}{k+1}\right\rfloor .
$$

Theorem 2.5. If $k \geq 1$ is an odd integer, and $G$ is a $2 r$-regular graph of even order $n$, then

$$
d_{s t}^{k}(G) \leq\left\lfloor\frac{k n}{k+3}\right\rfloor .
$$

Theorem 2.6. If $k \geq 2$ is an even integer, and $G$ is a $(2 r+1)$-regular graph of even order $n$, then

$$
d_{s t}^{k}(G) \leq\left\lfloor\frac{k n}{k+2}\right\rfloor .
$$

## 3. A General Upper Bound

As an application of the following known results and Theorems 2.1, 2.3 and 2.4, we derive a sharp upper bound on the signed total $(k, k)$-domatic number.

Proposition 3.1 [5]. If $G$ is a graph of order $n \geq 3$ and $k=n-1$ or $k=n-2$, then $\gamma_{s t}^{k}(G)=n$ and thus $d_{s t}^{k}(G)=1$.

Proposition 3.2 [5]. If $G$ is a graph with minimum degree $\delta(G) \geq k$, then $d_{s t}^{k}(G) \leq \delta(G)$.

Proposition 3.3 [5]. If $v$ is a vertex of a graph $G$ such that $d(v)$ is odd and $k$ is even or $d(v)$ is even and $k$ is odd, then

$$
d_{s t}^{k}(G) \leq \frac{k}{k+1} \cdot d(v)
$$

Proposition 3.4 [5]. If $G$ is graph such that such that $\delta(G)$ is odd and $k$ is even or $\delta(G)$ is even and $k$ is odd, then

$$
d_{s t}^{k}(G) \leq \frac{k}{k+1} \cdot \delta(G)
$$

Proposition 3.5 [5]. If $G$ is graph such that $k$ is odd and $d_{s t}^{k}(G)$ is even or $k$ is even and $d_{s t}^{k}(G)$ is odd, then

$$
d_{s t}^{k}(G) \leq \frac{k-1}{k} \cdot \delta(G) .
$$

Proposition 3.6 [5]. If $G$ is graph of minimum degree $\delta(G) \geq k+2$, then $d_{s t}^{k}(G) \geq k$.
Theorem 3.7. If $G$ is a graph of order $n \geq 4$ and minimum degree $\delta \geq k$, then $d_{s t}^{k}(G) \leq n-3$.

Proof. If $\delta \leq n-3$, then Proposition 3.2 implies the desired bound immediately.
Case 1. Assume that $\delta=n-2 \geq k$.
Subcase 1.1. Assume that $n-2=\delta<\Delta(G)=\Delta=n-1$. If $\delta$ is odd and $k$ is even, then it follows from Proposition 3.4 that $d_{s t}^{k}(G) \leq(k \delta) /(k+1)<n-2$ and thus $d_{s t}^{k}(G) \leq n-3$. If $\delta$ and $k$ are even, then $\Delta=\delta+1=n-1$ is odd. If $d(v)=\Delta$, then we deduce from Proposition 3.3 that

$$
d_{s t}^{k}(G) \leq \frac{k}{k+1} \cdot d(v)=\frac{k}{k+1}(n-1)<n-2
$$

when $k<n-2$ and so $d_{s t}^{k}(G) \leq n-3$ in that case. If $k=n-2$, then Proposition 3.1 leads to $d_{s t}^{k}(G)=1 \leq n-3$. If $\delta$ is even and $k$ is odd, then again Proposition 3.4 yields to $d_{s t}^{k}(G) \leq n-3$. If $\delta$ and $k$ are odd, then $\Delta$ is even. The desired bound follows as in the case that $\delta$ and $k$ are even.

Subcase 1.2. Assume that $\delta=\Delta=n-2$. The handshaking lemma implies that $n$ is even, and so $\delta$ is even too. If $k$ is odd, then Proposition 3.4 shows that $d_{s t}^{k}(G) \leq(k \delta) /(k+1)<n-2$ and thus $d_{s t}^{k}(G) \leq n-3$. If $k$ is even, then we conclude from Theorem 2.3 that

$$
d_{s t}^{k}(G) \leq \frac{k n}{k+2}<n-2
$$

when $k<n-2$ and so $d_{s t}^{k}(G) \leq n-3$ in that case. If $k=n-2$, then Proposition 3.1 leads to $d_{s t}^{k}(G)=1 \leq n-3$.

Case 2. Assume that $\delta=n-1 \geq k$.
Subcase 2.1. Assume that $n$ is even. Then $\delta=n-1$ is odd.
If $k$ is even, then Proposition 3.4 shows that

$$
d_{s t}^{k}(G) \leq \frac{k}{k+1} \cdot \delta=\frac{k}{k+1}(n-1)<n-2
$$

when $k<n-2$ and so $d_{s t}^{k}(G) \leq n-3$ in that case. If $k=n-2$, then $d_{s t}^{k}(G)=$ $1 \leq n-3$ by Proposition 3.1. As $k$ is even, $k=n-1$ is not possible.

