Discussiones Mathematicae
Graph Theory 35 (2015) 541–555
doi:10.7151/dmgt.1819

CHROMATIC SUMS FOR COLORINGS AVOIDING MONOCHROMATIC SUBGRAPHS[1]

Ewa Kubicka

University of Louisville, USA
e-mail: ewa@louisville.edu

Grzegorz Kubicki

University of Louisville, USA
University of Opole, Poland
e-mail: gkubicki@louisville.edu

and

Kathleen A. McKeon

Connecticut College, USA
e-mail: kamck@conncoll.edu

Abstract

Given graphs G and H, a vertex coloring $c : V(G) \to \mathbb{N}$ is an H-free coloring of G if no color class contains a subgraph isomorphic to H. The H-free chromatic number of G, $\chi(H, G)$, is the minimum number of colors in an H-free coloring of G. The H-free chromatic sum of G, $\Sigma(H, G)$, is the minimum value achieved by summing the vertex colors of each H-free coloring of G. We provide a general bound for $\Sigma(H, G)$, discuss the computational complexity of finding this parameter for different choices of H, and prove an exact formulas for some graphs G. For every integer k and for every graph H, we construct families of graphs, G_k with the property that k more colors than $\chi(H, G)$ are required to realize $\Sigma(H, G)$ for H-free colorings. More complexity results and constructions of graphs requiring extra colors are given for planar and outerplanar graphs.

Keywords: coloring, sum of colors, forbidden subgraphs.

2010 Mathematics Subject Classification: 05C15.

[1]An extended abstract was presented at the conference: Erdős Centennial, July 1-5, 2013, Budapest [20].

1. Introduction

Given a finite undirected simple graph $G = (V, E)$, a *k-coloring* is a partition of its vertex set V into independent sets $V_1, V_2, \ldots, V_k$. The *chromatic number* $\chi(G)$ is the minimum integer k for which such a partition exists. In a *proper coloring*, the subgraph $\langle V_i \rangle$ induced by each set of the partition does not contain K_2. Many generalizations of this classical coloring concept were born when this condition was relaxed. For a given graph property π, one can try to k-color G in such a way that each induced subgraph $\langle V_i \rangle$ has property π. Such generalizations were studied for different properties π: being acyclic or being a path of fixed length by Chartrand *et al.* [9], [10], and [11], being a cycle of fixed length [14, 22, 23], being a disjoint union of cliques by Albertson *et al.* [1]. A concept of generalized chromatic numbers defined in terms of partitions of V for additive hereditary properties was studied by Broere *et al.* [6]. A framework of such an approach comes from the earlier survey paper by Borowiecki *et al.* [4]. Broere and Mynhardt [5] introduced colorings in which color classes avoid a specific graph as an induced subgraph. In this paper we will use the terminology of H-free coloring adopted from paper [7] by Broersma *et al.* In fact, they consider the general problem of avoiding all subgraphs from a given family either as induced subgraphs (weakly colorings and chromatic numbers) or as noninduced subgraphs (strongly colorings and chromatic numbers). In this paper, we will consider only strongly colorings. The smallest possible number of colors in a coloring of G such that each color class does not contain noninduced H as a subgraph is called the *H-free chromatic number* and is denoted by $\chi(H, G)$. If H is a star $K_{1,d+1}$ then H-free coloring is equivalent to a d-defective coloring in which every color class induces a subgraph of maximum degree not exceeding d. This concept was introduced by Andrews and Jacobson [2] and studied, for example, by Archdeacon [3], Cowen *et al.* [12, 13], Harary and Jones [17, 18].

However, we are interested in minimizing not the number of colors but rather their sum. Given an H-free coloring c of G, the sum of colors over all vertices is $\Sigma(c) = \sum_{v \epsilon V(G)} c(v)$. The *$H$-free chromatic sum of G*, $\Sigma(H, G) = \min\{\Sigma(c)$: c is an H-free coloring of $G\}$. A coloring c for which $\Sigma(c) = \Sigma(H, G)$ is called a *best H-free coloring of G.* The concept of chromatic sum for proper colorings was introduced by Kubicka and Schwenk [19, 21]; one of the most interesting phenomena was the necessity for some graphs to use more colors than $\chi(G)$ in a best coloring. The paper by Erdős, Kubicka and Schwenk [15] provides examples of such graphs.

In the next section we observe some basic facts about H-free chromatic sum, provide a general upper bound for it and discuss some complexity issues proving that this problem is NP-complete for star-free colorings. The main results are in Section 3. We construct, for every integer k and every graph H, a graph H_k

with H-free chromatic number equal to 2 but requiring exactly k colors in any best H-free coloring. This construction is generalized to provide a graph G_k with a jump from $\chi(H, G_k) = p$ colors to $p + k$ colors in any best H-free coloring of G_k. In the last section, H-free chromatic numbers and sums are discussed for planar and outerplanar graphs. In particular, we prove that determining the P_3-free chromatic sum is NP-hard even for planar graphs and construct a family of maximal outerplanar graphs requiring many colors in any best H-free coloring for H a star or a cycle.

