Discussiones Mathematicae
Graph Theory 35 (2015) 533–539
doi:10.7151/dmgt.1818

PANCYCLICITY WHEN EACH CYCLE MUST PASS EXACTLY k HAMILTON CYCLE CHORDS

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Abstract

It is known that $\Theta(\log n)$ chords must be added to an n-cycle to produce a pancyclic graph; for vertex pancyclicity, where every vertex belongs to a cycle of every length, $\Theta(n)$ chords are required. A possibly 'intermediate' variation is the following: given k, $1 \le k \le n$, how many chords must be added to ensure that there exist cycles of every possible length each of which passes exactly k chords? For fixed k, we establish a lower bound of $\Omega\big(n^{1/k}\big)$ on the growth rate.

Keywords: extremal graph theory, pancyclic graph, Hamilton cycle.

2010 Mathematics Subject Classification: 05C38.

A simple graph G on n vertices is *pancyclic* if it has cycles of every length l, $3 \le l \le n$. The study of these graphs was initiated by Bondy's observation [1, 2] that, for non-bipartite graphs, sufficient conditions for Hamiltonicity can also be sufficient for pancyclicity. In general, we may distinguish, in a pancyclic graph G, a Hamilton cycle C; then the remaining edges of G form chords of C. We can then ask, given $k \leq l \leq n$ if, relative to C, a cycle of length l exists which uses exactly k chords. This suggests a k-chord analog of pancyclicity: do all possible cycle lengths occur when cycles must use exactly k-chords of a suitably chosen Hamilton cycle?

We accordingly define a function $c(n, k)$, $n \geq 6$, $k \geq 1$, to be the smallest number of chords which must be added to an n-cycle in order that cycles of all possible lengths may be found, each passing exactly k chords. No Hamilton cycle can use exactly one chord of another Hamilton cycle, so that when $k = 1$ cycle lengths must lie between 3 and $n - 1$. The function is undefined for $k > n$. We define the function for $n \geq 6$ because $n = 4, 5$ are too restrictive to be of interest to us.

Our aim in this paper is to investigate the growth of the function $c(n, k)$ as n increases, for fixed k.

Example 1. Label the vertices around the cycle C_6, in order, as $v_1, \ldots, v_6$. Add chords v_1v_3 and v_1v_4; the result is a pancyclic graph. It also has cycles of all lengths ≤ 5 each passing exactly one of the chords. If v_2v_6 is added then cycles exist of all lengths ≥ 3, each passing two chords. If two further chords, v_2v_4 and v_4v_6, are added then cycles exist of all lengths ≥ 3, each passing three chords. For 4-chord cycles we require six chords to be added, i.e., $c(6, 4) = 6$. Six suitably chosen chords are also sufficient for 5-chord and 6-chord cycles: $c(6, 5) = c(6, 6) = 6$.

Lemma 2. (1) $c(n, 1) = \left\lceil \dfrac{n-3}{2} \right\rceil$.

(2) $c(n, k) \geq k$, *with equality if and only if* $k = n$.

(3) $c(n, n - 1) = n$.

Proof. (1) follows from the observation that a chord in C_n forming a 1-chord cycle of length k automatically forms a 1-chord cycle of length $n + 2 - k$.

(2) is immediate from the definition of $c(n, k)$.

(3) Let G consist of an $(n - 1)$-cycle, together with an $(n - 1)$-chord cycle on the same vertices. Choose vertex v: let the chords at v be xv and yv and its adjacent cycle edges be uv and vw, with u, v, w, x, y appearing in clockwise order around the cycle. Replace v and its incident edges with two vertices v_u and v_w, with edges v_uv_w, uv_u, v_ww, xv_w and yv_u. The $(n - 1)$-chord cycle in G becomes an $(n-1)$-chord n-cycle. Add an n-th chord xv_u to give an $(n-1)$-chord $(n - 1)$-cycle. ■

Table 1 supplies some small values/bounds for $c(n, k)$. The lower bounds are supplied by Corollary 7; except for those values covered by Lemma 2, exact values and upper bounds were found by computer search.

		k										
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
n	**6**	2	3	5	6	6	6					
	7	2	3	5	6	6	7	7				
	8	3	4	5	6	6	7	8	8			
	9	3	4	5	6	7	8	8	9	9		
	10	4	4	5	6	≥ 6	≥ 7	≥ 8	≥ 9	10	10	
	11	4	4	≥ 5	≥ 6	≥ 7	≥ 7	≥ 8	≥ 9	≥ 10	11	11
	12	5	4	≥ 5	≥ 6	≥ 7	≥ 7	≥ 8	≥ 9	≥ 10	≥ 11	12
	13	5	4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 8	≥ 9	≥ 10	≥ 11	≥ 12

Table 1. Values of $c(n,k)$ for $6 \leq n \leq 13$ and $1 \leq k \leq 11$.