If $k$ is odd, then it follows from Theorem 2.4 that

$$
d_{s t}^{k}(G) \leq \frac{k}{k+1} \cdot n<n-1
$$

when $k<n-1$ and so $d_{s t}^{k}(G) \leq n-2$ when $k<n-1$. However, if $d_{s t}^{k}(G)=n-2$, then Proposition 3.5 leads to the contradiction

$$
n-2=d_{s t}^{k}(G) \leq \frac{k-1}{k} \cdot \delta=\frac{k-1}{k}(n-1)<n-2
$$

when $k<n-1$. Consequently, $d_{s t}^{k}(G) \leq n-3$ when $k<n-1$. In the case $k=n-1$, Proposition 3.1 yields to the desired bound.

Subcase 2.2. Assume that $n$ is odd. Then $\delta=n-1$ is even.
If $k$ is odd, then it follows from Proposition 3.4 that

$$
d_{s t}^{k}(G) \leq \frac{k}{k+1} \cdot \delta=\frac{k}{k+1}(n-1)<n-2
$$

when $k<n-2$ and so $d_{s t}^{k}(G) \leq n-3$ in that case. If $k=n-2$, then $d_{s t}^{k}(G)=$ $1 \leq n-3$, according to Proposition 3.1. As $k$ is odd, $k=n-1$ is not possible.

If $k$ is even, then we obtain by Theorem 2.1 that

$$
d_{s t}^{k}(G) \leq \frac{k}{k+1} \cdot n<n-1
$$

when $k<n-1$ and so $d_{s t}^{k}(G) \leq n-2$ when $k<n-1$. However, if $d_{s t}^{k}(G)=n-2$, then Proposition 3.5 leads to the contradiction

$$
n-2=d_{s t}^{k}(G) \leq \frac{k-1}{k} \cdot \delta=\frac{k-1}{k}(n-1)<n-2
$$

when $k<n-1$. Therefore $d_{s t}^{k}(G) \leq n-3$ when $k<n-1$. In the case $k=n-1$, again Proposition 3.1 yields to the desired bound.

Example 3.8. Let $n \geq 4$ be an integer. On the one hand it follows from Proposition 3.6 that $d_{s t}^{n-3}\left(K_{n}\right) \geq n-3$. On the other hand, Theorem 3.7 implies that $d_{s t}^{n-3}\left(K_{n}\right) \leq n-3$, and therefore we have $d_{s t}^{n-3}\left(K_{n}\right)=n-3$.

This example demonstrates that Theorem 3.7 is sharp. Next we present some further examples with equality in the bound of Theorem 3.7.

Example 3.9.1. Let $\left\{x_{1}, x_{2}, \ldots, x_{6}\right\}$ be the vertex set of the complete graph $K_{6}$, and let $f_{i}: V\left(K_{6}\right) \longrightarrow\{-1,1\}$ such that $f_{1}\left(x_{1}\right)=f_{1}\left(x_{2}\right)=-1$ and $f_{1}(x)=1$ otherwise, $f_{2}\left(x_{3}\right)=f_{2}\left(x_{4}\right)=-1$ and $f_{2}(x)=1$ otherwise, and $f_{3}\left(x_{5}\right)=f_{3}\left(x_{6}\right)=-1$ and $f_{3}(x)=1$ otherwise.

It follows that $\sum_{x \in N(y)} f_{i}(x) \geq 1$ for each vertex $y \in V\left(K_{6}\right)$ and $i=1,2,3$ and $f_{1}(x)+f_{2}(x)+f_{3}(x)=1$ for each vertex $x \in V\left(K_{6}\right)$. Therefore $d_{s t}^{1}\left(K_{6}\right) \geq 3$ and Theorem 3.7 yields to $d_{s t}^{1}\left(K_{6}\right)=3=n-3$.
2. Let $\left\{x_{1}, x_{2}, \ldots, x_{9}\right\}$ be the vertex set of the complete graph $K_{9}$, and let $f_{i}: V\left(K_{9}\right) \longrightarrow\{-1,1\}$ such that $f_{1}\left(x_{1}\right)=f_{1}\left(x_{2}\right)=f_{1}\left(x_{3}\right)=-1$ and $f_{1}(x)=1$ otherwise, $f_{2}\left(x_{4}\right)=f_{2}\left(x_{5}\right)=f_{2}\left(x_{6}\right)=-1$ and $f_{2}(x)=1$ otherwise, $f_{3}\left(x_{7}\right)=f_{3}\left(x_{8}\right)=f_{3}\left(x_{9}\right)=-1$ and $f_{3}(x)=1$ otherwise,
$f_{4}\left(x_{2}\right)=f_{4}\left(x_{3}\right)=f_{4}\left(x_{4}\right)=-1$ and $f_{4}(x)=1$ otherwise,
$f_{5}\left(x_{5}\right)=f_{5}\left(x_{6}\right)=f_{5}\left(x_{7}\right)=-1$ and $f_{5}(x)=1$ otherwise, and $f_{6}\left(x_{8}\right)=f_{6}\left(x_{9}\right)=f_{6}\left(x_{1}\right)=-1$ and $f_{6}(x)=1$ otherwise.

