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# DECOMPOSITION OF COMPLETE MULTIGRAPHS INTO STARS AND CYCLES

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**Abstract**

Let $k$ be a positive integer, $S_k$ and $C_k$ denote, respectively, a star and a cycle of $k$ edges. $\lambda K_n$ is the usual notation for the complete multigraph on $n$ vertices and in which every edge is taken $\lambda$ times. In this paper, we investigate necessary and sufficient conditions for the existence of the decomposition of $\lambda K_n$ into edges disjoint of stars $S_k$'s and cycles $C_k$'s.

**Keywords:** graph decomposition, complete multigraph, stars, cycles.

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## 1. Introduction

All graphs considered in this paper are finite and undirected, with no loops. Let $G$, $H$ and $F$ be three graphs. An $H$-decomposition of $G$ is a partition of the edge set of $G$ into copies of $H$. If $G$ has an $H$-decomposition, we say that $G$ is $H$-decomposable. An $(H, F)$-decomposition of $G$ is a partition of the edge set of $G$ into copies of $H$ and $F$ using at least one copy of each. If $G$ has an $(H, F)$-decomposition, we say that $G$ is $(H, F)$-decomposable (or $(H, F)$-multidecomposable).

In [20], Wilson stated his fundamental theorem on the existence of an $H$-decomposition of the complete graph $K_n$ for any fixed $H$ as long as the number of edges of $K_n$ is divisible by the number of edges of $H$ and $n$ is large enough. Since then, decomposition problems became an active research area. There have been several important research papers relating to various decompositions of different graphs. For example, the problem of the $H$-decomposition has been widely studied such as the decomposition of bipartite graphs into closed trails [8] and also the decomposition of complete multigraphs into crowns [10], paths [19], or cycles [7]. Moreover, the multidecomposition problems were also considered by several studies such as the multidecomposition of complete graph into cycles and stars [15] or paths and cycles [14, 13]. Another multidecomposition of bipartite graphs into subgraphs was considered in [16, 9].

Let $K_n$ be a complete graph of order $n$ and let $\lambda$ be a positive integer. We denote by $\lambda K_n$ the complete multigraph obtained by replacing each edge of $K_n$ by $\lambda$ parallel edges that have the same end-nodes. In [1, 2], Abueida and Daven gave necessary and sufficient conditions for decomposing $K_n$ into cycles of $k$ edges and stars of $k-1$ edges, for $k=4$ and $k=5$. Abueida and O'Neil [3] extended this decomposition for the complete multigraph $\lambda K_n$ when $k=3,4,5$, and they conjectured the result for any integer $k \geq 3$ and $n \geq k$. In [11], Priyadharsini and Muthusamy showed the above conjecture to be true for $n=k$.

More recently, Abueida and Lian [4] gave necessary and sufficient conditions for decomposing $K_n$ into cycles and stars of $k$ edges, for $n \geq 4k$ and $k$ even or $n$ odd. In our paper, we improve results on this decomposition and we extend it for the complete multigraph $\lambda K_n$. Thus, we present necessary and sufficient conditions for different cases as follows:

- $k$ is prime,
- $k$ divides either $n-1$, $n$ or $\lambda$,
- $n \geq 2k$ and $\lambda$ is even or $\gcd(\lambda, k) = 1$,
- $n \geq 4k$, independently of the parity of $n$ or $k$, thus improving the result of Abueida and Lian [4].

## 2. Preliminaries

### 2.1. Related works

We introduce here some results on a $C_k$-decomposition an dan $S_k$-decomposition that are useful for our proofs.