2. General Bounds and Complexity Results

For any graphs G and H, the most general bound on $\chi(H, G)$ results from the observation that any coloring in which each color class has fewer vertices than H is an H-free coloring. For example, if H is a nontrivial graph of order j then the smallest complete graph K_n with $\chi(H, K_n) = k$ is of order $n = (j-1)(k-1)+1$. This leads to the following bound which is obtained by assigning the colors with the smallest values to $\left\lfloor \frac{n}{|H|-1} \right\rfloor$ color classes with $|H| - 1$ vertices each.

Proposition 1. *If G has order n and H is a nontrivial graph of order j, then* $\Sigma(H, G) \leq \frac{\left\lfloor \frac{n}{j-1} \right\rfloor + 1}{2} \left(2n - (j-1) \left\lfloor \frac{n}{j-1} \right\rfloor\right)$.

While this clearly is a sharp bound for complete graphs, there are other families of graphs G for which equality also holds. For example, it was shown in [23] that when $H = K_1 + C_{j-1}$ (a wheel), $K_1 + P_{j-1}$ (a fan) or $tK_2 + K_1$ (a pinwheel) where j is odd and the edges of $\bar{G}$ form a 1-regular graph, no color class in an H-free coloring of G can contain more than $j - 1$ vertices. However, for such graphs G and H with $j = 2m$, we have a strict inequality in the bound since $\chi(H, G)$ and $\Sigma(H, G)$ are realized exactly when there are $\left\lfloor \frac{2m}{j} \right\rfloor$ color class with j vertices while as many of the remaining vertices as possible are in color classes with $j - 1$ vertices and, if necessary, there is one color class with fewer than $j - 1$ vertices.

Our next result shows, not surprisingly, that the problem of finding the H-free chromatic sum of a graph is as least as difficult as finding its H-free chromatic number.

Proposition 2. *The problem to determine whether or not an H-free chromatic sum of a graph is bounded by a given positive integer is reducible to a corresponding problem for an H-free chromatic number.*

Proof. For a given positive integer k and connected nontrivial graph H consider the decision problem: Is $\chi(H, G) \leq k$ for a given graph G of order n? If j is the order of H, define $G' \cong K_{(j-1)k} \times G$ and $k' = \frac{1}{2}k(k+1)(j-1)n$, where the symbol "$\times$" stands for the cartesian product of graphs. Then, the graph G has an H-free coloring with k colors if and only if $\Sigma(H, G') \leq k'$. In fact, if G can be H-free colored using k colors, we can color $(j-1)$ copies of G in G' in the same way and then cyclically shift all colors by 1 for the next $(j-1)$ copies, by 2 for next $(j-1)$ copies, ..., by $k-1$ for the last $(j-1)$ copies of G in G'. Then the sum of colors in G' is equal to k'. Conversely, if the sum of colors in G' is bounded by k', then we must use colors 1 through k in every copy of $K_{(j-1)k}$ because at most $(j-1)$ vertices of each copy can be monochromatic. Therefore, G' has an H-free coloring with k colors and the same holds for G. ■

Corollary 3. *If H is a fixed connected graph for which the H-free k-coloring problem is NP-hard for G belonging to some family of graphs, then the corresponding problem for the H-free chromatic sum is NP-hard as well.*

Since the problem of determining whether a graph has an r-defective coloring using at most k colors is NP-complete (for $k \geq 3, r \geq 0$), the problem of determining whether $\chi(K_{1,r+1}, G) \leq k$ is NP-complete and the problem of determining $\Sigma(K_{1,r+1}, G)$ is NP-hard. Notice that the case with $r = 0$ corresponds to standard proper coloring, the case with $r = 1$ to P_3-free coloring, and the case with $r = 2$ to $K_{1,3}$-free coloring.

Finding the H-free chromatic sum (for a fixed H) of a tree algorithmically can be done in linear time. We have pseudo-code for a program that computes the P_3-free chromatic sum for a tree. This is a straightforward algorithm which uses a preorder traversal of the vertices of a tree. For each vertex of the tree it keeps track of the three lowest sums for the subtree rooted at this vertex and two possible choices for the color of the vertex. Although the algorithm is standard, it has a lot of cumbersome details. This algorithm can be generalized to compute an H-free chromatic sum of a tree.

For a full binary (or d-ary) tree, we can find an exact formula for $\Sigma(H, T)$. For $d \geq 2$, a *d-ary* tree is a rooted tree in which each vertex has either 0 or d children. In a full d-ary tree, all leaves are at the same level. In a full d-ary tree of height h, the root is at level 1 and all leaves at level $h+1$.

