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# THE SATURATION NUMBER FOR THE LENGTH OF DEGREE MONOTONE PATHS

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#### Abstract

A degree monotone path in a graph G is a path P such that the sequence of degrees of the vertices in the order in which they appear on P is monotonic. The length (number of vertices) of the longest degree monotone path in G is denoted by mp(G). This parameter, inspired by the well-known Erdős-Szekeres theorem, has been studied by the authors in two earlier papers. Here we consider a saturation problem for the parameter mp(G). We call G saturated if, for every edge e added to G, mp(G + e) > mp(G), and we define h(n, k) to be the least possible number of edges in a saturated graph G on n vertices with mp(G) < k, while  $mp(G + e) \ge k$  for every new edge e.

We obtain linear lower and upper bounds for h(n, k), we determine exactly the values of h(n, k) for k = 3 and 4, and we present constructions of saturated graphs.

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### 1. INTRODUCTION

Given a graph G, a degree monotone path is a path  $v_1v_2\cdots v_k$  such that  $deg(v_1) \leq deg(v_2) \leq \cdots \leq deg(v_k)$  or  $deg(v_1) \geq deg(v_2) \geq \cdots \geq deg(v_k)$ . This notion,

inspired by the well-known Erdős-Szekeres theorem [7, 9], was introduced in [6] under the name of uphill and downhill path in relation to domination problems, also studied in [4, 5, 11]. In [6], the study of the parameter mp(G), which denotes the length of the longest degree monotone path in G, was specifically suggested. This parameter was studied by the authors in [2, 3], and among many results obtained, the parameter  $f(n,k) = \max\{|E(G)| : |V(G)| = n, mp(G) < k\}$  was also defined. It was shown that this is closely related to the Turán numbers  $t(n,k) = \max\{|E(G)| : |V(G)| = n, G$  contains no copy of  $K_k\}$ .

A general form of the Turán numbers with respect to a graph H is  $t(n, H) = \max\{|E(G)| : |V(G)| = n, G \text{ contains no copy of } H\}$ . The study of Turán numbers is undoubtedly considered as one of the fundamental problems in extremal graph and hypergraph theory [1].

The Turán number has a counter-part known as the saturation number with respect to a given graph H, defined as

$$sat(n, H) = \min\{|E(G)| : |V(G)| = n, G \text{ contains no copy of } H,$$
  
but  $G + e$  contains  $H$  for any edge added to  $G\}.$ 

Tuza and Kászonyi in [12] launched a systematic study of sat(n, H) following an earlier result by Erdős, Hajnal and Moon [8] who proved that  $sat(n, K_k) = {\binom{k-2}{2}} + (k-2)(n-k+2)$  with a unique graph attaining this bound, namely  $K_{k-2} + \overline{K_{n-k+2}}$ . For the current paper, it is worth noting that  $sat(n, P_k)$  (sat(n, k) for short) is known [12] for every k and n sufficiently large with respect to k, and in particular for n large enough, sat(n, k) = n(1-c(k)), where c(k) < 1 is a positive constant which depends only on k (for the exact value we refer the interested reader to [12]). For a survey and recent information about saturation, see [10].

In this spirit, we call a graph G saturated if mp(G+e) > mp(G) for all new edges e joining non-adjacent vertices in G. If it happens that  $mp(G+e) \ge k$  for all new edges e we sometimes refer to the saturated graph G as k-saturated. By convention we say that  $K_m$  is k-saturated for  $m \le k - 1$ . Then we define

$$h(n,k) = \min\{|E(G)| : |V(G)| = n, G \text{ is } k\text{-saturated}\}.$$

In Section 2, we prove linear lower and upper bounds for this parameter. In Section 3, we provide exact determination of h(n,k) for k = 3,4. In Section 4 we present several open problems concerning h(n,k) for  $k \ge 5$  as well as several other problems and conjectures.

### 2. General Lower and Upper Bounds

#### 2.1. Lower bounds

We begin by showing that sat(n, k) is a lower bound for h(n, k).

## **Proposition 2.1.** For $k \ge 2$ , $h(n,k) \ge sat(n,k)$ .

**Proof.** Clearly, if G is a graph realising  $sat(n, P_k) = sat(n, k)$ , this means that G does not contain a copy of  $P_k$ , and hence no degree monotone path of length k. But G + e contains  $P_k$ , but not necessarily a degree monotone path of length k. Hence  $h(n, k) \ge sat(n, k)$ .