**Theorem 1** [21]**.** *A necessary and sufficient condition for the existence of an $S_k$-decomposition of $\lambda K_n$ is that:*

- $\lambda n(n-1) \equiv 0\,[2k]$,
- $n \geq 2k$ *for* $\lambda = 1$,

- *$n \geq k+1$ for even $\lambda$,*
- *$n \geq k+1+k/\lambda$ for odd $\lambda \geq 3$.*

**Theorem 2** [6]. *Let $\lambda$, $n$ and $k$ be integers with $n,k \geq 3$ and $\lambda \geq 1$. There exists a decomposition of $\lambda K_n$ into cycles of $k$ edges if and only if $k \leq n$, $\lambda\ (n-1)$ is even and $k$ divides $\lambda n(n-1)/2$. There exists a decomposition of $\lambda K_n$ into cycles of $k$ edges and a perfect matching if and only if $k \leq n$, $\lambda\ (n-1)$ is odd and $k$ divides $\lambda n(n-1)/2-(n/2)$.*

**Theorem 3** ([5],[12]). *Let $n$ and $k$ be positive integers. $K_n$ has a $C_k$-decomposition if and only if $n$ is odd, $3 \leq k \leq n$, and $n(n-1) \equiv 0\,[2k]$.*

**Theorem 4** [21]. *Let $m$ and $n$ be integers with $m \geq n \geq 1$. Then $K_{m,n}$ is $S_k$-decomposable if and only if $m \geq k$ and $m \equiv 0\,[k]$ if $n < k$, $mn \equiv 0\,[k]$ if $n \geq k$.*

**Theorem 5** [17]. *For positive integers $m$, $n$, and $k$, the graph $K_{m,n}$ is $C_k$-decomposable if and only if $m$, $n$, and $k$ are even, $k \geq 4$, $\min\{m,n\} \geq k/2$, and $mn \equiv 0\,[k]$.*

### 2.2. Introductory results

Let $G$ be a graph. The order of $G$ is the cardinality of its vertex set and the size of the graph $G$ is the cardinality of its edge set. We begin with the following lemma to prove the necessary conditions when $\lambda K_n$ is $(S_k, C_k)$-decomposable.

**Lemma 6.** *Let $n \geq 3$ and $\lambda > 1$ be positive integers. If $\lambda K_n$ is $(S_k, C_k)$-decomposable, then $2 \leq k \leq n-1$ and $\lambda n(n-1) \equiv 0\,[2k]$.*

***Proof.*** Since the minimum length of a cycle and the maximum size of a star in $\lambda K_n$ are, respectively, 2 and $n-1$, so $2 \leq k \leq n-1$ is necessary. Since $\lambda K_n$ has $\lambda n(n-1)/2$ edges and each subgraph in a $(C_k, S_k)$-decomposition has $k$ edges, $k$ has to divide $\lambda n(n-1)/2$. ∎

As an introduction result, we show in the next proposition that the necessary conditions in Lemma 6 of the $(C_k, S_k)$-decomposition of $\lambda K_n$ are also sufficient in the special case when $k = 4$.

**Proposition 7.** *Let $n > 4$ and $\lambda > 1$ be positive integers. There exists a $(C_4, S_4)$-decomposition if and only if $\lambda n(n-1)/2 \equiv 0\,[4]$.*

***Proof.*** We distinguish two cases according to the parity of $\lambda$.

*Case* 1. $\lambda$ is odd. Since $\lambda n(n-1)/2 \equiv 0\,[4]$ and $\lambda$ is odd by assumption, $n(n-1) \equiv 0\,[8]$. We have two subcases.

*Subcase* 1a. $n$ is even. Since $n(n-1) \equiv 0\,[8]$ and $n$ is even, we obtain $n \equiv 0\,[8]$. Let $n = 8\alpha$ with $\alpha \geq 1$. Then $\lambda K_n$ can be decomposed into disjoint union of $\alpha$ copies of $\lambda K_8$ and disjoint union of $\alpha(\alpha-1)/2$ copies of $\lambda K_{8,8}$. Every $\lambda K_{8,8}$ can be decomposed into $S_4$ using Theorem 4. We now decompose each $\lambda K_8$ into $C_4$'s and $S_4$'s as follows. Note that $\lambda K_8 = K_8 \cup (\lambda - 1)K_8$. Since Theorem 1 implies that $K_8$ is $S_4$-decomposable and Theorem 2 guarantees that $(\lambda - 1)K_8$ is $C_4$-decomposable, we have $\lambda K_8$ is $(C_4, S_4)$-decomposable. Thus, $\lambda K_n$ is $(C_4, S_4)$-decomposable.

