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## PANCYCLICITY WHEN EACH CYCLE MUST PASS EXACTLY k HAMILTON CYCLE CHORDS

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#### Abstract

It is known that  $\Theta(\log n)$  chords must be added to an *n*-cycle to produce a pancyclic graph; for vertex pancyclicity, where every vertex belongs to a cycle of every length,  $\Theta(n)$  chords are required. A possibly 'intermediate' variation is the following: given  $k, 1 \leq k \leq n$ , how many chords must be added to ensure that there exist cycles of every possible length each of which passes exactly k chords? For fixed k, we establish a lower bound of  $\Omega(n^{1/k})$ on the growth rate.

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A simple graph G on n vertices is *pancyclic* if it has cycles of every length l,  $3 \le l \le n$ . The study of these graphs was initiated by Bondy's observation [1, 2]

that, for non-bipartite graphs, sufficient conditions for Hamiltonicity can also be sufficient for pancyclicity. In general, we may distinguish, in a pancyclic graph G, a Hamilton cycle C; then the remaining edges of G form chords of C. We can then ask, given  $k \leq l \leq n$  if, relative to C, a cycle of length l exists which uses exactly k chords. This suggests a k-chord analog of pancyclicity: do all possible cycle lengths occur when cycles must use exactly k-chords of a suitably chosen Hamilton cycle?

We accordingly define a function c(n, k),  $n \ge 6$ ,  $k \ge 1$ , to be the smallest number of chords which must be added to an *n*-cycle in order that cycles of all possible lengths may be found, each passing exactly k chords. No Hamilton cycle can use exactly one chord of another Hamilton cycle, so that when k = 1 cycle lengths must lie between 3 and n - 1. The function is undefined for k > n. We define the function for  $n \ge 6$  because n = 4, 5 are too restrictive to be of interest to us.

Our aim in this paper is to investigate the growth of the function c(n,k) as n increases, for fixed k.

**Example 1.** Label the vertices around the cycle  $C_6$ , in order, as  $v_1, \ldots, v_6$ . Add chords  $v_1v_3$  and  $v_1v_4$ ; the result is a pancyclic graph. It also has cycles of all lengths  $\leq 5$  each passing exactly one of the chords. If  $v_2v_6$  is added then cycles exist of all lengths  $\geq 3$ , each passing two chords. If two further chords,  $v_2v_4$  and  $v_4v_6$ , are added then cycles exist of all lengths  $\geq 3$ , each passing the chords to be added, i.e., c(6, 4) = 6. Six suitably chosen chords are also sufficient for 5-chord and 6-chord cycles: c(6, 5) = c(6, 6) = 6.

**Lemma 2.** (1)  $c(n,1) = \left\lceil \frac{n-3}{2} \right\rceil$ .

- (2)  $c(n,k) \ge k$ , with equality if and only if k = n.
- (3) c(n, n-1) = n.

**Proof.** (1) follows from the observation that a chord in  $C_n$  forming a 1-chord cycle of length k automatically forms a 1-chord cycle of length n + 2 - k.

(2) is immediate from the definition of c(n, k).

(3) Let G consist of an (n-1)-cycle, together with an (n-1)-chord cycle on the same vertices. Choose vertex v: let the chords at v be xv and yv and its adjacent cycle edges be uv and vw, with u, v, w, x, y appearing in clockwise order around the cycle. Replace v and its incident edges with two vertices  $v_u$ and  $v_w$ , with edges  $v_u v_w$ ,  $uv_u$ ,  $v_w w$ ,  $xv_w$  and  $yv_u$ . The (n-1)-chord cycle in G becomes an (n-1)-chord n-cycle. Add an n-th chord  $xv_u$  to give an (n-1)-chord (n-1)-cycle.

Table 1 supplies some small values/bounds for c(n,k). The lower bounds

		$\kappa$										
		1	<b>2</b>	3	4	5	6	7	8	9	10	11
$\boldsymbol{n}$	6	2	3	5	6	6	6					
	7	2	3	5	6	6	7	7				
	8	3	4	5	6	6	7	8	8			
	9	3	4	5	6	7	8	8	9	9		
	10	4	4	5	6	$\geq 6$	$\geq 7$	$\geq 8$	$\geq 9$	10	10	
	11	4	4	$\geq 5$	$\geq 6$	$\geq 7$	$\geq 7$	$\geq 8$	$\geq 9$	$\geq 10$	11	11
	12	5	4	$\geq 5$	$\geq 6$	$\geq 7$	$\geq 7$	$\geq 8$	$\geq 9$	$\geq 10$	$\geq 11$	12
	13	5	4	$\geq 5$	$\geq 6$	$\geq 7$	$\geq 8$	$\geq 8$	$\geq 9$	$\geq 10$	$\geq 11$	$\geq 12$

are supplied by Corollary 7; except for those values covered by Lemma 2, exact values and upper bounds were found by computer search.

