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UPPER BOUNDS ON THE SIGNED TOTAL (k, k)-DOMATIC NUMBER OF GRAPHS

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Abstract

Let G be a graph with vertex set $V(G)$, and let $f : V(G) \longrightarrow \{-1, 1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the neighborhood of v, then f is a signed total k-dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq k$ for each $x \in V(G)$, is called a signed total (k, k)-dominating family (of functions) on G. The maximum number of functions in a signed total (k, k)-dominating family on G is the signed total (k, k)-domatic number of G.

In this article we mainly present upper bounds on the signed total (k, k)-domatic number, in particular for regular graphs.

Keywords: signed total (k, k)-domatic number, signed total k-dominating function, signed total k-domination number, regular graphs.

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1. Terminology and Introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants (see, for example the monographs by Haynes, Hedetniemi and Slater [1, 2]). In this paper we continue the investigations of the signed total (k, k)-domatic number, introduced by Sheikholeslami and Volkmann [5] in 2010.

We consider finite, undirected and simple graphs G with vertex set $V(G)$. The *order* $n = n(G)$ of a graph G is the number of its vertices. If v is a vertex of the graph G, then $N(v) = N_G(v)$ is the *open neighborhood* of v, i.e., the set of all vertices adjacent to v. The number $d_G(v) = d(v) = |N(v)|$ is the *degree* of the vertex $v \in V(G)$, and $\delta(G)$ and $\Delta(G)$ are the *minimum degree* and *maximum degree* of G, respectively. A graph G is *regular of degree* r if $\delta(G) = \Delta(G) = r$. Such graphs are called *r-regular*. The *complete graph* of order n is denoted by K_n. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$.

If $k \geq 1$ is an integer, then the *signed total k-dominating function* was defined by Wang [6] as a two-valued function $f : V(G) \longrightarrow \{-1, 1\}$ such that $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$. The sum $f(V(G))$ is called the *weight* $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed total k-dominating functions f on G, is called the *signed total k-domination number* of G, denoted by $\gamma_{st}^k(G)$. The special case $k = 1$ was defined and investigated by Zelinka [7] in 2001. Further information on this parameter can be found in the article [3] by Henning.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq k$ for each vertex $x \in V(G)$, is called in [5] a *signed total (k, k)-dominating family* on G. The maximum number of functions in a signed total (k, k)-dominating family on G is the *signed total (k, k)-domatic number* of G, denoted by $d_{st}^k(G)$. As the assumption $\delta(G) \geq k$ is necessary, we always assume that when we discuss $\gamma_{st}^k(G)$ or $d_{st}^k(G)$, all graphs involved satisfy $\delta(G) \geq k$. The special case $k = 1$ of the signed total (k, k)-domatic number was defined and investigated by Henning [4] in 2006.

In this paper we continue the studies of the signed total (k, k)-domatic number, which is an extension of the classical signed total domatic number. First we present upper bounds on $d_{st}^k(G)$ for regular graphs in terms of order. As an application of some of these upper bounds and some known results, we prove that $d_{st}^k(G) \leq n - 3$ for each graph G of order $n \geq 4$. For the complete graph K_n we show that $d_{st}^{n-3}(K_n) = n - 3$, and therefore this bound is sharp.

2. Regular Graphs

Throughout this section, if f is a signed total k-dominating function on a graph G, then we let P and M denote the sets of those vertices in G which are assigned under f the values 1 and -1, respectively. Thus $|P| + |M| = n(G)$.

Theorem 2.1. *If $k \geq 2$ is an even integer, and G is a $2r$-regular graph of odd order $n = 2q + 1 \geq 3$, then*

$$d_{st}^k(G) \leq \left\lfloor \frac{kn}{k+1} \right\rfloor.$$

In addition, if $2r < (nk)/(k + 1)$, then

$$d_{st}^k(G) \leq \left\lfloor \frac{kn}{k+3} \right\rfloor.$$

Proof. If f is an arbitrary signed total k-dominating function on G, then we show that

$$|P| \geq q + \frac{k+2}{2} \tag{1}$$

and

$$|P| \geq q + \frac{k+4}{2} \tag{2}$$

when $2r < (nk)/(k+1)$. The condition $\sum_{x\in N(y)} f(x) \geq k$ for each vertex $y \in V(G)$ implies that each vertex $u \in P$ is adjacent to at most $(2r-k)/2$ vertices in M and each vertex $v \in M$ is adjacent to at least $(2r+k)/2$ vertices in P. Therefore we obtain