*Subcase* 1b. $n$ is odd. Since $n$ is odd and $n(n-1) \equiv 0\,[8]$ by assumption, $n - 1 \equiv 0\,[8]$. Let $n - 1 = 8\alpha$. Since the degree of each vertex of $\lambda K_n$ equals to $\lambda(n-1)$ and is divisible by 4, we take one vertex and decompose all its incident edges into $2\lambda\alpha$ stars of 4 edges. The remaining graph is $\lambda K_{n-1}$ with $n - 1 = 8\alpha$. In this case, we use the same method as in the previous Subcase 1a for $\lambda K_n$ with $n = 8\alpha$.

*Case* 2. $\lambda$ is even. Recall that $n > 4$. We give the $(C_4, S_4)$-decomposition of $\lambda K_n$ as follows according to values of $n$.

$n = 5$: Note that $\lambda K_5 = \lambda S_4 \cup \lambda K_4$. Since $\lambda$ is even we decompose $\lambda K_4$ into $C_4$, by Theorem 2. Thus, $\lambda K_5$ is $(C_4, S_4)$-decomposable.

$n = 6$ or $n = 7$: We have $n(n-1) \equiv 0\,[2]$ and $\lambda n(n-1) \equiv 0\,[8]$ by assumption. Consequently, $\lambda \equiv 0\,[4]$. Then we take incident edges of one vertex and decompose them into $S_4$'s. The remaining graph is either $\lambda K_5$ when $n = 6$ or $\lambda K_6$ when $n = 7$. Both remaining graphs are $C_4$-decomposable using Theorem 2.

$n = 8$: Since $\lambda$ is even, $\lambda K_8$ can be written as the disjoint union of $2K_8$'s. Now we give the $(C_4, S_4)$-decomposition of $2K_8$. Each $2K_4$ is decomposed into $C_4$'s by Theorem 2 and $2K_{4,4}$ is decomposed into $S_4$'s using Theorem 4. Since each $2K_8$ is $(C_4, S_4)$-decomposable we have $\lambda K_8$ is also $(C_4, S_4)$-decomposable.

$n \geq 9$: Note that $\lambda K_n = \lambda K_4 \cup \lambda K_{n-4} \cup \lambda K_{4,n-4}$. Observe that $|E(\lambda K_4)|$ and $|E(\lambda K_{4,n-4})|$ are divisible by 4. By assumption $|E(K_n)|$ is a multiple of 4, so $|E(\lambda K_{n-4})|$ is also a multiple of 4. We decompose $\lambda K_4$ into cycles of 4 edges using Theorem 2 with $\lambda$ even. $\lambda K_{4,n-4}$ is $S_4$-decomposable using Theorem 4. Since $\lambda$ is even, we decompose $\lambda K_{n-4}$ into $C_4$ using Theorem 2. Thus, we conclude that $\lambda K_n$ is $(S_4, C_4)$-decomposable. ∎

## 3. Decomposition of $\lambda K_n$ When $n \geq 4k$ or $n \geq 2k$ and $\lambda$ Even

In this section, we prove some lemmas and theorems, each of them treating a special case of decomposition of $\lambda K_n$ into $S_k$'s and $C_k$'s.

The next proposition proves that $\lambda K_n$ is $(S_k, C_k)$-decomposable for all $n \geq 4k$ and $\lambda = 1$, so we complete the missing cases in [4] when $n \geq 4k$.

**Proposition 8.** *Let $n$ and $k$ be positive integers such that $n \geq 4k$ and $n(n-1)/2 \equiv 0\,[k]$. Then the graph $K_n$ is $(S_k, C_k)$-decomposable.*

***Proof.*** Let $n = qk + r$, where $q$ and $r$ are integers with $0 \leq r < k$ and $q \geq 4$. Note that $K_n = K_{qk+r} = K_{2k} \cup K_{(q-2)k+r} \cup K_{2k,(q-2)k+r}$.