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Table 1. Values of c(n,k) for  $6 \le n \le 13$  and  $1 \le k \le 11$ .

Our aim is to compare c(n, k) with the number of chords required for pancyclicity and for *vertex pancyclicity*, in which each vertex must lie on a cycle of every length.

The following lower bound is stated without proof in [1].

**Theorem 3.** In a pancyclic graph G on n vertices the number of edges is not less than  $n - 1 + \log_2(n - 1)$ .

For the sake of completeness we observe that Theorem 3 follows immediately from the following lemma.

**Lemma 4.** Suppose p chords are added to  $C_n$ ,  $n \ge 3$ . Then the number N(n, p) of cycles in the resulting graph satisfies

$$\binom{p+2}{2} \le N(n,p) \le 2^{p+1} - 1.$$

**Proof.** Embed  $C_n$  convexly in the plane. Suppose the chords added to  $C_n$  are, in order of inclusion,  $e_1, e_2, \ldots, e_p$ . Say that  $e_i$  intersects  $e_j$  if these edges cross each other when added to the embedding of  $C_n$ . Let  $n_i$  be the number of new cycles obtained with  $e_i$  is added. Then  $n_i$  satisfies:

1.  $n_i \ge i + 1$ , the minimum occurring if and only if the  $e_j$  are pairwise nonintersecting for  $j \le i$ ; 2.  $n_i \leq 2^i$ , the maximum occurring if and only if  $e_i$  intersects with  $e_j$  for all j < i, giving  $n_i = \sum_{j=0}^i {i \choose j}$ .

Now 
$$1 + \sum_{i=1}^{p} (i+1) \le 1 + \sum_{i=1}^{p} n_i \le 1 + \sum_{i=1}^{p} 2^i$$
 and the result follows.

The exact value of the minimum number of edges in an *n*-vertex pancyclic graph has been calculated for small *n* by George *et al.* [5] and Griffin [6]. For  $3 \leq n \leq 14$ , the lower bound in Theorem 3 is exact; however, it can be seen that, for n = 15, 16, we must add four chords to  $C_n$  to achieve pancyclicity while the argument in the proof of Lemma 4 can only account for three.

As regards an upper bound on the number of chords required for pancyclicity, [1] again asserts  $O(\log n)$ , again without a proof. A log *n* construction has been given by Sridharan [7]. Together with Theorem 3 this gives an 'exact' growth rate for pancyclicity: it is achieved by adding  $\Theta(\log n)$  chords to  $C_n$ .

In contrast, vertex pancyclicity, in which every vertex lies in a cycle of every length has been shown by Broersma [3] to require  $\Theta(n)$  edges to be added to  $C_n$ . Our question is: where between  $\log n$  and n does c(n,k) lie? For fixed k, we find a lower bound strictly between the two:  $\Omega(n^{1/k})$ .

Let us for the moment restrict to  $k \geq 3$ . Suppose we add p chords to  $C_n$ ,  $3 \leq k \leq p \leq \binom{n}{2} - n$ . Suppose that these p added chords include a k-cycle. We will use K(k,p), defined for  $k \geq 3$ , to denote the maximum number of k-chord cycles that can be created in the resulting graph. Then  $1 \leq K(k,p)$  by definition and  $K(k,p) \leq 2^{p+1} - 1$  by Lemma 4. By lowering this upper bound we can increase the lower bound on c(n,k).

# **Theorem 5.** $K(k,p) \leq \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k}.$

We will use the following Lemma to prove Theorem 5.

**Lemma 6.** Suppose that a set X of chords is added to  $C_n$ . In the resulting graph the maximum number of cycles passing all edges in X is 1 if X contains adjacent chords,

2 if no two chords of X are adjacent.

**Proof.** Let G be the graph resulting from adding the chords of X to  $C_n$ . We may assume without loss of generality that G has no vertices of degree 2, since such vertices may be contracted out. For a given cycle in G passing all chords of X, let H denote the intersection of this cycle with the  $C_n$ . Then H consists of isolated vertices and disjoint edges, and H is completely determined once any of these vertices or edges is fixed. If two chords are adjacent this fixes an isolated vertex of H; if no two chords are adjacent then there is a maximum of two ways in which a single edge of H may be fixed.