Recall that for fixed k and large n, sat(n,k) = n(1-c(k)) < n. We now strengthen Proposition 2.1 to show that for  $k \ge 4$ ,  $h(n,k) \ge n$ . First we prove a lemma, and subsequently a corollary, which will then be used in the main proof.

**Lemma 2.2.** Let G be a connected graph with a vertex u of degree 1 and a vertex v of maximum degree  $\Delta \ge 2$  which are not adjacent. Then  $mp(G+uv) \le mp(G)$ , namely G is not saturated.

**Proof.** Let H = G + uv and let P be a path in H which realizes mp(H). Let  $u^*$  and  $v^*$  be the vertices u and v as they appear in H.

If  $\Delta = 2$ , then clearly G is a path on  $k \ge 4$  vertices and mp(G) = k - 1, and if we take u to be the first vertex of the path, and v to be the  $(k - 1)^{th}$  vertex, then mp(H) = k - 1 = mp(G).

So we may assume  $\Delta \geq 3$ . Now, if  $u^*$  and  $v^*$  are not on P, then P is degree monotone in G and hence  $mp(H) \leq mp(G)$ . If  $v^*$  is on P but  $u^*$  is not, then  $v^*$  must be the last vertex on P, and hence the path P with  $v^*$  replaced by v is also degree monotone in G and  $mp(H) \leq mp(G)$ . Similarly, if  $u^*$  is on P but  $v^*$ is not, then  $u^*$  must be the first vertex on P, since clearly  $u^*$  cannot be in the "middle" of the path as then the next vertex on P must be  $v^*$ , which is not on P. Then the path P in G with  $u^*$  replaced by u is also degree monotone in Gand again  $mp(H) \leq mp(G)$ . If  $u^*$  is the last vertex on the path, then clearly Pis not maximal as  $P \cup \{v^*\}$  via the edge  $u^*v^*$  is a longer degree monotone path, contradicting maximality of P.

So the only remaining case to consider is when  $u^*$  and  $v^*$  are both on P. Then clearly  $v^*$  is the last vertex on P. If  $u^*$  is the first vertex, then either  $P = u^*v^*$  and  $mp(H) = 2 \leq mp(G)$ , or the path P is degree monotone in G too. If  $u^*$  is not the first vertex, then the next vertex on P must be  $v^*$  which is the last vertex. Hence, in this case, all predecessors of  $u^*$  on P must have degree at most 2. But if the first vertex y in P has degree 1, then, in G, the path  $y \cdots u$  is disconnected from the rest of G, which is impossible. Therefore deg(y) = 2 and y has a neighbour w which must have degree greater than 2 (note that w may be equal to  $v^*$  but cannot be any other vertex on P). But then, the path  $u \cdots yw$  is degree monotone in G and is of the same length as P, and hence  $mp(H) \leq mp(G)$ .

Lemma 2.2 is best possible with respect to the adjacency condition between minimum degrees and maximum degrees because if the minimum degree is greater than 1, and a vertex u of minimum degree is not adjacent to vertex v, then mp(G + uv) may be larger than mp(G). As an example, consider a graph  $G_n$  made up of the cycle  $C_{2n}$ ,  $n \geq 3$ , with vertices labelled  $v_1, v_2, \ldots, v_{2n}$ , and a vertex w connected to vertices  $v_1, v_3, v_5, \ldots, v_{2n-1}$ . Thus w has degree  $\Delta = n$  and  $\delta = 2$ , and  $mp(G_n) = 3$ . The vertices of degree 2 are not connected to w, but connecting any such vertex to w by an edge e gives  $mp(G_n + e) = 5$ . In fact, these graphs are 5-saturated even though they have non-adjacent vertices of maximum degree  $\Delta \geq 3$  and minimum degree  $\delta = 2$ .

**Corollary 2.3.** Let T be a tree on  $n \ge 3$  vertices. Then T is saturated for a degree monotone path if and only if  $T = K_{1,n-1}$ .

**Proof.** Suppose first  $mp(T) \ge 3$ . Then clearly T is not a star, hence there is a leaf not connected to a vertex of maximum degree and by Lemma 2.2, T is not saturated.

So suppose mp(T) = 2. If not all leaves are adjacent to the same vertex of maximum degre, then again by Lemma 2.2, T is not saturated. Hence T must be a star  $K_{1,n-1}$ .

