# PANCYCLICITY WHEN EACH CYCLE MUST PASS EXACTLY $k$ HAMILTON CYCLE CHORDS 

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#### Abstract

It is known that $\Theta(\log n)$ chords must be added to an $n$-cycle to produce a pancyclic graph; for vertex pancyclicity, where every vertex belongs to a cycle of every length, $\Theta(n)$ chords are required. A possibly 'intermediate' variation is the following: given $k, 1 \leq k \leq n$, how many chords must be added to ensure that there exist cycles of every possible length each of which passes exactly $k$ chords? For fixed $k$, we establish a lower bound of $\Omega\left(n^{1 / k}\right)$ on the growth rate.


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A simple graph $G$ on $n$ vertices is pancyclic if it has cycles of every length $l$, $3 \leq l \leq n$. The study of these graphs was initiated by Bondy's observation [1, 2]
that, for non-bipartite graphs, sufficient conditions for Hamiltonicity can also be sufficient for pancyclicity. In general, we may distinguish, in a pancyclic graph $G$, a Hamilton cycle $C$; then the remaining edges of $G$ form chords of $C$. We can then ask, given $k \leq l \leq n$ if, relative to $C$, a cycle of length $l$ exists which uses exactly $k$ chords. This suggests a $k$-chord analog of pancyclicity: do all possible cycle lengths occur when cycles must use exactly $k$-chords of a suitably chosen Hamilton cycle?

We accordingly define a function $c(n, k), n \geq 6, k \geq 1$, to be the smallest number of chords which must be added to an $n$-cycle in order that cycles of all possible lengths may be found, each passing exactly $k$ chords. No Hamilton cycle can use exactly one chord of another Hamilton cycle, so that when $k=1$ cycle lengths must lie between 3 and $n-1$. The function is undefined for $k>n$. We define the function for $n \geq 6$ because $n=4,5$ are too restrictive to be of interest to us.

Our aim in this paper is to investigate the growth of the function $c(n, k)$ as $n$ increases, for fixed $k$.

Example 1. Label the vertices around the cycle $C_{6}$, in order, as $v_{1}, \ldots, v_{6}$. Add chords $v_{1} v_{3}$ and $v_{1} v_{4}$; the result is a pancyclic graph. It also has cycles of all lengths $\leq 5$ each passing exactly one of the chords. If $v_{2} v_{6}$ is added then cycles exist of all lengths $\geq 3$, each passing two chords. If two further chords, $v_{2} v_{4}$ and $v_{4} v_{6}$, are added then cycles exist of all lengths $\geq 3$, each passing three chords. For 4 -chord cycles we require six chords to be added, i.e., $c(6,4)=6$. Six suitably chosen chords are also sufficient for 5 -chord and 6 -chord cycles: $c(6,5)=c(6,6)=$ 6.

Lemma 2. (1) $c(n, 1)=\left\lceil\frac{n-3}{2}\right\rceil$.
(2) $c(n, k) \geq k$, with equality if and only if $k=n$.
(3) $c(n, n-1)=n$.

Proof. (1) follows from the observation that a chord in $C_{n}$ forming a 1-chord cycle of length $k$ automatically forms a 1 -chord cycle of length $n+2-k$.
(2) is immediate from the definition of $c(n, k)$.
(3) Let $G$ consist of an $(n-1)$-cycle, together with an $(n-1)$-chord cycle on the same vertices. Choose vertex $v$ : let the chords at $v$ be $x v$ and $y v$ and its adjacent cycle edges be $u v$ and $v w$, with $u, v, w, x, y$ appearing in clockwise order around the cycle. Replace $v$ and its incident edges with two vertices $v_{u}$ and $v_{w}$, with edges $v_{u} v_{w}, u v_{u}, v_{w} w, x v_{w}$ and $y v_{u}$. The $(n-1)$-chord cycle in $G$ becomes an ( $n-1$ )-chord $n$-cycle. Add an $n$-th chord $x v_{u}$ to give an ( $n-1$ )-chord ( $n-1$ )-cycle.