$$|P| \cdot \frac{2r-k}{2} \geq (2q+1-|P|)\frac{2r+k}{2}$$

and thus

$$|P| \geq \frac{(2r+k)(2q+1)}{4r}. \tag{3}$$

If we suppose that $|P| \leq q + \frac{k}{2}$, then the last inequality leads to

$$q + \frac{k}{2} \geq |P| \geq \frac{(2r+k)(2q+1)}{4r}.$$

It follows that $r > q$. This is a contradiction to the hypothesis $r \leq q$, and thus (1) is proved. If we suppose in the case $2r < (nk)/(k+1)$ that $|P| \leq q + \frac{k+2}{2}$, then (3) leads to the contradiction $2r(k+1) \geq k(2q+1) = kn$. Hence (2) is proved too.

Now let $\{f_1, f_2, \ldots, f_d\}$ be a signed total (k,k)-dominating family on G such that $d = d^k_{st}(G)$. Since $\sum_{i=1}^{d} f_i(u) \leq k$ for every $u \in V(G)$, each of these sums contains at least $\lceil (d-k)/2 \rceil$ summands of value -1. Applying this and inequality (1), we see that the sum

$$\sum_{x\in V(G)} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x\in V(G)} f_i(x) \tag{4}$$

contains at least $(2q+1)\lceil (d-k)/2 \rceil$ summands of value -1 and at least $d(q + (k+2)/2)$ summands of value 1. As the sum (4) consists of exactly $d(2q+1)$ summands, it follows that

$$(2q+1)\frac{d-k}{2} + d\left(q + \frac{k+2}{2}\right) \leq (2q+1)\left\lceil \frac{d-k}{2} \right\rceil + d\left(q + \frac{k+2}{2}\right) \leq d(2q+1).$$

We deduce that

$$(2q+1)(d-k) + d(2q+k+2) \leq 2d(2q+1)$$

and thus $d(k+1) \leq k(2q+1)$. This yields to the first bound immediately. Using (2) and (4) instead of (1) and (4), we obtain the second bound analogously. ■

Example 3.10 will demonstrate that the first bound in Theorem 2.1 is sharp. If k is odd in Theorem 2.1, then we can improve the upper bound on the signed total (k,k)-domatic number.

Theorem 2.2. *If $k \geq 1$ is an odd integer, and G is a $2r$-regular graph of odd order $n = 2q+1 \geq 3$, then*

$$d_{st}^k(G) \leq \left\lfloor \tfrac{kn}{k+2} \right\rfloor.$$

In addition, if $2r < (n(k+1))/(k+2)$, then

$$d_{st}^k(G) \leq \left\lfloor \tfrac{kn}{k+4} \right\rfloor.$$

Proof. If f is an arbitrary signed total k-dominating function on G, then we show that

$$|P| \geq q + \frac{k+3}{2} \tag{5}$$

and

$$|P| \geq q + \frac{k+5}{2} \tag{6}$$

when $2r < (n(k+1))/(k+2)$. As G is $2r$-regular and k is odd, the condition $\sum_{x \in N(y)} f(x) \geq k$ leads to $\sum_{x \in N(y)} f(x) \geq k+1$ for each vertex $y \in V(G)$. This implies that each vertex $u \in P$ is adjacent to at most $(2r-1-k)/2$ vertices in M and each vertex $v \in M$ is adjacent to at least $(2r+k+1)/2$ vertices in P. Therefore we obtain

$$|P| \cdot \frac{2r-1-k}{2} \geq (2q+1-|P|)\frac{2r+1+k}{2}$$

and thus

$$|P| \geq \frac{(2r+k+1)(2q+1)}{4r}. \tag{7}$$

If we suppose that $|P| \leq q + \frac{k+1}{2}$, then the last inequality leads to

$$q + \frac{k+1}{2} \geq |P| \geq \frac{(2r+k+1)(2q+1)}{4r}.$$

It follows that $r > q$. This is a contradiction to the hypothesis $r \leq q$, and thus (5) is proved. If we suppose in the case $2r < (n(k+1))/(k+2)$ that $|P| \leq q + \frac{k+3}{2}$, then (7) leads to the contradiction $2r(k+2) \geq (k+1)(2q+1) = n(k+1)$. Hence (6) is proved too.