Indeed,  $K_{1,n-1}$  is saturated and  $mp(K_{1,n-1}) = 2$  while  $mp(K_{1,n-1} + e) = 3$  for every edge  $e \notin E(K_{1,n-1})$ .

**Theorem 2.4.** For  $n \ge 3$  and  $k \ge 4$ ,  $h(n,k) \ge n$ .

**Proof.** We may assume that  $n \ge k$  for otherwise, trivially,  $K_n$  is saturated having  $\binom{n}{2} \ge n$  edges for  $n \ge 3$ .

So let G be a graph on  $n \ge k$  vertices realizing h(n,k),  $k \ge 4$ . If G is connected, then by Corollary 2.3, G is not a tree, and hence  $|E(G)| \ge n$  as required.

So we may assume that G is not connected, and let  $G_1, G_2, \ldots, G_t$  be the connected components of G. Again, by Corollary 2.3, we infer that every component on at least three vertices is not a tree and hence must have at least  $|V(G_j)|$  edges.

If there are two components  $G_i$  and  $G_j$  on at most two vertices, adding an edge joining these two components does not create a degree monotone path of length 4 or more, contradicting the fact that G is saturated.

If there is just one component on at most two vertices, then one can connect one vertex of this component to a vertex of maximum degree in another component, and again no degree monotone path of length four or more is created, contradicting the fact that G is saturated.

Hence

$$|E(G)| = \sum_{i=1} |E(G_i)| \ge \sum_{i=1} |V(G_i)| = n,$$

and therefore  $h(n,k) \ge n$  for  $n \ge 3$  and  $k \ge 4$ .

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### 2.2. Upper bounds

We now give a linear upper bound for h(n, k). We consider separately k odd and k even.

First, we recall the definition of the Cartesian product  $G \Box H$  for two graphs G and H. The vertex set of the product is  $V(G) \times V(H)$ . Two vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent if either  $u_1$  and  $u_2$  are adjacent in G and  $v_1 = v_2$ , or  $v_1, v_2$  are adjacent in H and  $u_1 = u_2$ .

**Theorem 2.5.** If  $k \geq 3$  is an odd integer, then  $h(n,k) \leq \frac{n(3k-1)}{12}$  for  $n \equiv 0 \pmod{\frac{3(k-1)}{2}}$ .

**Proof.** Consider the graph  $G = P_3 \Box K_t$  for  $k \ge 3$  odd and  $t = \frac{k-1}{2}$ . Clearly,  $|V(G)| = \frac{3(k-1)}{2}$  and  $|E(G)| = \frac{3(k-1)(k-3)}{8} + \frac{2(k-1)}{2} = \frac{(k-1)(3k-1)}{8}$ . For k = 3 (so t = 1) this is simply  $P_3$  and  $mp(P_3) = 2$ , while for k = 5 (so t = 2) this gives the graph  $G = P_3 \Box K_2$ , which is  $C_6$  plus one edge joining two antipodal vertices and clearly mp(G) = 4.

We now show that this graph, which has mp(G) = k - 1, is saturated. In  $G = P_3 \Box K_t$ , let the top t vertices be  $u_1, \ldots, u_t$ , all having degree t, the middle vertices  $v_1, \ldots, v_t$  all having degree t + 1, and the bottom vertices  $w_1, \ldots, w_t$  all having degree t. It is clear that mp(G) = 2t = k - 1, taking for example the path  $u_1 \cdots u_t v_t \cdots v_1$ . Because of the symmetry of G, we only need to check the addition of the edges  $u_1v_2, v_1w_2$  and  $u_1w_1$ .

- If the edge  $u_1v_2$  is added, then the path  $w_1 \cdots w_t v_t \cdots v_3 v_1 u_1 v_2$  has exactly t + t 2 + 3 = 2t + 1 = k vertices.
- If the edge  $v_1w_2$  is added, then the path  $u_1 \cdots u_t v_t \cdots v_2 w_2 v_1$  has exactly t + t 1 + 2 = 2t + 1 = k vertices.
- If the edge  $u_1w_1$  is added, then the path  $u_2 \cdots u_t v_t \cdots v_1 w_1 u_1$  has exactly t 1 + t + 2 = 2t + 1 = k vertices.

Hence G is saturated with mp(G) = k - 1.