Our aim is to compare $c(n,k)$ with the number of chords required for pancyclicity and for *vertex pancyclicity*, in which each vertex must lie on a cycle of every length.

The following lower bound is stated without proof in [1].

Theorem 3. *In a pancyclic graph G on n vertices the number of edges is not less than $n-1+\log_2(n-1)$.*

For the sake of completeness we observe that Theorem 3 follows immediately from the following lemma.

Lemma 4. *Suppose p chords are added to C_n, $n \geq 3$. Then the number $N(n,p)$ of cycles in the resulting graph satisfies*

$$\binom{p+2}{2} \leq N(n,p) \leq 2^{p+1}-1.$$

Proof. Embed C_n convexly in the plane. Suppose the chords added to C_n are, in order of inclusion, $e_1, e_2, \ldots, e_p$. Say that e_i intersects e_j if these edges cross each other when added to the embedding of C_n. Let n_i be the number of new cycles obtained with e_i is added. Then n_i satisfies:

1. $n_i \geq i+1$, the minimum occurring if and only if the e_j are pairwise non-intersecting for $j \leq i$;

2. $n_i \leq 2^i$, the maximum occurring if and only if e_i intersects with e_j for all $j < i$, giving $n_i = \sum_{j=0}^{i} \binom{i}{j}$.

Now $1 + \sum_{i=1}^{p} (i+1) \leq 1 + \sum_{i=1}^{p} n_i \leq 1 + \sum_{i=1}^{p} 2^i$ and the result follows. ■

The exact value of the minimum number of edges in an n-vertex pancyclic graph has been calculated for small n by George *et al.* [5] and Griffin [6]. For $3 \leq n \leq 14$, the lower bound in Theorem 3 is exact; however, it can be seen that, for $n = 15, 16$, we must add four chords to C_n to achieve pancyclicity while the argument in the proof of Lemma 4 can only account for three.

As regards an upper bound on the number of chords required for pancyclicity, [1] again asserts $O(\log n)$, again without a proof. A $\log n$ construction has been given by Sridharan [7]. Together with Theorem 3 this gives an 'exact' growth rate for pancyclicity: it is achieved by adding $\Theta(\log n)$ chords to C_n.

In contrast, *vertex pancyclicity*, in which every vertex lies in a cycle of every length has been shown by Broersma [3] to require $\Theta(n)$ edges to be added to C_n. Our question is: where between $\log n$ and n does $c(n,k)$ lie? For fixed k, we find a lower bound strictly between the two: $\Omega(n^{1/k})$.

Let us for the moment restrict to $k \geq 3$. Suppose we add p chords to C_n, $3 \leq k \leq p \leq \binom{n}{2} - n$. Suppose that these p added chords include a k-cycle. We will use $K(k,p)$, defined for $k \geq 3$, to denote the maximum number of k-chord cycles that can be created in the resulting graph. Then $1 \leq K(k,p)$ by definition and $K(k,p) \leq 2^{p+1} - 1$ by Lemma 4. By lowering this upper bound we can increase the lower bound on $c(n,k)$.

Theorem 5. $K(k,p) \leq \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k}$.

We will use the following Lemma to prove Theorem 5.

Lemma 6. *Suppose that a set X of chords is added to C_n. In the resulting graph the maximum number of cycles passing all edges in X is*
1 *if X contains adjacent chords,*
2 *if no two chords of X are adjacent.*

Proof. Let G be the graph resulting from adding the chords of X to C_n. We may assume without loss of generality that G has no vertices of degree 2, since such vertices may be contracted out. For a given cycle in G passing all chords of X, let H denote the intersection of this cycle with the C_n. Then H consists of isolated vertices and disjoint edges, and H is completely determined once any of these vertices or edges is fixed. If two chords are adjacent this fixes an isolated vertex of H; if no two chords are adjacent then there is a maximum of two ways in which a single edge of H may be fixed. ■

***Proof of Theorem* 5.** By definition of $K(k,p)$ we must use a set, say S, of k chords to create a k-cycle. We add new chords to S, one by one. On adding the r-th additional chord, $1 \leq r \leq p-k$, we ask how many k-chord cycles use this chord. For any such a cycle the previous $r-1$ chords will be split between S and non-S chords: with i chords from S being used, $0 \leq i \leq k-1$, this can happen in

$$\binom{k}{i}\binom{r-1}{k-i-1}$$

ways. Since $i > 1$ forces two adjacent chords in S to be used, summing over i, according to Lemma 6, and then over r gives

$$K(k,p) \leq 1 + \sum\nolimits_{r=1}^{p-k} \left(2\sum\nolimits_{i=0}^{1} \binom{k}{i}\binom{r-1}{k-i-1} + \sum\nolimits_{i=2}^{k-1} \binom{k}{i}\binom{r-1}{k-i-1}\right).$$