Clearly, $|E(K_{2k})|$ and $|E(K_{2k,(q-2)k+r})|$ are multiples of $k$. Thus $((q-2)k + r)((q-2)k + r - 1)/2$ is also a multiple of $k$. We distinguish two cases according to the parity of $k$.

*Case* 1. $k$ is odd. It follows that $K_{(q-2)k+r}$ is $S_k$-decomposable by Theorem 1, since $(q-2)k + r \geq 2k$, and $K_{2k,(q-2)k+r}$ is also $S_k$-decomposable by Theorem 4.

We write $K_{2k} = K_k \cup K_k \cup K_{k,k}$. Now, it is clear that each copy of $K_k$ is $C_k$-decomposable when $k$ is odd by Theorem 3, and $K_{k,k}$ is $S_k$-decomposable by Theorem 4.

*Case* 2. $k$ is even. In this case, $K_{2k}$ is $S_k$-decomposable by Theorem 1. If $n$ is even, then $(q-2)k + r$ is even. So, we can decompose $K_{2k,(q-2)k+r}$ into $C_k$ using Theorem 5. Since $q \geq 4$, $(q-2)k + r \geq 2k$. Consequently, $K_{(q-2)k+r}$ is $S_k$-decomposable by Theorem 1. Conversely, if $n$ is odd, then $(q-2)k + r$ is odd. Using Theorem 3, $K_{(q-2)k+r}$ can be decomposed into cycles of $k$ edges, and $K_{2k,(q-2)k+r}$ is $S_k$-decomposable by Theorem 4. Thus, we conclude that $\lambda K_n$ is $(S_k, C_k)$-decomposable when $\lambda = 1$. ■

In the rest of this section, we will focus on complete multigraph $\lambda K_n$, where $\lambda > 1$. The following lemma gives sufficient conditions for decomposing $\lambda K_n$ into $C_k$'s and $S_k$'s, where $\lambda > 1$ is odd and $n \geq 4k$.

**Lemma 9.** *Let $n$, $k$ and $\lambda > 1$ be positive integers such that $n \geq 4k$ and $\lambda$ is odd. If $\lambda n(n-1)/2 \equiv 0\,[k]$, then $\lambda K_n$ is $(C_k, S_k)$-decomposable.*

***Proof.*** Let $n = qk + r$, where $q$ and $r$ are integers with $0 \leq r < k$ and $q \geq 4$. Note that

$$\begin{aligned}\lambda K_n &= \lambda K_{qk+r} = \lambda K_{2k} \cup \lambda K_{(q-2)k+r} \cup \lambda K_{2k,(q-2)k+r}\\ &= (\lambda - 1)K_{2k} \cup K_{2k} \cup \lambda K_{(q-2)k+r} \cup \lambda K_{2k,(q-2)k+r}.\end{aligned}$$

$|E(\lambda K_{2k})|$ and $|E(\lambda K_{2k,(q-2)k+r})|$ are multiples of $k$. Using argument that $|E(\lambda K_n)|$ is a multiple of $k$, i.e., $\lambda n(n-1)$ is divisible by $k$, we obtain $\lambda((q-2)k + r)((q-2)k + r - 1)/2 \equiv 0\,[k]$. Since $(\lambda - 1)(2k - 1)$ is even and $2k \geq k$ we have $(\lambda - 1)K_{2k}$ is $C_k$-decomposable by Theorem 2. Theorem 1 for $K_{2k}$ with $\lambda = 1$ implies that $K_{2k}$ is $S_k$-decomposable. We now decompose $\lambda K_{(q-2)k+r}$.