We now consider two disjoint copies of G,  $G_1$  and  $G_2$ . We label this graph 2G and show that this graph is also saturated. Again labelling the vertices of G as above, by the symmetry of G we only need to consider the addition of the edges joining  $u_t$  in  $G_1$  to  $u_1$  in  $G_2$ ,  $u_t$  in  $G_1$  to  $v_1$  in  $G_2$ , and  $v_t$  in  $G_1$  to  $v_1$  in  $G_2$ .

- If the edge joining  $u_t$  in  $G_1$  to  $u_1$  in  $G_2$  is added, then the path  $u_1 \cdots u_t$  in  $G_1$  followed by  $u_1v_1 \cdots v_t$  in  $G_2$  has exactly t + t + 1 = 2t + 1 = k vertices.
- If the edge joining  $u_t$  in  $G_1$  to  $v_1$  in  $G_2$  is added, then the path  $v_1 \cdots v_t u_t$ in  $G_1$  followed by  $v_1 \cdots v_t$  in  $G_2$  has exactly t + 1 + t = 2t + 1 = k vertices.

• If the edge joining  $v_t$  in  $G_1$  to  $v_1$  in  $G_2$  is added, then the path  $u_t \cdots u_1 v_1 \cdots v_t$  in  $G_1$  followed by  $v_1$  in  $G_2$  has exactly 2t + 1 = k vertices.

Hence 2G is saturated, and clearly this also applies to  $p \ge 3$  disjoint copies of G, pG. Now pG has  $n = p \frac{3(k-1)}{2}$  vertices and  $p \frac{(k-1)(3k-1)}{8}$  edges. Hence, for  $n \equiv 0 \pmod{\frac{3(k-1)}{2}}$ , the number of edges is  $\frac{n(3k-1)}{12}$ , as stated.

**Lemma 2.6.** Let G be a saturated graph with mp(G) = k. Consider the graph H = G + v, where v is a new vertex connected to all the vertices of G. Then mp(H) = k + 1, and H is saturated.

**Proof.** Consider the graph H. Then deg(v) = |V(G)| and v has maximum degree. So any degree monotone path in G can be extended in H by including vertex v, and hence mp(H) = mp(G) + 1 = k + 1.

Now, since G is saturated, adding any edge e creates a degree monotone path of length k + 1, and hence, adding the same edge in H creates a path of length k + 2. The only edges which can be added in H are those that can be added in G, and hence H is saturated with mp(H) = k + 1, as required.

This lemma together with Theorem 2.5 leads to the following result.

**Theorem 2.7.** For even  $k, k \ge 4, h(n,k) \le \frac{n(3k+8)(k-2)}{4(3k-4)}$  for  $n \equiv 0 \pmod{\frac{3k-4}{2}}$ .

**Proof.** In Theorem 2.5 we proved that  $G = P_3 \Box K_t$ , where  $t = \frac{j-1}{2}$ , has mp(G) = j-1, and G is saturated for  $j \ge 3$  and j odd. Now by Lemma 2.6, H = G + v has mp(H) = j + 1 (even) and is saturated. Then H has  $\frac{3(j-1)}{2} + 1 = \frac{3j-1}{2}$  vertices and  $\frac{(j-1)(3j-1)}{8} + \frac{3(j-1)}{2} = \frac{(3j+11)(j-1)}{8}$  edges. Now let k = j + 1, and hence we have  $\frac{3k-4}{2}$  vertices and  $\frac{(3k+8)(k-2)}{2}$  edges.

We now consider two disjoint copies of H,  $H_1$  and  $H_2$  and call this graph 2H. We need only consider edges which involve the new vertex of degree  $\frac{3(k-2)}{2}$ , which has the largest degree, as other edges have the same effect as they have in 2G. If we connect the vertex of degree  $\frac{3(k-2)}{2}$  in  $H_1$  to that of the same degree in  $H_2$ , then we can take a path of length k - 1 in  $H_1$  ending with the vertex of maximum degree, and then move to the vertex in  $H_2$ , giving a path of length k. If we connect the vertex of degree  $\frac{3(k-2)}{2}$  in  $H_1$  to one of degree  $\frac{k}{2}$  in  $H_2$ , then we take a path of length k - 1 in  $H_2$  ending with the vertex connected to the vertex in  $H_1$ , and then move to this vertex in  $H_1$  to give a degree monotone path of length k. Finally, if we connect the vertex of degree  $\frac{3(k-2)}{2}$  in  $H_1$  to one of degree  $\frac{4k+2}{2}$  in  $H_2$ , then we can take a degree monotone path in  $H_2$  of length k - 1 ending with the vertex in  $H_2$  to give a degree monotone path of length k. I ending with the vertex connected to  $H_2$ , and then the vertex in  $H_2$  to give a degree monotone path of length k - 1 ending with the vertex connected to  $H_2$ , and then the vertex in  $H_2$  to give a degree monotone path of length k - 1 ending with the vertex connected to  $H_2$ , and then the vertex in  $H_2$  to give a degree monotone path of length k - 1 ending with the vertex connected to  $H_2$ , and then the vertex in  $H_2$  to give a degree monotone path of length k in 2H.