If $\{f_1, f_2, \ldots, f_d\}$ is a signed total (k,k)-dominating family on G such that $d = d_{st}^k(G)$, then the proof of the desired bounds is similar to that of the proof of Theorem 2.1. ∎

The proofs of the next upper bounds for regular graphs are analogous to that of Theorems 2.1 and 2.2.

Theorem 2.3. *If $k \geq 2$ is an even integer, and G is a $2r$-regular graph of even order n, then*

$$d_{st}^{k}(G) \leq \left\lfloor \tfrac{kn}{k+2} \right\rfloor .$$

Theorem 2.4. *If $k \geq 1$ is an odd integer, and G is a $(2r+1)$-regular graph of even order n, then*

$$d_{st}^{k}(G) \leq \left\lfloor \tfrac{kn}{k+1} \right\rfloor .$$

Theorem 2.5. *If $k \geq 1$ is an odd integer, and G is a $2r$-regular graph of even order n, then*

$$d_{st}^{k}(G) \leq \left\lfloor \tfrac{kn}{k+3} \right\rfloor .$$

Theorem 2.6. *If $k \geq 2$ is an even integer, and G is a $(2r+1)$-regular graph of even order n, then*

$$d_{st}^{k}(G) \leq \left\lfloor \tfrac{kn}{k+2} \right\rfloor .$$

3. A General Upper Bound

As an application of the following known results and Theorems 2.1, 2.3 and 2.4, we derive a sharp upper bound on the signed total (k, k)-domatic number.

Proposition 3.1 [5]**.** *If G is a graph of order $n \geq 3$ and $k = n-1$ or $k = n-2$, then $\gamma_{st}^{k}(G) = n$ and thus $d_{st}^{k}(G) = 1$.*

Proposition 3.2 [5]**.** *If G is a graph with minimum degree $\delta(G) \geq k$, then $d_{st}^{k}(G) \leq \delta(G)$.*

Proposition 3.3 [5]**.** *If v is a vertex of a graph G such that $d(v)$ is odd and k is even or $d(v)$ is even and k is odd, then*

$$d_{st}^{k}(G) \leq \tfrac{k}{k+1} \cdot d(v).$$

Proposition 3.4 [5]**.** *If G is graph such that such that $\delta(G)$ is odd and k is even or $\delta(G)$ is even and k is odd, then*

$$d_{st}^{k}(G) \leq \tfrac{k}{k+1} \cdot \delta(G).$$

Proposition 3.5 [5]**.** *If G is graph such that k is odd and $d_{st}^{k}(G)$ is even or k is even and $d_{st}^{k}(G)$ is odd, then*

$$d_{st}^{k}(G) \leq \tfrac{k-1}{k} \cdot \delta(G).$$

Proposition 3.6 [5]**.** *If G is graph of minimum degree $\delta(G) \geq k+2$, then $d_{st}^{k}(G) \geq k$.*

Theorem 3.7. *If G is a graph of order $n \geq 4$ and minimum degree $\delta \geq k$, then $d_{st}^{k}(G) \leq n-3$.*

Proof. If $\delta \le n-3$, then Proposition 3.2 implies the desired bound immediately.

Case 1. Assume that $\delta = n-2 \ge k$.

Subcase 1.1. Assume that $n-2 = \delta < \Delta(G) = \Delta = n-1$. If δ is odd and k is even, then it follows from Proposition 3.4 that $d_{st}^k(G) \le (k\delta)/(k+1) < n-2$ and thus $d_{st}^k(G) \le n-3$. If δ and k are even, then $\Delta = \delta+1 = n-1$ is odd. If $d(v) = \Delta$, then we deduce from Proposition 3.3 that

$$d_{st}^k(G) \le \frac{k}{k+1} \cdot d(v) = \frac{k}{k+1}(n-1) < n-2$$

when $k < n-2$ and so $d_{st}^k(G) \le n-3$ in that case. If $k = n-2$, then Proposition 3.1 leads to $d_{st}^k(G) = 1 \le n-3$. If δ is even and k is odd, then again Proposition 3.4 yields to $d_{st}^k(G) \le n-3$. If δ and k are odd, then Δ is even. The desired bound follows as in the case that δ and k are even.