Theorem 4. (a) *If T_h is a full binary tree of height h, then $\Sigma(P_3, T_h) = \left\lfloor \frac{2^{h+3}-4}{3} \right\rfloor$.*
(b) *If T_h is a full d-ary tree of height h, then*

$$\Sigma(K_{1,d+1}, T_h) = \frac{d^{h+1}-1}{d-1} + \frac{d^{h+1} - d^{(h+1)(\bmod 3)}}{d^3-1}.$$

Proof. Both proofs are by induction on h, the height of the tree T_h.

For (a), it is straightforward to prove that for h even, T_h has the unique best P_3-free coloring; in fact this coloring is a proper coloring in which all leaves have color 1. For h odd, a similar proper coloring is a best P_3-free coloring, but there are other best colorings that are P_3-free and not proper (see Figure 1).

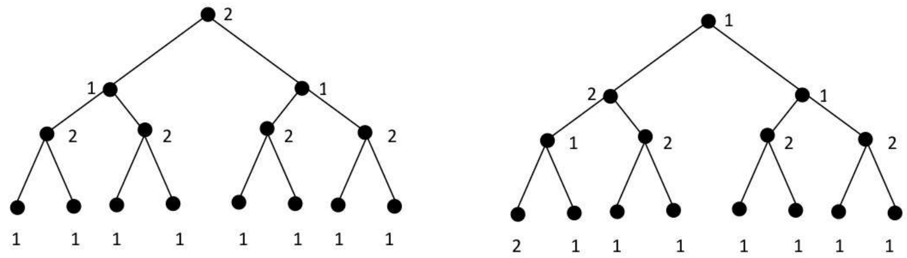

Figure 1. An example of an odd height binary tree and two different best P_3-free colorings.

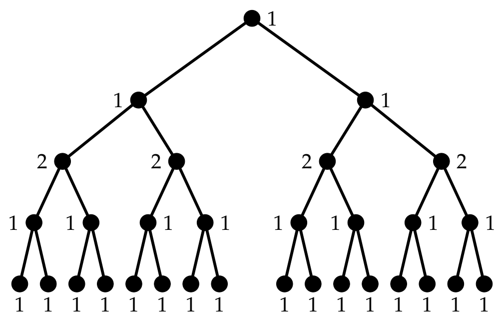

Figure 2. A best $K_{1,3}$-free coloring of the full binary tree T_4.

Since the full binary tree T_h has 2^{i-1} vertices at each level i, straightforward computations show that $\Sigma(P_3, T_h) = \left\lfloor \frac{2^{h+3}-4}{3} \right\rfloor$.

For (b), the coloring c that assigns color 1 to the vertices on levels $h+1, h, h-2, h-3, \ldots$ and color 2 to the vertices on levels $h-1, h-4, \ldots$ is a best $K_{1,d+1}$-free coloring. The details of a standard inductive proof are omitted. The formula for $\Sigma(K_{1,d+1}, T_h)$ is computed using the fact that, for each $i = 1, 2, \ldots, h+1$, level i of a full d-ary tree contains d^{i-1} vertices. ■

3. Graphs that Require More Colors in Their Best H-Free Colorings

Note that in all examples we provided so far, a best H-free coloring uses $\chi(H, G)$ colors. However, in some instances, a best H-free coloring may require more than $\chi(H, G)$ colors. This is addressed in the following theorem.

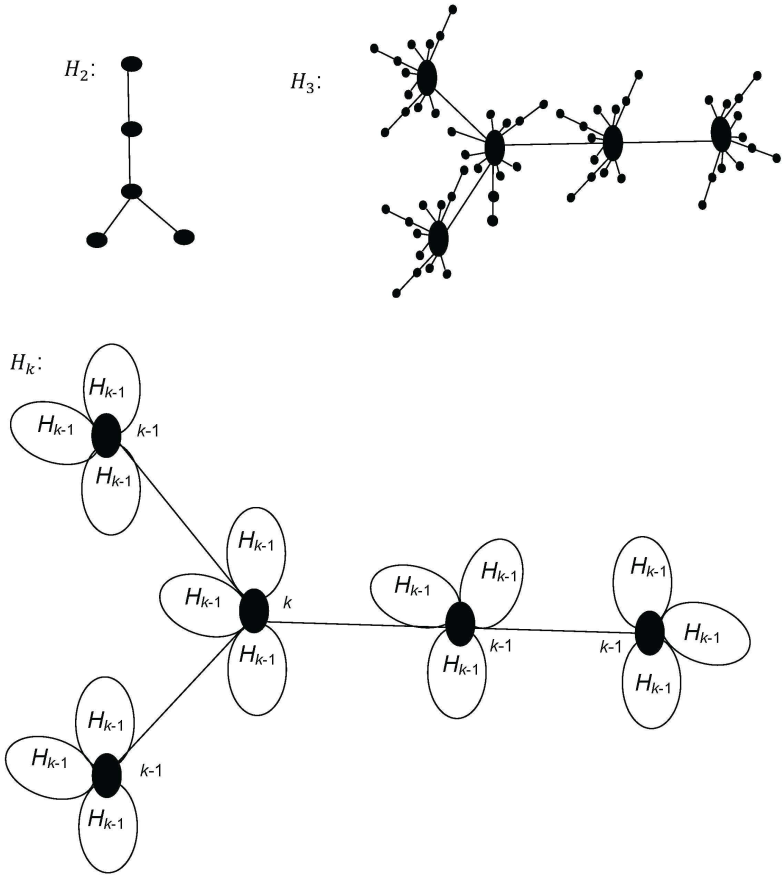

Figure 3. Graphs H_2, H_3, and H_k.