Hence 2H is saturated, and this also applies to  $p \ge 3$  disjoint copies of H, pH. This graph has  $n = p\frac{3k-4}{2}$  vertices and  $p\frac{(3k+8)(k-2)}{8}$  edges. Hence for  $n \equiv 0 \pmod{\frac{3k-4}{2}}$ , the number of edges is  $\frac{n(3k+8)(k-2)}{4(3k-4)}$ , as stated.

Next We show, as an example, how to extend the results given in Theorems 2.5 and 2.7, to the case where  $n \not\equiv 0 \pmod{f(k)}$ , where f(k) is the modulus given in these theorems. We will demonstrate it in the case k = 5.

**Proposition 2.8.** For  $n \ge 8$ ,  $h(n,5) \le \frac{7n+c(n \pmod{6})}{6}$ , where  $c(n \pmod{6}) = \{0,35,16,27,8,28\}$  for  $n \equiv 0,1,2,3,4,5 \pmod{6}$ , respectively.

**Proof.** Consider the graphs  $G = P_3 \Box K_2$ ,  $H = K_5 - e$  for  $e \in E(K_5)$  and  $K_4$ , which are sturated for k = 5 and clearly  $mp(G) = mp(H) = mp(K_4) = 4$ . Every integer  $n \ge 8$  can be represented in the form 6x + 5y + 4z with x, y, z non-negative integers. Hence x copies of G, y copies of H and z copies of  $K_4$  produce graphs for every  $n \ge 8$ . It is easy to check that any graph made up of two vertex disjoint copies of any combination of G, H and  $K_4$  is also saturated, and hence any combination of vertex disjoint copies of these graphs is saturated.

Hence any graph made up of a disjoint combination of any number of these three graphs is saturated.

For  $n \equiv 0 \pmod{6}$ , the result follows immediately by substituting k = 5 in Theorem 2.5.

For  $n \equiv 1 \pmod{6}$ , we take the graph made up of  $\frac{n-13}{6}$  copies of G, two copies  $K_4$  and one copy of H. The graph thus obtained is saturated and has  $\frac{7(n-13)}{6} + 12 + 9 = \frac{7n+35}{6}$  edges. For  $n \equiv 2 \pmod{6}$ , we take the graph made up of  $\frac{n-8}{6}$  copies of G and two

For  $n \equiv 2 \pmod{6}$ , we take the graph made up of  $\frac{n-8}{6}$  copies of G and two copies  $K_4$ . The graph thus obtained is saturated and has  $\frac{7(n-8)}{6} + 12 = \frac{7n+16}{6}$  edges.

For  $n \equiv 3 \pmod{6}$ , we take the graph made up of  $\frac{n-9}{6}$  copies of G, one copy of  $K_4$  and one copy of H. The graph thus obtained is saturated and has  $\frac{7(n-9)}{6} + 6 + 9 = \frac{7n+27}{6}$  edges. For  $n \equiv 4 \pmod{6}$ , we take the graph made up of  $\frac{n-4}{6}$  copies of G and one

For  $n \equiv 4 \pmod{6}$ , we take the graph made up of  $\frac{n-4}{6}$  copies of G and one copy of  $K_4$ . The graph thus obtained is saturated and has  $\frac{7(n-4)}{6} + 6 = \frac{7n+8}{6}$  edges.

For  $n \equiv 5 \pmod{6}$ , we take the graph made up of  $\frac{n-5}{6}$  copies of G and one copy of H. The graph thus obtained is saturated and has  $\frac{7(n-5)}{6} + 9 = \frac{7n+28}{6}$  edges.