Table 1 supplies some small values/bounds for $c(n, k)$. The lower bounds
are supplied by Corollary 7; except for those values covered by Lemma 2, exact values and upper bounds were found by computer search.

|  |  | $\boldsymbol{k}$ |  |  |  |  |  |  |  |  |  |  |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\mathbf{1 0}$ | $\mathbf{1 1}$ |
| $\boldsymbol{n}$ | $\mathbf{6}$ | 2 | 3 | 5 | 6 | 6 | 6 |  |  |  |  |  |
|  | $\mathbf{7}$ | 2 | 3 | 5 | 6 | 6 | 7 | 7 |  |  |  |  |
|  | $\mathbf{8}$ | 3 | 4 | 5 | 6 | 6 | 7 | 8 | 8 |  |  |  |
|  | $\mathbf{9}$ | 3 | 4 | 5 | 6 | 7 | 8 | 8 | 9 | 9 |  |  |
| $\mathbf{1 0}$ | 4 | 4 | 5 | 6 | $\geq 6$ | $\geq 7$ | $\geq 8$ | $\geq 9$ | 10 | 10 |  |  |
| $\mathbf{1 1}$ | 4 | 4 | $\geq 5$ | $\geq 6$ | $\geq 7$ | $\geq 7$ | $\geq 8$ | $\geq 9$ | $\geq 10$ | 11 | 11 |  |
| $\mathbf{1 2}$ | 5 | 4 | $\geq 5$ | $\geq 6$ | $\geq 7$ | $\geq 7$ | $\geq 8$ | $\geq 9$ | $\geq 10$ | $\geq 11$ | 12 |  |
| $\mathbf{1 3}$ | 5 | 4 | $\geq 5$ | $\geq 6$ | $\geq 7$ | $\geq 8$ | $\geq 8$ | $\geq 9$ | $\geq 10$ | $\geq 11$ | $\geq 12$ |  |

Table 1. Values of $c(n, k)$ for $6 \leq n \leq 13$ and $1 \leq k \leq 11$.
Our aim is to compare $c(n, k)$ with the number of chords required for pancyclicity and for vertex pancyclicity, in which each vertex must lie on a cycle of every length.

The following lower bound is stated without proof in [1].
Theorem 3. In a pancyclic graph $G$ on $n$ vertices the number of edges is not less than $n-1+\log _{2}(n-1)$.

For the sake of completeness we observe that Theorem 3 follows immediately from the following lemma.

Lemma 4. Suppose $p$ chords are added to $C_{n}, n \geq 3$. Then the number $N(n, p)$ of cycles in the resulting graph satisfies

$$
\binom{p+2}{2} \leq N(n, p) \leq 2^{p+1}-1 .
$$

Proof. Embed $C_{n}$ convexly in the plane. Suppose the chords added to $C_{n}$ are, in order of inclusion, $e_{1}, e_{2}, \ldots, e_{p}$. Say that $e_{i}$ intersects $e_{j}$ if these edges cross each other when added to the embedding of $C_{n}$. Let $n_{i}$ be the number of new cycles obtained with $e_{i}$ is added. Then $n_{i}$ satisfies:

1. $n_{i} \geq i+1$, the minimum occurring if and only if the $e_{j}$ are pairwise nonintersecting for $j \leq i$;
2. $n_{i} \leq 2^{i}$, the maximum occurring if and only if $e_{i}$ intersects with $e_{j}$ for all $j<i$, giving $n_{i}=\sum_{j=0}^{i}\binom{i}{j}$.
Now $1+\sum_{i=1}^{p}(i+1) \leq 1+\sum_{i=1}^{p} n_{i} \leq 1+\sum_{i=1}^{p} 2^{i}$ and the result follows.
The exact value of the minimum number of edges in an $n$-vertex pancyclic graph has been calculated for small $n$ by George et al. [5] and Griffin [6]. For $3 \leq n \leq 14$, the lower bound in Theorem 3 is exact; however, it can be seen that, for $n=15,16$, we must add four chords to $C_{n}$ to achieve pancyclicity while the argument in the proof of Lemma 4 can only account for three.

As regards an upper bound on the number of chords required for pancyclicity, [1] again asserts $O(\log n)$, again without a proof. A $\log n$ construction has been given by Sridharan [7]. Together with Theorem 3 this gives an 'exact' growth rate for pancyclicity: it is achieved by adding $\Theta(\log n)$ chords to $C_{n}$.