We have $q \geq 4$. Then $(q-2)k + r \geq 2k + r$ implies that $(q-2)k + r \geq 2k \geq 3k/2 + 1$ for any $k \geq 2$. Given that $\lambda \geq 2$ we obtain $3k/2 + 1 \geq k + 1 + k/\lambda$, so $(q-2)k+r \geq k+1+k/\lambda$. Using Theorem 1 when $\lambda$ is odd, since $(q-2)k+r \geq k+$

$1+k/\lambda$, we have $\lambda K_{(q-2)k+r}$ is $S_k$-decomposable. Note that $\lambda K_{2k,(q-2)k+r}$ can be decomposed into $\lambda$ copies of $K_{2k,(q-2)k+r}$. Since $K_{2k,(q-2)k+r}$ is $S_k$-decomposable by Theorem 4, so is $\lambda K_{2k,(q-2)k+r}$. Thus $\lambda K_n$ is $(C_k, S_k)$-decomposable. ∎

In the following lemmas, we will give sufficient conditions of the decomposition of $\lambda K_n$ into $C_k$'s and $S_k$'s, where $n \geq 2k$ and $\lambda$ is even or $\gcd(\lambda, k) = 1$.

**Lemma 10.** *Let $n$, $k$ and $\lambda$ be positive integers such that $\lambda$ is even. For all $n \geq 2k$, if $\lambda n(n-1)/2 \equiv 0\,[k]$, then $\lambda K_n$ is $(C_k, S_k)$-decomposable.*

***Proof.*** Let $n = qk + r$, where $q$ and $r$ are integers with $0 \leq r < k$ and $q \geq 2$. Note that $\lambda K_n = \lambda K_{qk+r} = \lambda K_{(q-1)k} \cup \lambda K_{k+r} \cup \lambda K_{(q-1)k,k+r}$.

Obviously, $|E(\lambda K_{(q-1)k})|$ and $|E(\lambda K_{(q-1)k,k+r})|$ are multiples of $k$. Thus, $\lambda(k+r)(k+r-1)/2 \equiv 0\,[k]$ from the assumption that $\lambda n(n-1)/2$ is divisible by $k$. $\lambda K_{(q-1)k}$ and $\lambda K_{k+r}$ are $C_k$-decomposable by Theorem 2 because $\lambda$ is even, $(q-1)k \geq k$ and $k + r \geq k$ by assumption. Note that $\lambda K_{(q-1)k,k+r}$ can be decomposed into $\lambda$ copies of $K_{(q-1)k,k+r}$. Since $K_{(q-1)k,k+r}$ is $S_k$-decomposable by Theorem 4, so is $\lambda K_{(q-1)k,k+r}$. Thus, $\lambda K_n$ is $(C_k, S_k)$-decomposable. ∎

**Lemma 11.** *Let $n$, $k$ and $\lambda > 1$ be positive integers such that* $\gcd(\lambda, k) = 1$. *For all $n \geq 2k$, if $\lambda n(n-1)/2 \equiv 0\,[k]$, then $\lambda K_n$ is $(C_k, S_k)$-decomposable.*

***Proof.*** From the previous lemma, we only have to examine the case when $\lambda$ is odd. We can decompose $\lambda K_n$ as an edge disjoint union of $(\lambda - 1)K_n$ and $K_n$. Since $\gcd(\lambda, k) = 1$, we have $|E(K_n)| \equiv 0\,[k]$. It is clear that $(\lambda - 1)K_n$ has a $(C_k, S_k)$-decomposition by Lemma 10. Now we decompose $K_n$ into stars of size $k$ by Theorem 1, since $n \geq 2k$. Thus $\lambda K_n$ is $(C_k, S_k)$-decomposable. ∎

Using Proposition 8 and Lemmas 9, 10 and 11, we obtain the following theorem.

**Theorem 12.** *Let $n$, $k$ and $\lambda$ be positive integers. If $\lambda n(n-1)/2 \equiv 0\,[k]$ and*

- *$n \geq 4k$, or*
- *$n \geq 2k$ and $\lambda > 1$ is even or* $\gcd(\lambda, k) = 1$,

*then $\lambda K_n$ is $(C_k, S_k)$-decomposable.*

## 4. Decomposition of $\lambda K_n$ When $k$ Is Prime or Divides Either $n-1$, $n$ or $\lambda$

One can easily check that $\lambda K_n$ is $(C_2, S_2)$-decomposable if and only if $n > 2$, $\lambda > 1$ and $\lambda n(n-1) \equiv 0\,[4]$. Thus, we admit the following lemma without proof.