Theorem 5. *For every integer $k \geq 2$ and every connected graph H, there is a graph H_k with $\chi(H, H_k) = 2$ and for which every best H-free coloring of H_k must use k colors. Furthermore, color k is assigned to exactly one vertex of H_k.*

Proof. The graphs H_k are constructed recursively and the proof of the theorem is by induction. In addition, we prove that after assigning color k to the root r_k, the best H-free coloring of H_k is unique. Let $H_2 = H$. Any best H-free coloring of H_2 assigns color 2 to exactly one vertex and color 1 to every other vertex. Let us fix one such coloring, say c, and call the vertex r_2 with $c(r_2) = 2$ the root of H_2. Now, for $k \geq 3$, suppose H_{k-1} has the property that $\chi(H, H_{k-1}) = 2$ and every best H-free coloring assigns color $k - 1$ to exactly one vertex, namely r_{k-1}, the root of H_{k-1}. Take a copy of H to form the skeleton of H_k and then identify the roots of three copies of H_{k-1} with each vertex of the skeleton, as illustrated in Figure 3. Of course $\chi(H, H_k) = 2$ because we can color the skeleton using two colors; color 2 only for its root, and on each copy of H_{k-1}, that can be H-free colored with two colors, switch colors 1 and 2, if necessary to have an agreement of colors on the skeleton. This is not a best coloring, however. We claim that any best H-free coloring of H_k requires color k on exactly one vertex of the skeleton. To prove this, suppose c is a coloring of H_k such that each copy of H_{k-1} has been colored with a best H-free coloring and each vertex of the skeleton of H_k has been assigned color $k-1$. Let S represent the sum of the vertex colors in this coloring. Increasing by one the color of a vertex on the skeleton will produce an H-free coloring. Without loss of generality, assign color k to r_k, which corresponds to r_2 in H. Thus, $\Sigma(H, H_k) \leq S + 1$.

Now suppose that c' is a best H-free coloring of H_k and u is a vertex on its skeleton with $c'(u) \leq k-2$. Since H_{k-1} satisfies the theorem, color $k-1$ must be assigned to a vertex other than u in each of the three copies of H_{k-1} rooted at u. This results in an additional cost of at least 2 in coloring this subgraph of H_k and $\Sigma(c') \geq S + 2$, which contradicts the assumption that c' is a best H-free coloring of H_k. Thus, every best H-free coloring of H_k must have exactly one vertex assigned color k and that vertex must be on the skeleton. Then the remaining vertices of the skeleton must be colored with $k-1$ and each copy of H_{k-1} (except its root that is already colored) is colored according to its unique best coloring given by inductive assumption. If H has n vertices, then the order $|H_k|$ of the graph H_k satisfies the following recurrence relation: $|H_2| = n$ and $|H_k| = 3n|H_{k-1}| - 2n$ for $k \geq 3$, which implies that $|H_k| = \frac{(3n)^{k-1}(n-1)+2n}{3n-1}$. For the chromatic sums, from the recursive equation $\Sigma(H, H_2) = n+1$ and $\Sigma(H, H_k) = [\Sigma(H, H_{k-1}) - (k-1)]3n + n(k-1) + 1$ for $k \geq 3$, we get $\Sigma(H, H_k) = \frac{n(3n+4)(3n)^{k-2}-2n-1}{3n-1} - 2n\sum_{i=3}^{k} i(3n)^{k-i}$. ■

The graph H_k in the proof of Theorem 5 has $\chi(H, H_k) = 2$ but requires k colors in any best coloring. We can modify this construction to get a graph G_k with $\chi(H, G_k) = p$ that needs k additional colors to achieve the H-free chromatic sum.

Theorem 6. *For all integers $p \geq 2$ and $k \geq 1$ and every connected graph H, there is a graph G_k with $\chi(H, G_k) = p$ for which every best H-free coloring of G_k must use $p + k$ colors.*