Then $\sum_{x \in N(y)} f_{i}(x) \geq 2$ for each vertex $y \in V\left(K_{9}\right)$ and $i=1,2, \ldots, 6$ and $\sum_{i=1}^{6} f_{i}(x)=2$ for each vertex $x \in V\left(K_{9}\right)$. Therefore $d_{s t}^{2}\left(K_{9}\right) \geq 6$ and Theorem 3.7 implies that $d_{s t}^{2}\left(K_{9}\right)=6=n-3$.
3. Let $\left\{x_{1}, x_{2}, \ldots, x_{12}\right\}$ be the vertex set of the complete graph $K_{12}$, and let $f_{i}: V\left(K_{12}\right) \longrightarrow\{-1,1\}$ such that
$f_{1}\left(x_{1}\right)=f_{1}\left(x_{2}\right)=f_{1}\left(x_{3}\right)=f_{1}\left(x_{4}\right)=-1$ and $f_{1}(x)=1$ otherwise,
$f_{2}\left(x_{5}\right)=f_{2}\left(x_{6}\right)=f_{2}\left(x_{7}\right)=f_{2}\left(x_{8}\right)=-1$ and $f_{2}(x)=1$ otherwise,
$f_{3}\left(x_{9}\right)=f_{3}\left(x_{10}\right)=f_{3}\left(x_{11}\right)=f_{3}\left(x_{12}\right)=-1$ and $f_{3}(x)=1$ otherwise,
$f_{4}\left(x_{2}\right)=f_{4}\left(x_{3}\right)=f_{4}\left(x_{4}\right)=f_{4}\left(x_{5}\right)=-1$ and $f_{4}(x)=1$ otherwise,
$f_{5}\left(x_{6}\right)=f_{5}\left(x_{7}\right)=f_{5}\left(x_{8}\right)=f_{5}\left(x_{9}\right)=-1$ and $f_{5}(x)=1$ otherwise, $f_{6}\left(x_{10}\right)=f_{6}\left(x_{11}\right)=f_{6}\left(x_{12}\right)=f_{6}\left(x_{1}\right)=-1$ and $f_{6}(x)=1$ otherwise, $f_{7}\left(x_{3}\right)=f_{7}\left(x_{4}\right)=f_{7}\left(x_{5}\right)=f_{7}\left(x_{6}\right)=-1$ and $f_{7}(x)=1$ otherwise, $f_{8}\left(x_{7}\right)=f_{8}\left(x_{8}\right)=f_{8}\left(x_{9}\right)=f_{8}\left(x_{10}\right)=-1$ and $f_{8}(x)=1$ otherwise, and $f_{9}\left(x_{11}\right)=f_{9}\left(x_{12}\right)=f_{9}\left(x_{1}\right)=f_{9}\left(x_{2}\right)=-1$ and $f_{9}(x)=1$ otherwise.

So $\sum_{x \in N(y)} f_{i}(x) \geq 3$ for each vertex $y \in V\left(K_{12}\right)$ and $i=1,2, \ldots, 9$ and $\sum_{i=1}^{9} f_{i}(x)=3$ for each vertex $x \in V\left(K_{12}\right)$. Therefore $d_{s t}^{3}\left(K_{12}\right) \geq 9$ and Theorem 3.7 leads to $d_{s t}^{3}\left(K_{12}\right)=9=n-3$.