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**Proof of Theorem 5.** By definition of K(k, p) we must use a set, say S, of k chords to create a k-cycle. We add new chords to S, one by one. On adding the r-th additional chord,  $1 \leq r \leq p - k$ , we ask how many k-chord cycles use this chord. For any such a cycle the previous r-1 chords will be split between S and non-S chords: with i chords from S being used,  $0 \le i \le k-1$ , this can happen in

$$\binom{k}{i}\binom{r-1}{k-i-1}$$

ways. Since i > 1 forces two adjacent chords in S to be used, summing over i, according to Lemma 6, and then over r gives

$$K(k,p) \le 1 + \sum_{r=1}^{p-k} \left( 2\sum_{i=0}^{1} \binom{k}{i} \binom{r-1}{k-i-1} + \sum_{i=2}^{k-1} \binom{k}{i} \binom{r-1}{k-i-1} \right).$$

This simplifies (e.g. using symbolic algebra software such as Maple) to give the result.

**Corollary 7.** For given positive integers k and n, with  $3 \le k \le n$  and  $n \ge 6$ , the value of c(n, k) is not less than the largest root of the following polynomial in p:

$$\Pi(p;n,k) = \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k} - n + k - 1.$$

**Proof.** Suppose that, with n and k fixed, we add p chords to  $C_n$  and create cycles of all lengths  $\geq k$ , each passing k chords. Then  $n - k + 1 \leq K(k, p)$ . So p must satisfy  $0 \leq \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k} - n + k - 1$ . The right-hand side of this inequality is a polynomial in p which has positive slope at its largest root, so that c(n,k) cannot be less than this root.

We finally extend our analysis to include the cases k = 1, 2.

**Corollary 8.** Let  $n \ge 6$  be a positive integer. Then for  $k \ge 1$  fixed, c(n,k) is of order  $\Omega(n^{1/k})$ .

**Proof.** For k = 1 the required linear bound was provided in Lemma 2.

For k = 2 an analysis similar to that used in the proof of Theorem 5 shows that the number of 2-chord cycles which may be created by adding p chords to  $C_n$ is at most  $p^2 - p - 1$ . So to have 2-chord cycles of all lengths from 3 to n we require  $p^2 - p - 1 \ge n - 2$ . In this case we can solve explicitly to get the bound  $p \ge \frac{1}{2} \left( 1 + \sqrt{4n - 3} \right).$ Now suppose  $k \ge 3$ . In order to have all k-chord cycles of all lengths be-

tween k and n we must have

$$n-k+1 \le \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k} \le f(k)p^k,$$

for some function f(k). Therefore  $p^k \ge (n-k+1)/f(k)$  so, for k fixed,  $p = \Omega(n^{1/k})$ .

**Remark 1.** We are suggesting that the value of c(n, k) may be 'intermediate' between pancyclicity and vertex pancyclicity in the sense that the number of chords it requires to be added to  $C_n$  may lie between  $\log n$  and n. Thus far we have only a lower bound in support of our suggestion. Moreover, a comparison of the growth orders,  $\Omega(\log n)$  as opposed to  $\Omega(n^{1/k})$ , suggests that this is very much a 'for large n' type result. The equation  $\ln n = n^{1/k}$  has two positive real solutions for  $k \geq 3$ , given in terms of the two real branches of the Lambert W function [4]. In particular  $\ln n$  exceeds  $n^{1/k}$  for  $n > e^{-kW_{-1}(-1/k)}$ , and this bound grows very fast with k. To give a specific example, k = 10, the log bound exceeds the 10-th root bound until the number of vertices exceeds about  $3.4 \times 10^{15}$ . Until then, so far as our analysis goes, we might expect 'most' pancyclic graphs to be 10-chord pancyclic. However we suggest that, in the long term, a guarantee of this implication, analogous to Hamiltonicity guaranteeing pancyclicity, will not be found.

**Remark 2.** We would like to know if c(n, k) is monotonically increasing in n. However, it is still open even whether pancyclicity is monotonic in the number of chords requiring to be added to  $C_n$  (the question is investigated in [6]). We believe that c(n, k) it is not increasing in k and c(n, 1) > c(n, 2) for n = 12, 13 confirms this in a limited sense. Our  $n^{1/k}$  lower bound instead suggests the possibility that c(n, k) is convex for fixed n, as a function of k.



Figure 1. No 4-cycle uses exactly 1 chord of the bold-edge Hamilton cycle.

**Remark 3.** We observe that, unlike pancyclicity, the property of having cycles of all lengths each passing k chords is not an invariant of a graph: it depends on the initial choice of a Hamilton cycle. For example, in Figure 1, there are cycles of all lengths  $\leq 9$  each passing exactly one of the c(10, 1) = 4 chords of the outer

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cycle but there is no 4-cycle passing exactly one chord of the bold-edge Hamilton cycle.

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