Proof. Let H be a connected nontrivial graph of order j and H_{k+1} the graph constructed in the proof of Theorem 5, i.e., $\chi(H, H_k) = 2$ but every best H-free coloring of H_{k+1} uses $k+1$ colors with color $k+1$ used once. Construct G_k by taking a copy of the complete graph $K_{(j-1)(p-1)+1}$ as the skeleton and then identify the roots of $p+k$ copies of H_{k+1} with each vertex of the skeleton. Of course $\chi(H, G_k) = p$ because p colors are required to H-free color the skeleton and the rest of vertices of G_k can be 2-colored. We claim that the following is a best H-free coloring of G_k: take a best coloring of each copy of H_{k+1} with colors 1 through k on the vertices different then the roots; use color $k+1$ for $j-1$ roots, color $k+2$ for $j-1$ roots, etc., color $k+p-1$ for $j-1$ roots and color $k+p$ for the remaining root (the last vertex on the skeleton). This is clearly an H-free coloring of G_k that uses $k+p$ colors. Assume that the sum of colors in it is S_k. Suppose that there is a best coloring c of G_k in which some vertex v on the skeleton has $c(v) < k+1$. This causes a maximum saving of $k+p-1$ on v but a loss of at least one for each of $k+p$ copies of H_{k+1} rooted at v producing a sum larger than S_k, a contradiction. ∎

4. Chromatic Numbers and Chromatic Sums of Planar and Outerplanar Graphs

We will summarize here some known results about H-free chromatic numbers for planar and outerplanar graphs. As we will see all these numbers are small, but in contrast, some planar or outerplanar graphs will require arbitrary many colors for their best H-free colorings. A *linear forest* is a disjoint union of paths. Goddard [16] and Poh [24] proved independently that every planar graph has a partition of its vertex set into three subsets such that every subset induces a linear forest. Therefore, if $\Delta(H) \geq 3$, then for every planar graph G we have $\chi(H,G) \leq 3$. Using results of Thomassen [25] and Burstein [8], Broersma *et al.* [7] observed that if H contains a cycle, then for every planar graph G we have $\chi(H,G) \leq 2$. Let us observe that if H is a planar graph or a tree, then for every integer k the graph H_k from Theorem 3 is planar (respectively, a tree). Therefore, for a family of planar graphs we have the following results.

Proposition 7. (1) *If H is a planar graph containing a cycle, then for every planar graph G, $\chi(H,G) \leq 2$, but for every $k \geq 3$, there exists a planar graph requiring k colors in any best H-free coloring of G.*

(2) *If H is a tree different from a path, then for every planar graph G, $\chi(H,G) \leq 3$, but for every $k \geq 4$, there exists a planar graph* (*in fact a tree*) *requiring k colors in any best H-free coloring of G.*

(3) *If $H = K_2$, then by the Four Color Theorem, for every planar graph G, $\chi(H,G) \leq 4$, but for every $k \geq 5$, there exists a planar graph* (*a tree*) *requiring k colors in any best H-free coloring of G.*

Returning to complexity issues, Broersma *et al.* [7] proved that the following problems are NP-hard for planar graphs.

1. Let H be a tree with at least two edges. Deciding whether a planar graph G has a H-free 2-coloring is NP-hard.

2. For any $k \geq 2$ deciding whether a planar graph G has P_k-free 3-coloring is NP-hard.

We will use result 1 for $H = P_3$ to prove that P_3-free chromatic sum for planar graphs is NP-hard.

Theorem 8. *The following decision problem is NP-complete. For a given integer k and a planar graph G, is $\Sigma(P_3, G) \leq k$?*

Proof. We perform a polynomial reduction to the following decision problem: Is $\chi(P_3, G) \leq 2$ for a planar graph G?
Notice first that the graph $K_4 - e$ has a P_3-free chromatic sum equal to 6 and each of the two colorings presented in Figure 4 is a best P_3-free coloring.

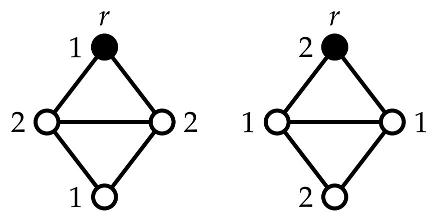

Figure 4. Two best P_3-free colorings of $K_4 - e$.

Having a planar graph G of order n, take n copies of the graph $K_4 - e$ and identify the root r of each copy with one vertex of G. The obtained graph G' of order $4n$ is P_3-free 2-colorable if and only if G is P_3-free 2-colorable. Moreover, $\Sigma(P_3, G') = 6n$ if and only if G is 2-colorable. Therefore, by taking $k = 6n$, the positive answer to the decision problem: $\Sigma(P_3, G') \leq k$? is equivalent to the problem: $\chi(P_3, G) = 2$? ■

A graph G is *outerplanar* if it can be embedded in the plane so that every vertex of G lies on the boundary of the exterior region. A well known characterization of outerplanar graphs states that G is outerplanar if and only if G contains no subgraph homeomorphic from K_4 or $K_{2,3}$, i.e. does not contain a subgraph that can be obtained from K_4 or $K_{2,3}$ by a sequence of edge subdivisions. Broere and Mynhardt [5] proved by induction that every outerplanar graph has a partition of its vertex set into two subsets such that the subgraph induced by each subset is a linear forest. We observe in the next lemma that the same result can be achieved by a constructive proof using a concept of "layers" based on the distance. Let us recall that G is a *maximal outerplanar graph* if G is outerplanar but $G + uv$ is not for any nonadjacent vertices $u, v \in V(G)$. In maximal outerplanar graphs, all interior regions are triangles. Such graphs are also 2-connected.