This simplifies (e.g. using symbolic algebra software such as Maple) to give the result. ■

Corollary 7. For given positive integers k and n, with $3 \leq k \leq n$ and $n \geq 6$, the value of $c(n,k)$ is not less than the largest root of the following polynomial in p:

$$\Pi(p;n,k) = \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k} - n + k - 1.$$

Proof. Suppose that, with n and k fixed, we add p chords to C_n and create cycles of all lengths $\geq k$, each passing k chords. Then $n-k+1 \leq K(k,p)$. So p must satisfy $0 \leq \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k} - n + k - 1$. The right-hand side of this inequality is a polynomial in p which has positive slope at its largest root, so that $c(n,k)$ cannot be less than this root. ■

We finally extend our analysis to include the cases $k = 1, 2$.

Corollary 8. Let $n \geq 6$ be a positive integer. Then for $k \geq 1$ fixed, $c(n,k)$ is of order $\Omega(n^{1/k})$.

Proof. For $k = 1$ the required linear bound was provided in Lemma 2.

For $k = 2$ an analysis similar to that used in the proof of Theorem 5 shows that the number of 2-chord cycles which may be created by adding p chords to C_n is at most $p^2 - p - 1$. So to have 2-chord cycles of all lengths from 3 to n we require $p^2 - p - 1 \geq n - 2$. In this case we can solve explicitly to get the bound $p \geq \frac{1}{2}\left(1 + \sqrt{4n-3}\right)$.

Now suppose $k \geq 3$. In order to have all k-chord cycles of all lengths between k and n we must have

$$n - k + 1 \leq \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k} \leq f(k)p^k,$$

for some function $f(k)$. Therefore $p^k \geq (n-k+1)/f(k)$ so, for k fixed, $p = \Omega(n^{1/k})$. ■

Remark 1. We are suggesting that the value of $c(n,k)$ may be 'intermediate' between pancyclicity and vertex pancyclicity in the sense that the number of chords it requires to be added to C_n may lie between $\log n$ and n. Thus far we have only a lower bound in support of our suggestion. Moreover, a comparison of the growth orders, $\Omega(\log n)$ as opposed to $\Omega(n^{1/k})$, suggests that this is very much a 'for large n' type result. The equation $\ln n = n^{1/k}$ has two positive real solutions for $k \geq 3$, given in terms of the two real branches of the Lambert W function [4]. In particular $\ln n$ exceeds $n^{1/k}$ for $n > e^{-kW_{-1}(-1/k)}$, and this bound grows very fast with k. To give a specific example, $k = 10$, the log bound exceeds the 10-th root bound until the number of vertices exceeds about 3.4×10^{15}. Until then, so far as our analysis goes, we might expect 'most' pancyclic graphs to be 10-chord pancyclic. However we suggest that, in the long term, a guarantee of this implication, analogous to Hamiltonicity guaranteeing pancyclicity, will not be found.

Remark 2. We would like to know if $c(n,k)$ is monotonically increasing in n. However, it is still open even whether pancyclicity is monotonic in the number of chords requiring to be added to C_n (the question is investigated in [6]). We believe that $c(n,k)$ it is not increasing in k and $c(n,1) > c(n,2)$ for $n = 12, 13$ confirms this in a limited sense. Our $n^{1/k}$ lower bound instead suggests the possibility that $c(n,k)$ is convex for fixed n, as a function of k.

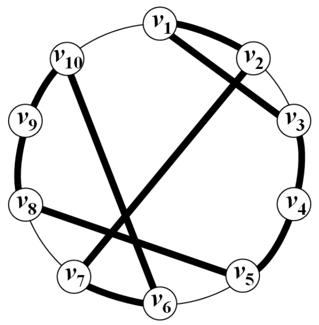

Figure 1. No 4-cycle uses exactly 1 chord of the bold-edge Hamilton cycle.

Remark 3. We observe that, unlike pancyclicity, the property of having cycles of all lengths each passing k chords is not an invariant of a graph: it depends on the initial choice of a Hamilton cycle. For example, in Figure 1, there are cycles of all lengths ≤ 9 each passing exactly one of the $c(10,1) = 4$ chords of the outer cycle but there is no 4-cycle passing exactly one chord of the bold-edge Hamilton cycle.

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Received 14 April 2014
Revised 7 November 2014
Accepted 7 November 2014