Example 3.9 leads us to a more general result.
Example 3.10. If $k \geq 1$ is an integer and $n=3(k+1)$, then $d_{s t}^{k}\left(K_{n}\right)=n-3$.
Proof. Let $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be the vertex set of the complete graph $K_{n}$, and let $f_{i}: V\left(K_{n}\right) \longrightarrow\{-1,1\}$ such that
$f_{1}\left(x_{1}\right)=f_{1}\left(x_{2}\right)=\cdots=f_{1}\left(x_{k+1}\right)=-1$ and $f_{1}(x)=1$ otherwise,
$f_{2}\left(x_{k+2}\right)=f_{2}\left(x_{k+3}\right)=\cdots=f_{2}\left(x_{2 k+2}\right)=-1$ and $f_{2}(x)=1$ otherwise, $f_{3}\left(x_{2 k+3}\right)=f_{3}\left(x_{2 k+4}\right)=\cdots=f_{3}\left(x_{3 k+3}\right)=-1$ and $f_{3}(x)=1$ otherwise, $f_{4}\left(x_{2}\right)=f_{4}\left(x_{3}\right)=\cdots=f_{4}\left(x_{k+2}\right)=-1$ and $f_{4}(x)=1$ otherwise,
$f_{3 k-2}\left(x_{k}\right)=f_{3 k-2}\left(x_{k+1}\right)=\cdots=f_{3 k-2}\left(x_{2 k}\right)=-1$ and $f_{3 k-2}(x)=1$ otherwise, $f_{3 k-1}\left(x_{2 k+1}\right)=f_{3 k-1}\left(x_{2 k+2}\right)=\cdots=f_{3 k-1}\left(x_{3 k+1}\right)=-1$ and $f_{3 k-1}(x)=1$ otherwise,
$f_{3 k}\left(x_{3 k+2}\right)=f_{3 k}\left(x_{3 k+3}\right)=f_{3 k}\left(x_{1}\right)=\cdots=f_{3 k}\left(x_{k-1}\right)=-1$ and $f_{3 k}(x)=1$ otherwise.

It is straightforward to verify that $\sum_{x \in N(y)} f_{i}(x) \geq k$ for each vertex $y \in$ $V\left(K_{n}\right)$ and $i=1,2, \ldots, 3 k$ and $\sum_{i=1}^{3 k} f_{i}(x)=k$ for each vertex $x \in V\left(K_{n}\right)$. Therefore $d_{s t}^{k}\left(K_{n}\right) \geq 3 k$ and thus it follows from Theorem 3.7 that $d_{s t}^{k}\left(K_{n}\right)=$ $3 k=n-3$.

Notice that Example 3.10 also demonstrates that Theorems 2.1 and 2.4 are sharp. Finally, we give some examples of non-complete graphs with equality in the inequality of Theorem 3.7.

Example 3.11. Let $u, v$ and $w$ be three distinct vertices of the complete graph $K_{n}$.

1. Let $G_{5}=K_{5}-u v$, and let $f_{i}: V\left(G_{5}\right) \longrightarrow\{-1,1\}$ such that $f_{1}(u)=-1$ and $f_{1}(x)=1$ for $x \neq u$ and $f_{2}(x)=1$ for each $x \in V\left(G_{5}\right)$.

Then it is easy to see that $\sum_{x \in N(y)} f_{i}(x) \geq 2$ for each vertex $y \in V\left(G_{5}\right)$ and $i=1,2$ and $f_{1}(x)+f_{2}(x) \leq 2$ for each vertex $x \in V\left(G_{5}\right)$. Therefore $d_{s t}^{2}\left(G_{5}\right) \geq 2$ and Theorem 3.7 shows that $d_{s t}^{2}\left(G_{5}\right)=2=n-3$.
2. Let $G_{6}=K_{6}-u v$, and let $f_{i}: V\left(G_{6}\right) \longrightarrow\{-1,1\}$ such that $f_{1}(u)=-1$ and $f_{1}(x)=1$ for $x \neq u$, $f_{2}(v)=-1$ and $f_{2}(x)=1$ for $x \neq v$ and $f_{3}(x)=1$ for each $x \in V\left(G_{6}\right)$.

Then $\sum_{x \in N(y)} f_{i}(x) \geq 3$ for each vertex $y \in V\left(G_{6}\right)$ and $i=1,2,3$ and $f_{1}(x)+f_{2}(x)+f_{3}(x) \leq 3$ for each vertex $x \in V\left(G_{6}\right)$. Therefore $d_{s t}^{3}\left(G_{6}\right) \geq 3$ and Theorem 3.7 leads to $d_{s t}^{3}\left(G_{6}\right)=3=n-3$.
3. Let $G_{7}=K_{7}-\{u v, u w, v w\}$, and let $f_{i}: V\left(G_{7}\right) \longrightarrow\{-1,1\}$ such that
$f_{1}(u)=-1$ and $f_{1}(x)=1$ for $x \neq u$,
$f_{2}(v)=-1$ and $f_{2}(x)=1$ for $x \neq v$, $f_{3}(w)=-1$ and $f_{3}(x)=1$ for $x \neq w$ and $f_{4}(x)=1$ for each $x \in V\left(G_{7}\right)$.

Then $\sum_{x \in N(y)} f_{i}(x) \geq 4$ for each vertex $y \in V\left(G_{7}\right)$ and $i=1,2,3,4$ and $f_{1}(x)+f_{2}(x)+f_{3}(x)+f_{4}(x) \leq 4$ for each vertex $x \in V\left(G_{7}\right)$. Therefore $d_{s t}^{4}\left(G_{7}\right) \geq 4$ and Theorem 3.7 shows that $d_{s t}^{4}\left(G_{7}\right)=4=n-3$.

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