Lemma 9. *Let G be a maximal outerplanar graph and $x \in V(G)$. Let $V_k(x) = \{v \in V(G) : d(v,x) = k\}$ be the set of vertices whose distance from x is k. Then the induced graph $\langle V_k(x)\rangle$ contains neither $K_{1,3}$ nor a cycle C_l for any $l \geq 3$.*

Proof. Suppose that $\langle V_k(x)\rangle$ contains a copy of C_l, $l \geq 3$. Pick three consecutive vertices a, b, c on that cycle (see Figure 5).

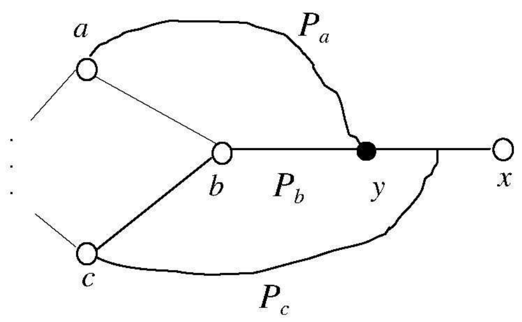

Figure 5. A color class cannot contain a cycle.

Let P_a, P_b, P_c be shortest $x-a$, $x-b$, $x-c$ paths, respectively, in G. The vertices a, b and c cannot be internal vertices of these paths. Let y be a vertex belonging to at least two of these paths that is farthest from x (y could be x, if P_a, P_b, P_c are internally disjoint). It is easy to see that $\langle V_k(x) \cup P_a \cup P_b \cup P_c\rangle$ contains a subgraph homeomorphic from K_4, which cannot happen in outerplanar graphs. The proof that $\langle V_k(x)\rangle$ does not contain a copy of $K_{1,3}$ is similar and therefore omitted. ∎

If, for a maximal outerplanar graph G and a vertex $x \in V(G)$, we define $C_1 = \{v \in V(G) : d(v,x) \text{ is odd}\}$ and $C_2 = \{v \in V(G) : d(v,x) \text{ is even}\}$, then in the induced subgraphs $\langle C_1\rangle$ and $\langle C_2\rangle$, there are no edges between vertices with different distance from x. Therefore, Lemma 1 implies that $\langle C_1\rangle$ and $\langle C_2\rangle$ are both linear forests. This observation gives the following result.

Corollary 10. *If G is a connected outerplanar graph and H is a connected graph that is not a path, then $\chi(H,G) \leq 2$.*

Proof. Assume that G is a connected outerplanar graph. Add edges to G until we get a maximal outerplanar graph M. Pick a vertex $x \in V(M) = V(G)$. Let $C_1 = \{v \in V(M) : d(v,x) \text{ is odd}\}$ and $C_2 = \{v \in V(M) : d(v,x) \text{ is even}\}$. Assigning color i to all vertices in C_i defines an H-free coloring of M that produces also an H-free coloring of G. ∎

Of course, for a particular graph H, connected and different from a path, it is easy to decide whether $\chi(H,G) = 1$ or $\chi(H,G) = 2$; therefore, for outerplanar graphs the H-free chromatic number problem is trivially solvable. The $K_{1,3}$-free coloring presented in the last theorem is usually not the one producing the chromatic sum. We would like to find a partition that is unbalanced in order to minimize the number of occurences of color 2. For example, the graph G presented in Figure 6 has the $K_{1,3}$-free chromatic sum equal to 14 and contains a long path induced by vertices colored by 1.

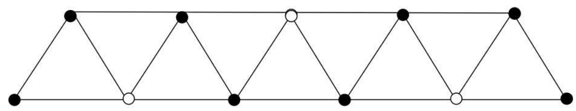

Figure 6. Maximal outerplanar graph with $K_{1,3}$-free chromatic sum 14. Black vertices have color 1.

However, as we will see from the next theorem, a partition of the vertex set into two subsets (and using two colors) is sometimes not sufficient. The purpose of the next theorem is the construction of a family of maximal outerplanar graphs that require many colors to achieve the $K_{1,3}$-free chromatic sum. In this construction, we will repeatedly use a *fan attaching* procedure that is described below. Let uv be an edge lying on the boundary of the exterior region of a maximal outerplanar graph G. A p-fan attachment at uv around u is obtained by adding a path on vertices $v_1, v_2, \ldots, v_p$ (with v_1 adjacent to v) and all edges between them and u (see Figure 7).

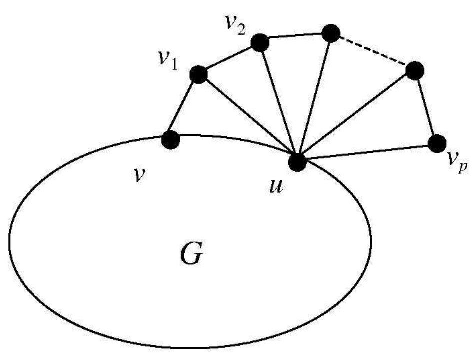

Figure 7. Construction of a p-fan attachement at uv around u.