**Lemma 13.** *Let $n > 2$ and $\lambda > 1$ be positive integers. There exists a decomposition of $\lambda K_n$ into copies of $S_2$ and copies of $C_2$ if and only if $\lambda n(n-1)/2$ is even.*

In Lemmas 14–16, we will show the sufficient conditions of the decomposition of $\lambda K_n$ into $C_k$'s and $S_k$'s when $n = k+1$, $n = 2k+1$ and $n = 3k+1$, respectively, with $k \geq 3$.

**Lemma 14.** *Let $n = k+1$, $\lambda > 1$ and $k \geq 3$ be positive integers. There exists a decomposition of $\lambda K_n$ into copies of $S_k$ and $C_k$ if and only if $\lambda k(k-1)/2 \equiv 0\,[k]$.*

***Proof.*** We split the proof into two cases as follows.

*Case* 1. $k$ is odd or $\lambda$ is even. By assumption, $n = k+1$ and the degree of each vertex of $\lambda K_n$ is $\lambda k$. We use one vertex in order to construct $\lambda$ stars of $k$ edges. The remaining graph is $\lambda K_{n-1}$. Since $k$ is odd or $\lambda$ is even and we have $n-1 = k$, we obtain $\lambda(n-2) = \lambda(k-1)$ is always even and $\lambda k(k-1)/2 \equiv 0\,[k]$, so by Theorem 2 $\lambda K_{n-1}$ is $C_k$-decomposable. Thus, $\lambda K_n$ is $(S_k, C_k)$-decomposable.

*Case* 2. $k$ is even and $\lambda$ is odd. This subcase does not exist because by assumption $\lambda k(k-1)/2 \equiv 0\,[k]$, which implies $\lambda(k-1)$ to be even, a contradiction. The opposite implication is easy to prove. ∎

**Lemma 15.** *Let $n = 2k+1$ and $\lambda > 1$ be positive integers, and let $k$ be a positive integer, $k \geq 3$. There exists a decomposition of $\lambda K_n$ into copies of $S_k$ and $C_k$ for any $k$.*

***Proof.*** The number of edges in $\lambda K_{2k+1}$, $\lambda k(2k+1)$, is a multiple of $k$. We decompose $\lambda K_{2k+1}$ as follows : $\lambda K_{2k+1} = (\lambda-1)K_{2k+1} \cup K_{2k+1}$. Clearly, $|E((\lambda-1)K_{2k+1})|$ and $|E(K_{2k+1})|$ are multiples of $k$. We decompose $(\lambda-1)K_{2k+1}$ into $C_k$'s and $K_{2k+1}$ into $S_k$'s. Hence $\lambda K_{2k+1}$ is $(S_k, C_k)$-decomposable. ∎

**Lemma 16.** *Let $n = 3k+1$, $\lambda > 1$ and $k \geq 3$ be positive integers. There exists a decomposition of $\lambda K_n$ into copies of $S_k$ and $C_k$ if and only if $3\lambda k(3k-1)/2 \equiv 0\,[k]$.*

***Proof.*** We split the proof into two cases as follows:

*Case* 1. $\lambda$ is even. This case is solved by Lemma 10.

*Case* 2. $\lambda$ is odd. If $k$ is odd, then $\lambda K_{3k+1} = \lambda K_{2k+1} \cup \lambda K_k \cup \lambda K_{2k+1,k}$. By Lemma 15, $\lambda K_{2k+1}$ is $(S_k, C_k)$-decomposable. $\lambda K_k$ can be decomposed into $C_k$'s, and $\lambda K_{2k+1,k}$ is $S_k$-decomposable.