Subcase 1.2. Assume that $\delta = \Delta = n-2$. The handshaking lemma implies that n is even, and so δ is even too. If k is odd, then Proposition 3.4 shows that $d_{st}^k(G) \le (k\delta)/(k+1) < n-2$ and thus $d_{st}^k(G) \le n-3$. If k is even, then we conclude from Theorem 2.3 that

$$d_{st}^k(G) \le \frac{kn}{k+2} < n-2$$

when $k < n-2$ and so $d_{st}^k(G) \le n-3$ in that case. If $k = n-2$, then Proposition 3.1 leads to $d_{st}^k(G) = 1 \le n-3$.

Case 2. Assume that $\delta = n-1 \ge k$.

Subcase 2.1. Assume that n is even. Then $\delta = n-1$ is odd.

If k is even, then Proposition 3.4 shows that

$$d_{st}^k(G) \le \frac{k}{k+1} \cdot \delta = \frac{k}{k+1}(n-1) < n-2$$

when $k < n-2$ and so $d_{st}^k(G) \le n-3$ in that case. If $k = n-2$, then $d_{st}^k(G) = 1 \le n-3$ by Proposition 3.1. As k is even, $k = n-1$ is not possible.

If k is odd, then it follows from Theorem 2.4 that

$$d_{st}^k(G) \le \frac{k}{k+1} \cdot n < n-1$$

when $k < n-1$ and so $d_{st}^k(G) \le n-2$ when $k < n-1$. However, if $d_{st}^k(G) = n-2$, then Proposition 3.5 leads to the contradiction

$$n-2 = d_{st}^k(G) \le \frac{k-1}{k} \cdot \delta = \frac{k-1}{k}(n-1) < n-2$$

when $k < n - 1$. Consequently, $d_{st}^k(G) \leq n - 3$ when $k < n - 1$. In the case $k = n - 1$, Proposition 3.1 yields to the desired bound.

Subcase 2.2. Assume that n is odd. Then $\delta = n - 1$ is even.

If k is odd, then it follows from Proposition 3.4 that

$$d_{st}^k(G) \leq \frac{k}{k+1} \cdot \delta = \frac{k}{k+1}(n-1) < n-2$$

when $k < n - 2$ and so $d_{st}^k(G) \leq n - 3$ in that case. If $k = n - 2$, then $d_{st}^k(G) = 1 \leq n - 3$, according to Proposition 3.1. As k is odd, $k = n - 1$ is not possible.

If k is even, then we obtain by Theorem 2.1 that

$$d_{st}^k(G) \leq \frac{k}{k+1} \cdot n < n - 1$$

when $k < n-1$ and so $d_{st}^k(G) \leq n-2$ when $k < n-1$. However, if $d_{st}^k(G) = n-2$, then Proposition 3.5 leads to the contradiction

$$n - 2 = d_{st}^k(G) \leq \frac{k-1}{k} \cdot \delta = \frac{k-1}{k}(n-1) < n-2$$

when $k < n-1$. Therefore $d_{st}^k(G) \leq n-3$ when $k < n-1$. In the case $k = n-1$, again Proposition 3.1 yields to the desired bound. ■

Example 3.8. Let $n \geq 4$ be an integer. On the one hand it follows from Proposition 3.6 that $d_{st}^{n-3}(K_n) \geq n - 3$. On the other hand, Theorem 3.7 implies that $d_{st}^{n-3}(K_n) \leq n - 3$, and therefore we have $d_{st}^{n-3}(K_n) = n - 3$.

This example demonstrates that Theorem 3.7 is sharp. Next we present some further examples with equality in the bound of Theorem 3.7.

Example 3.9. 1. Let $\{x_1, x_2, \ldots, x_6\}$ be the vertex set of the complete graph K_6, and let $f_i : V(K_6) \longrightarrow \{-1, 1\}$ such that
$f_1(x_1) = f_1(x_2) = -1$ and $f_1(x) = 1$ otherwise,
$f_2(x_3) = f_2(x_4) = -1$ and $f_2(x) = 1$ otherwise, and
$f_3(x_5) = f_3(x_6) = -1$ and $f_3(x) = 1$ otherwise.