The graph obtained by a p-fan attachment is also maximal outerplanar.

Theorem 11. *For every natural number k, $k \geq 3$, there exists a maximal outerplanar graph G_k requiring at least k colors to achieve the $K_{1,3}$-free chromatic sum.*

Proof. To construct G_k, we use induction on k. For abbreviation, we introduce $\Sigma_k = \Sigma(K_{1,3}, G_k)$. For $k = 3$, we start with a copy of $K_4 - e$ on vertices x, y, z, w. Attach to it a 5-fan at xz around x, a 5-fan at wy around w, a 4-fan at yx around y, and a 4-fan at zw around z producing the graph G_3 of order 22 (see Figure 8).

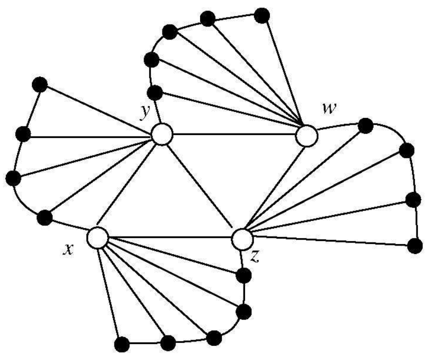

Figure 8. Maximal outerplanar graph requiring three colors to achieve the $K_{1,3}$-free chromatic sum.

If we color all attached vertices in G_3 by 1, vertices x, z, and w by 2, and the vertex y by 3, then we will get the $K_{1,3}$-free coloring with the sum of colors $18 \times 1 + 3 \times 2 + 3 = 27$. If only two colors are used for G_3, then at least one of the vertices x, y, z, or w must be colored with 1. Since the degree of this vertex is 8, at least six of its neighbors must be colored with 2. Then the sum of the colors would be at least $6 \times 2 + 16 \times 1 = 28$. Therefore, the $K_{1,3}$-free chromatic sum Σ_3 of G_3 is at most 27 and requires at least 3 colors.

Suppose that, for $k \geq 3$, G_k is an maximal outerplanar graph of order n_k requiring at least k colors to achieve the $K_{1,3}$-free chromatic sum Σ_k. If less than k colors are used for a $K_{1,3}$-free coloring of G_k, then the sum of colors is at least $\Sigma_k + 1$. The graph G_{k+1} is obtained from G_k by going along all edges on the exterior boundary of G_k and attaching a p-fan to each edge around each consecutive vertex of the boundary, where $p = \Sigma_k$. The order of G_{k+1} is $n_{k+1} = n_k + n_k\Sigma_k = n_k(\Sigma_k + 1)$. The vertices on attached fans have degree 2 or 3, but each vertex of the *core* graph (the vertices present in G_k) now will have degree at least $2 + \Sigma_k + 1 = \Sigma_k + 3$. Color the vertices of G_{k+1} as follows: all new attached vertices get color 1, each vertex of the core G_k gets the color one more than in the optimal coloring of the graph G_k. This is a $K_{1,3}$-free coloring c with the sum $\Sigma(c) = \Sigma_k + n_k + n_k\Sigma_k = \Sigma_k(n_k + 1) + n_k$. Suppose that at most k colors are used to color the vertices of G_{k+1}. If there is a vertex, say x, of the core graph G_k with color 1, then at least $\Sigma_k + 1$ of its neighbors must be colored with colors larger than 1 (otherwise $K_{1,3}$ in color 1 would be present in G_{k+1}). But then the sum of colors is at least $2(\Sigma_k + 1) + 1(n_{k+1} - \Sigma_k - 1) = 2(\Sigma_k + 1) + n_k(\Sigma_k + 1) - (\Sigma_k + 1) = n_k(\Sigma_k + 1) + \Sigma_k + 1 = \Sigma(c) + 1$, so it cannot be optimal.

If none of the vertices of G_k has color 1, then the colors used for G_k are from the range $[2, k]$. The best we can do is to take the coloring of G_k with $k - 1$ colors (with the sum at least $\Sigma_k + 1$) and increase the color of each of the n_k vertices by 1. Then the sum of the colors of the core graph G_k must be at least $\Sigma_k + 1 + n_k$ and with all other vertices colored with 1, the sum would be at least $\Sigma_k + 1 + n_k + n_k(\Sigma_k) = \Sigma(c) + 1$ and cannot be optimal. Therefore, $\Sigma_{k+1} \leq \Sigma(c)$ and at least $k + 1$ colors are needed to obtain a $K_{1,3}$-free chromatic sum of G_k. ■

The construction from the proof of Theorem 11 can be easily generalized for other forbidden subgraphs. For example, if $H = K_{1,r}$ with $r \geq 4$, we can use similar fan attachments to a core graph that is a maximal outerplanar graph obtained from H by adding $r - 1$ edges. Similarly, for $H = C_l$, where $l \geq 3$, one can start with the core graph being a triangulation of C_l. In fact, for $l = 4$ the core graph will be the same as for the $K_{1,3}$-free coloring from Theorem 11, i.e. $K_4 - e$.