In contrast, vertex pancyclicity, in which every vertex lies in a cycle of every length has been shown by Broersma [3] to require $\Theta(n)$ edges to be added to $C_{n}$. Our question is: where between $\log n$ and $n$ does $c(n, k)$ lie? For fixed $k$, we find a lower bound strictly between the two: $\Omega\left(n^{1 / k}\right)$.

Let us for the moment restrict to $k \geq 3$. Suppose we add $p$ chords to $C_{n}$, $3 \leq k \leq p \leq\binom{ n}{2}-n$. Suppose that these $p$ added chords include a $k$-cycle. We will use $K(k, p)$, defined for $k \geq 3$, to denote the maximum number of $k$-chord cycles that can be created in the resulting graph. Then $1 \leq K(k, p)$ by definition and $K(k, p) \leq 2^{p+1}-1$ by Lemma 4. By lowering this upper bound we can increase the lower bound on $c(n, k)$.
Theorem 5. $K(k, p) \leq\binom{ p}{k}+k\binom{p-k}{k-1}+\binom{p-k}{k}$.
We will use the following Lemma to prove Theorem 5.
Lemma 6. Suppose that a set $X$ of chords is added to $C_{n}$. In the resulting graph the maximum number of cycles passing all edges in $X$ is
1 if $X$ contains adjacent chords,
2 if no two chords of $X$ are adjacent.
Proof. Let $G$ be the graph resulting from adding the chords of $X$ to $C_{n}$. We may assume without loss of generality that $G$ has no vertices of degree 2 , since such vertices may be contracted out. For a given cycle in $G$ passing all chords of $X$, let $H$ denote the intersection of this cycle with the $C_{n}$. Then $H$ consists of isolated vertices and disjoint edges, and $H$ is completely determined once any of these vertices or edges is fixed. If two chords are adjacent this fixes an isolated vertex of $H$; if no two chords are adjacent then there is a maximum of two ways in which a single edge of $H$ may be fixed.

Proof of Theorem 5. By definition of $K(k, p)$ we must use a set, say $S$, of $k$ chords to create a $k$-cycle. We add new chords to $S$, one by one. On adding the $r$-th additional chord, $1 \leq r \leq p-k$, we ask how many $k$-chord cycles use this chord. For any such a cycle the previous $r-1$ chords will be split between $S$ and non- $S$ chords: with $i$ chords from $S$ being used, $0 \leq i \leq k-1$, this can happen in

$$
\binom{k}{i}\binom{r-1}{k-i-1}
$$

ways. Since $i>1$ forces two adjacent chords in $S$ to be used, summing over $i$, according to Lemma 6 , and then over $r$ gives

$$
K(k, p) \leq 1+\sum_{r=1}^{p-k}\left(2 \sum_{i=0}^{1}\binom{k}{i}\binom{r-1}{k-i-1}+\sum_{i=2}^{k-1}\binom{k}{i}\binom{r-1}{k-i-1}\right) .
$$

This simplifies (e.g. using symbolic algebra software such as Maple) to give the result.

Corollary 7. For given positive integers $k$ and $n$, with $3 \leq k \leq n$ and $n \geq 6$, the value of $c(n, k)$ is not less than the largest root of the following polynomial in $p$ :

$$
\Pi(p ; n, k)=\binom{p}{k}+k\binom{p-k}{k-1}+\binom{p-k}{k}-n+k-1 .
$$

Proof. Suppose that, with $n$ and $k$ fixed, we add $p$ chords to $C_{n}$ and create cycles of all lengths $\geq k$, each passing $k$ chords. Then $n-k+1 \leq K(k, p)$. So $p$ must satisfy $0 \leq\binom{ p}{k}+k\binom{p-k}{k-1}+\binom{p-k}{k}-n+k-1$. The right-hand side of this inequality is a polynomial in $p$ which has positive slope at its largest root, so that $c(n, k)$ cannot be less than this root.