It follows that $\sum_{x \in N(y)} f_i(x) \geq 1$ for each vertex $y \in V(K_6)$ and $i = 1, 2, 3$ and $f_1(x) + f_2(x) + f_3(x) = 1$ for each vertex $x \in V(K_6)$. Therefore $d_{st}^1(K_6) \geq 3$ and Theorem 3.7 yields to $d_{st}^1(K_6) = 3 = n - 3$.

2. Let $\{x_1, x_2, \ldots, x_9\}$ be the vertex set of the complete graph K_9, and let $f_i : V(K_9) \longrightarrow \{-1, 1\}$ such that
$f_1(x_1) = f_1(x_2) = f_1(x_3) = -1$ and $f_1(x) = 1$ otherwise,
$f_2(x_4) = f_2(x_5) = f_2(x_6) = -1$ and $f_2(x) = 1$ otherwise,
$f_3(x_7) = f_3(x_8) = f_3(x_9) = -1$ and $f_3(x) = 1$ otherwise,

$f_4(x_2) = f_4(x_3) = f_4(x_4) = -1$ and $f_4(x) = 1$ otherwise,
$f_5(x_5) = f_5(x_6) = f_5(x_7) = -1$ and $f_5(x) = 1$ otherwise, and
$f_6(x_8) = f_6(x_9) = f_6(x_1) = -1$ and $f_6(x) = 1$ otherwise.

Then $\sum_{x\in N(y)} f_i(x) \ge 2$ for each vertex $y \in V(K_9)$ and $i = 1, 2, \ldots, 6$ and $\sum_{i=1}^{6} f_i(x) = 2$ for each vertex $x \in V(K_9)$. Therefore $d_{st}^2(K_9) \ge 6$ and Theorem 3.7 implies that $d_{st}^2(K_9) = 6 = n - 3$.

3. Let $\{x_1, x_2, \ldots, x_{12}\}$ be the vertex set of the complete graph K_{12}, and let $f_i : V(K_{12}) \longrightarrow \{-1, 1\}$ such that
$f_1(x_1) = f_1(x_2) = f_1(x_3) = f_1(x_4) = -1$ and $f_1(x) = 1$ otherwise,
$f_2(x_5) = f_2(x_6) = f_2(x_7) = f_2(x_8) = -1$ and $f_2(x) = 1$ otherwise,
$f_3(x_9) = f_3(x_{10}) = f_3(x_{11}) = f_3(x_{12}) = -1$ and $f_3(x) = 1$ otherwise,
$f_4(x_2) = f_4(x_3) = f_4(x_4) = f_4(x_5) = -1$ and $f_4(x) = 1$ otherwise,
$f_5(x_6) = f_5(x_7) = f_5(x_8) = f_5(x_9) = -1$ and $f_5(x) = 1$ otherwise,
$f_6(x_{10}) = f_6(x_{11}) = f_6(x_{12}) = f_6(x_1) = -1$ and $f_6(x) = 1$ otherwise,
$f_7(x_3) = f_7(x_4) = f_7(x_5) = f_7(x_6) = -1$ and $f_7(x) = 1$ otherwise,
$f_8(x_7) = f_8(x_8) = f_8(x_9) = f_8(x_{10}) = -1$ and $f_8(x) = 1$ otherwise, and
$f_9(x_{11}) = f_9(x_{12}) = f_9(x_1) = f_9(x_2) = -1$ and $f_9(x) = 1$ otherwise.

So $\sum_{x\in N(y)} f_i(x) \ge 3$ for each vertex $y \in V(K_{12})$ and $i = 1, 2, \ldots, 9$ and $\sum_{i=1}^{9} f_i(x) = 3$ for each vertex $x \in V(K_{12})$. Therefore $d_{st}^3(K_{12}) \ge 9$ and Theorem 3.7 leads to $d_{st}^3(K_{12}) = 9 = n - 3$.

Example 3.9 leads us to a more general result.

Example 3.10. If $k \ge 1$ is an integer and $n = 3(k + 1)$, then $d_{st}^k(K_n) = n - 3$.