References

[1] M. Albertson, R. Jamison, S. Hedetniemi and S. Locke, *The subchromatic number of a graph*, Discrete Math. **74** (1989) 33–49.
doi:10.1016/0012-365X(89)90196-9

[2] J. Andrews and M. Jacobson, *On a generalization of chromatic number*, Congr. Numer. **47** (1985) 33–48.

[3] D. Archdeacon, *A note on defective colorings of graphs in surfaces*, J. Graph Theory **11** (1987) 517–519.
doi:10.1002/jgt.3190110408

[4] M. Borowiecki. I. Broere, M. Frick, P. Mihók and G. Semanišin, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.
doi:10.7151/dmgt.1037

[5] I. Broere and M. Mynhardt, *Generalized colorings of outer-planar and planar graphs*, Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York (1985) 151–161.

[6] I. Broere, S. Dorfing and E. Jonck, *Generalized chromatic numbers and hereditary properties of graphs*, Discuss. Math. Graph Theory **22** (2002) 259–270. doi:10.7151/dmgt.1174

[7] H. Broersma, F.V. Fomin, J. Kratochvil and G.J. Woeginger, *Planar graph coloring avoiding monochromatic subgraphs: trees, and paths make it difficult*, Algorithmica **44** (2006) 343–361. doi:10.1007/s00453-005-1176-8

[8] M.I. Burstein, *The bi-colorability of planar hypergraphs*, Sakharth. SSR Mecn. Akad. Moambe **78** (1975) 293–296.

[9] G. Chartrand, D. Geller and S. Hedetniemi, *A generalization of the chromatic number*, Proc. Cambridge Philos. Soc. **64** (1968) 265–271. doi:10.1017/S0305004100042808

[10] G. Chartrand and H. Kronk, *The point-arboricity of planar graphs*, J. London Math. Soc. **44** (1969) 612–616. doi:10.1112/jlms/s1-44.1.612

[11] G. Chartrand, H. Kronk and C. Wall, *The point-arboricity of a graph*, Israel J. Math. **6** (1968) 169–175. doi:10.1007/BF02760181

[12] L.J. Cowen, R.H. Cowen and D.R. Woodall, *Defective colorings of graphs in surfaces; partitions into subgraphs of bounded valency*, J. Graph Theory **10** (1986) 187–195. doi:10.1002/jgt.3190100207

[13] L.J. Cowen, W. Goddard and C.E. Jesurum, *Defective colorings revisited*, J. Graph Theory **24** (1997) 205–219. doi:10.1002/(SICI)1097-0118(199703)24:3⟨205::AID-JGT2⟩3.0.CO;2-T

[14] K. Dargen and K. Fraughnaugh, *Conditional chromatic numbers with forbidden cycles*, Linear Algebra Appl. **217** (1985) 53–66. doi:10.1016/0024-3795(94)00139-5

[15] P. Erdős, E. Kubicka and A.J. Schwenk, *Graphs that require many colors to achieve their chromatic sum*, Proc. Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing **71** (1990) 17–28.

[16] W. Goddard, *Acyclic colorings of planar graphs*, Discrete Math. **91** (1991) 91–94. doi:10.1016/0012-365X(91)90166-Y

[17] F. Harary and K. Fraughnaugh (Jones), *Conditional colorability* II: *bipartite variations*, Congr. Numer. **50** (1985) 205–218.

[18] F. Harary and K. Fraughnaugh (Jones), *Degree conditional bipartition numbers in graphs*, Congr. Numer. **55** (1986) 39–50.

[19] E. Kubicka, The chromatic sums and efficient tree algorithms, Ph.D. Thesis, Western Michigan University (1989).

[20] E. Kubicka, G. Kubicki and K. McKeon, *Chromatic sums for colorings avoiding monochromatic subgraphs*, Electron. Notes Discrete Math. **43** (2013) 247–254. doi:10.1016/j.endm.2013.07.041

[21] E. Kubicka and A. Schwenk, *Introduction to chromatic sums*, Congr. Numer. **71** (1990) 7–28.

[22] K. McKeon, *Generalized chromatic numbers of graphs with bipartite complements*, Congr. Numer. **197** (2009) 97–105.

[23] K. McKeon and H. Pham, *Generalized chromatic numbers of graphs with prescribed complements*, Congr. Numer. **191** (2008) 71–79.

[24] K.S. Poh, *On the linear vertex-arboricity of a planar graph*, J. Graph Theory **14** (1990) 73–75.
doi:10.1002/jgt.3190140108

[25] C. Thomassen, *Decomposing a planar graph into degenerate graphs*, J. Combin. Theory Ser. B **65** (1995) 305–314.
doi:10.1006/jctb.1995.1057

Received 27 February 2013
Revised 10 November 2014
Accepted 10 November 2014