If $k$ is even, then $3\lambda(3k+1)$ is not even, so this case cannot exist. The opposite implication is easy to prove. ∎

In the following proposition, we prove that for any $k$ that divides $n$ or $n-1$, $\lambda K_n$ is $(S_k, C_k)$-decomposable.

**Proposition 17.** *For integers $k, n$ and $\lambda$ with $\lambda > 1$ and $2 \leq k \leq n+1$, if $n \equiv 0, 1\,[k]$ and $\lambda n(n-1)/2 \equiv 0\,[k]$, then $\lambda K_n$ is $(S_k, C_k)$-decomposable.*

***Proof.*** In the case when $n = k+1$, $n = 2k+1$ and $n = 3k+1$ we use Lemmas 13, 14, 15 and 16, respectively.

By Theorem 12, if $n = \alpha k + 1$ or $n = \alpha k$ with $\alpha \geq 4$, then $\lambda K_n$ is $(S_k, C_k)$-decomposable. To complete the proof, we study the cases when $n = 2k$ and $n = 3k$.

$n = 2k$: When $\lambda$ is even, $\lambda K_n$ is $(S_k, C_k)$-decomposable by Lemma 10. When $\lambda$ is odd, observe that $\lambda K_{2k} = (\lambda - 1)K_{2k} \cup K_{2k}$. $(\lambda - 1)K_{2k}$ is $C_k$-decomposable by Theorem 2 and $K_{2k}$ is $S_k$-decomposable by Theorem 1.

$n = 3k$: If $\lambda$ is even, then we have $\lambda K_n$ is $(S_k, C_k)$-decomposable by Lemma 10. If $\lambda$ is odd and $k$ is odd, then $\lambda K_{3k} = (\lambda - 1)K_{3k} \cup K_{3k}$, since $|E(K_{3k})|$ and $|E((\lambda - 1)K_{3k})|$ are multiples of $k$. Thus, $(\lambda - 1)K_{3k}$ is $C_k$-decomposable by Theorem 2, and $K_{3k}$ is $S_k$-decomposable by Theorem 1. On the other hand, if $\lambda$ is odd and $k$ is even, then it is sufficient to show that $\lambda 3k(3k-1) \equiv 0\,[2k]$ is not true in this case. So, when $\lambda$ is odd, $k$ must be also odd. ■

In the following proposition, we will show the decomposition of $\lambda K_n$ into $S_k$'s and $C_k$'s when $\lambda$ is a multiple of $k$.

**Proposition 18.** *For integers $k$ and $n$ with $2 \leq k \leq n-1$, if $\lambda \equiv 0\,[k]$, then $\lambda K_n$ is $(S_k, C_k)$-decomposable.*

***Proof.*** Since $n \geq k+1$, we distinguish two cases.

*Case* 1. $n \geq k+2$. $\lambda \equiv 0\,[k]$ implies that the degree of each vertex of $\lambda K_n$ is multiple of $k$. Thus, we can construct stars $S_k$ using each vertex of the multigraph. First we decompose incident edges of some vertex into $S_k$'s in a circular manner as illustrated in Example 19. This process is repeated until the remaining graph is a $\lambda K_m$, where $m = k+1$ if $k$ is even, and $m = k$ if $k$ is odd.

If $k$ is odd, then the remaining graph is $\lambda K_k$ and $\lambda k(k-1)/2 \equiv 0\,[k]$, which implies that $\lambda K_k$ can be decomposed into cycles of size $k$ by Theorem 2. If $k$ is even, then the remaining graph is $\lambda K_{k+1}$, which has $\lambda k(k+1)/2$ edges. Thus the number of edges is divisible by $k$. Since $\lambda k$ is even, the graph $\lambda K_{k+1}$ can be decomposed into cycles of size $k$ by Theorem 2. Hence, $\lambda K_n$ is $(S_k, C_k)$-decomposable.