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be the vertex set of the complete graph K_n, and let $f_i : V(K_n) \longrightarrow \{-1, 1\}$ such that
$f_1(x_1) = f_1(x_2) = \cdots = f_1(x_{k+1}) = -1$ and $f_1(x) = 1$ otherwise,
$f_2(x_{k+2}) = f_2(x_{k+3}) = \cdots = f_2(x_{2k+2}) = -1$ and $f_2(x) = 1$ otherwise,
$f_3(x_{2k+3}) = f_3(x_{2k+4}) = \cdots = f_3(x_{3k+3}) = -1$ and $f_3(x) = 1$ otherwise,
$f_4(x_2) = f_4(x_3) = \cdots = f_4(x_{k+2}) = -1$ and $f_4(x) = 1$ otherwise,
. . . .
$f_{3k-2}(x_k) = f_{3k-2}(x_{k+1}) = \cdots = f_{3k-2}(x_{2k}) = -1$ and $f_{3k-2}(x) = 1$ otherwise,
$f_{3k-1}(x_{2k+1}) = f_{3k-1}(x_{2k+2}) = \cdots = f_{3k-1}(x_{3k+1}) = -1$ and $f_{3k-1}(x) = 1$ otherwise,
$f_{3k}(x_{3k+2}) = f_{3k}(x_{3k+3}) = f_{3k}(x_1) = \cdots = f_{3k}(x_{k-1}) = -1$ and $f_{3k}(x) = 1$ otherwise.

It is straightforward to verify that $\sum_{x\in N(y)} f_i(x) \ge k$ for each vertex $y \in V(K_n)$ and $i = 1, 2, \ldots, 3k$ and $\sum_{i=1}^{3k} f_i(x) = k$ for each vertex $x \in V(K_n)$. Therefore $d_{st}^k(K_n) \ge 3k$ and thus it follows from Theorem 3.7 that $d_{st}^k(K_n) = 3k = n - 3$. ■

Notice that Example 3.10 also demonstrates that Theorems 2.1 and 2.4 are sharp. Finally, we give some examples of non-complete graphs with equality in the inequality of Theorem 3.7.

Example 3.11. Let u, v and w be three distinct vertices of the complete graph K_n.

1. Let $G_5 = K_5 - uv$, and let $f_i : V(G_5) \longrightarrow \{-1, 1\}$ such that
$f_1(u) = -1$ and $f_1(x) = 1$ for $x \neq u$ and $f_2(x) = 1$ for each $x \in V(G_5)$.

Then it is easy to see that $\sum_{x \in N(y)} f_i(x) \geq 2$ for each vertex $y \in V(G_5)$ and $i = 1, 2$ and $f_1(x) + f_2(x) \leq 2$ for each vertex $x \in V(G_5)$. Therefore $d_{st}^2(G_5) \geq 2$ and Theorem 3.7 shows that $d_{st}^2(G_5) = 2 = n - 3$.

2. Let $G_6 = K_6 - uv$, and let $f_i : V(G_6) \longrightarrow \{-1, 1\}$ such that
$f_1(u) = -1$ and $f_1(x) = 1$ for $x \neq u$,
$f_2(v) = -1$ and $f_2(x) = 1$ for $x \neq v$
and $f_3(x) = 1$ for each $x \in V(G_6)$.

Then $\sum_{x \in N(y)} f_i(x) \geq 3$ for each vertex $y \in V(G_6)$ and $i = 1, 2, 3$ and $f_1(x) + f_2(x) + f_3(x) \leq 3$ for each vertex $x \in V(G_6)$. Therefore $d_{st}^3(G_6) \geq 3$ and Theorem 3.7 leads to $d_{st}^3(G_6) = 3 = n - 3$.

3. Let $G_7 = K_7 - \{uv, uw, vw\}$, and let $f_i : V(G_7) \longrightarrow \{-1, 1\}$ such that
$f_1(u) = -1$ and $f_1(x) = 1$ for $x \neq u$,
$f_2(v) = -1$ and $f_2(x) = 1$ for $x \neq v$,
$f_3(w) = -1$ and $f_3(x) = 1$ for $x \neq w$
and $f_4(x) = 1$ for each $x \in V(G_7)$.

Then $\sum_{x \in N(y)} f_i(x) \geq 4$ for each vertex $y \in V(G_7)$ and $i = 1, 2, 3, 4$ and $f_1(x)+f_2(x)+f_3(x)+f_4(x) \leq 4$ for each vertex $x \in V(G_7)$. Therefore $d_{st}^4(G_7) \geq 4$ and Theorem 3.7 shows that $d_{st}^4(G_7) = 4 = n - 3$.

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