We finally extend our analysis to include the cases $k=1,2$.
Corollary 8. Let $n \geq 6$ be a positive integer. Then for $k \geq 1$ fixed, $c(n, k)$ is of order $\Omega\left(n^{1 / k}\right)$.
Proof. For $k=1$ the required linear bound was provided in Lemma 2.
For $k=2$ an analysis similar to that used in the proof of Theorem 5 shows that the number of 2 -chord cycles which may be created by adding $p$ chords to $C_{n}$ is at most $p^{2}-p-1$. So to have 2-chord cycles of all lengths from 3 to $n$ we require $p^{2}-p-1 \geq n-2$. In this case we can solve explicitly to get the bound $p \geq \frac{1}{2}(1+\sqrt{4 n-3})$.

Now suppose $k \geq 3$. In order to have all $k$-chord cycles of all lengths between $k$ and $n$ we must have

$$
n-k+1 \leq\binom{ p}{k}+k\binom{p-k}{k-1}+\binom{p-k}{k} \leq f(k) p^{k}
$$

for some function $f(k)$. Therefore $p^{k} \geq(n-k+1) / f(k)$ so, for $k$ fixed, $p=$ $\Omega\left(n^{1 / k}\right)$.

Remark 1. We are suggesting that the value of $c(n, k)$ may be 'intermediate' between pancyclicity and vertex pancyclicity in the sense that the number of chords it requires to be added to $C_{n}$ may lie between $\log n$ and $n$. Thus far we have only a lower bound in support of our suggestion. Moreover, a comparison of the growth orders, $\Omega(\log n)$ as opposed to $\Omega\left(n^{1 / k}\right)$, suggests that this is very much a 'for large $n$ ' type result. The equation $\ln n=n^{1 / k}$ has two positive real solutions for $k \geq 3$, given in terms of the two real branches of the Lambert $W$ function [4]. In particular $\ln n$ exceeds $n^{1 / k}$ for $n>e^{-k W_{-1}(-1 / k)}$, and this bound grows very fast with $k$. To give a specific example, $k=10$, the $\log$ bound exceeds the 10 -th root bound until the number of vertices exceeds about $3.4 \times 10^{15}$. Until then, so far as our analysis goes, we might expect 'most' pancyclic graphs to be 10 -chord pancyclic. However we suggest that, in the long term, a guarantee of this implication, analogous to Hamiltonicity guaranteeing pancyclicity, will not be found.

Remark 2. We would like to know if $c(n, k)$ is monotonically increasing in $n$. However, it is still open even whether pancyclicity is monotonic in the number of chords requiring to be added to $C_{n}$ (the question is investigated in [6]). We believe that $c(n, k)$ it is not increasing in $k$ and $c(n, 1)>c(n, 2)$ for $n=12,13$ confirms this in a limited sense. Our $n^{1 / k}$ lower bound instead suggests the possibility that $c(n, k)$ is convex for fixed $n$, as a function of $k$.


Figure 1 . No 4 -cycle uses exactly 1 chord of the bold-edge Hamilton cycle.

Remark 3. We observe that, unlike pancyclicity, the property of having cycles of all lengths each passing $k$ chords is not an invariant of a graph: it depends on the initial choice of a Hamilton cycle. For example, in Figure 1, there are cycles of all lengths $\leq 9$ each passing exactly one of the $c(10,1)=4$ chords of the outer

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cycle but there is no 4-cycle passing exactly one chord of the bold-edge Hamilton cycle.

## References

[1] J.A. Bondy, Pancyclic graphs I, J. Combin. Theory Ser. B 11 (1971) 80-84. doi:10.1016/0095-8956(71)90016-5
[2] J.A. Bondy, Pancyclic graphs: recent results, infinite and finite sets, in : Colloq. Math. Soc. János Bolyai, Keszthely, Hungary (1973) 181-187.
[3] H.J. Broersma, A note on the minimum size of a vertex pancyclic graph, Discrete Math. 164 (1997) 29-32. doi:10.1016/S0012-365X(96)00040-4
[4] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, On the Lambert W function, Adv. Comput. Math. 5 (1996) 329-359. doi:10.1007/BF02124750
[5] J.C. George, A.M. Marr and W.D. Wallis, Minimal pancyclic graphs, J. Combin. Math. Combin. Comput. 86 (2013) 125-133.
[6] S. Griffin, Minimal pancyclicity, preprint, arxiv.org/abs/1312.0274, 2013.
[7] M.R. Sridharan, On an extremal problem concerning pancyclic graphs, J. Math. Phys. Sci. 12 (1978) 297-306.