*Case* 2. $n = k+1$. Since the degree of each vertex is $\lambda k$, we decompose the incident edges of one vertex into $\lambda$ copies of $S_k$. The remaining graph is $\lambda K_k$. By assumption, $\lambda K_k$ has number of edges divisible by $k$, which implies that $\lambda k(k-1)/2 \equiv 0\,[k]$. Since $\lambda(k-1)$ is even, we decompose $\lambda K_k$ into copies of $C_k$ using Theorem 2. ■

Example 19, illustrated by Figure 1, shows how Proposition 18 is applied to a graph $3K_5$.

**Example 19.** $(S_3, C_3)$-decomposition of a graph $\lambda K_n$ with $n = 5$ and $\lambda = 3$ is as follows.

- Consider the graph $3K_5$. Since $\lambda \equiv 0\,[3]$ we have $|E(3K_5)| \equiv 0\,[k]$.
- Taking on a vertex of $3K_5$ called $v$, we decompose the graph into $\lambda(n-1)/k = 4$ stars by rotation. This rotation is applied to all the incident edges of the considered node $v$ (Figure 1 illustrates rotation construction).
- The remaining graph is $3K_4$. The same rotation construction is applied to finding $\lambda(n-2)/k = 3$ stars. This rotation construction is applied until the remaining graph is $3K_k (k = 3)$.
- The remaining graph $3K_3$ can be decomposed into 3 copies of $C_3$.

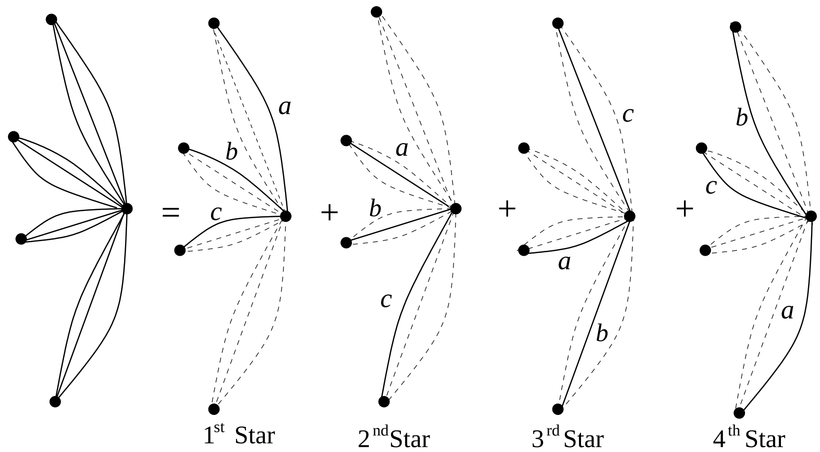


Figure 1. Rotation construction of stars. The edges of a star are labeled by $a$, $b$ and $c$.

In the following corrolary, we will investigate the problem of decomposing $\lambda K_n$ into $S_k$'s and $C_k$'s for each prime number $k$.

**Corollary 20.** *Let $n$ and $\lambda > 1$ be positive integers, and let $k$ be a positive prime number. There exists a $(C_k, S_k)$-decomposition of $\lambda K_n$ if and only if $n \geq k + 1$ and $\lambda n(n-1)/2 \equiv 0\,[k]$.*

***Proof.*** We show that the necessary conditions given in Lemma 6 are also sufficient. $\lambda n(n-1)/2$ is a multiple of $k$ and $k$ is a prime number, so we distinguish three cases according to the multiplicity of $n$, $n-1$ and $\lambda$.

When $k$ divides $n$ or $k$ divides $n-1$, these cases are proved in Proposition 17. When $k$ divides $\lambda$, this case is proved in Proposition 18. ∎

The following theorem is a direct consequence of Propositions 17 and 18, and Corollary 20.

**Theorem 21.** *Let $n$, $k$ and $\lambda > 1$ be positive integers. Then $\lambda K_n$ is $(S_k, C_k)$-decomposable if $\lambda n(n-1)/2 \equiv 0\,[k]$ and*

- *$k$ is prime, or*
- *$k$ divides either $n-1$, $n$ or $\lambda$.*

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