Note: Applying the technique demonstrated in Proposition 2.8, we can extend Theorems 2.5 and 2.7 to cover all  $n \ge (k-1)(k-2)$ , and we state it rather crudely as follows.

1. For odd  $k, k \ge 3$ , and  $n \ge (k-1)(k-2), h(n,k) \le \frac{n(3k-1)}{12} + O(k^2)$ .

2. For even 
$$k, k \ge 4$$
, and  $n \ge (k-1)(k-2), h(n,k) \le \frac{n(3k+8)(k-2)}{4(3k-4)} + O(k^2)$ .

3. Determination of h(n,k) for k = 2, 3, 4.

First we determine the exact value of h(n, 2) and h(n, 3).

**Proposition 3.1.** (1) h(n,2) = 0. (2)  $h(n,3) = \frac{n}{2}$  for *n* even, while  $h(n,3) = \frac{n+1}{2}$  for *n* odd.

**Proof.** 1. mp(G) = 1 if and only if G is a graph with no edges, and any edge we add gives mp(G + e) = 2.

2. By Proposition 2.1,  $h(n,3) \ge sat(n,3) = \lfloor \frac{n}{2} \rfloor$ . Consider even. Let G be made up of  $\frac{n}{2}$  copies of  $K_2$ . This is the only graph which achieves sat(n,3). Clearly mp(G) = 2, and adding any edge will create a copy of  $P_4$  so mp(G+e) = 3.

Now if n is odd, then the graph G made up of  $\lfloor \frac{n}{2} \rfloor$  copies of  $K_2$ , and one copy of  $K_1$  achieves sat(n,3), and is the only such graph. Again mp(G) = 2. If we add an edge joining two vertices from disjoint copies of  $K_2$ , then we get a copy of  $P_4$  and mp(G+e) = 3. However, if we add a vertex joining a vertex from  $K_2$  to the vertex in  $K_1$ , then this gives a copy of  $P_3$ , and mp(G+e) = 2, hence  $h(n,3) \ge sat(n,3) + 1$ .

Consider the graph G made up of  $\frac{n-3}{2}$  copies of  $K_2$ , and a single copy of  $P_3$ . Again it is clear that mp(G) = 2. Adding an edge joining two vertices from disjoint copies of  $K_2$  then we get a copy of  $P_4$  and mp(G + e) = 3, while adding an edge joining a vertex from  $K_2$  to one in  $P_3$  gives mp(G + e) = 4. The number of edges in this graph is  $\frac{n+1}{2} = sat(n,3) + 1$ , as stated.

We now determine the exact value of h(n, 4). For this we need another lemma.

**Lemma 3.2.** Let G be a saturated connected graph with  $|E(G)| \leq |V(G)|$  and  $2 \leq mp(G) \leq 3$ . Then

- (1) If mp(G) = 2, then  $G = K_{1,\Delta}$ , and for  $\Delta \ge 2$ , mp(G + e) = 3, for every  $e \notin E(G)$ .
- (2) If mp(G) = 3, then  $G = K_3$ , which is saturated by definition.

**Proof.** Let G be such a graph. Then since  $|E(G)| \leq |V(G)|$ , G is either a tree or is unicyclic.

If G is a tree such that all leaves are adjacent to the same vertex which has maximum degree, that is  $G = K_{1,\Delta}$ , then mp(G) = 2 and, in case  $\Delta \ge 2$ , adding any edge between two leaves u and v gives mp(G + uv) = 3. If G is a tree but not  $K_{1,\Delta}$ , then there is a leaf u and a vertex v of maximum degree which are not adjacent, and hence by Lemma 2.2, G is not saturated.

So suppose G is unicyclic. Then it cannot be a simple cycle  $C_n$  on  $n \ge 4$  vertices, since otherwise  $mp(C_n) = n \ge 4$ . Observe that  $C_3 = K_3$  is saturated by definition. So G is unicyclic with at least one leaf if the cycle has at least four vertices.

Suppose mp(G) = 2. If there are at least two vertices on the cycle which have branches attached, then on one of these branches (including the vertex on the cycle) there must be a vertex of maximum degree, and on the other branch there must be a leaf not connected to this vertex of maximum degree, and hence by Lemma 2.2 G is not saturated. So there is precisely one vertex on the cycle with degree greater than two, which means that mp(G) > 2, a contradiction.

So now suppose mp(G) = 3. If there are at least two vertices on the cycle which have branches attached, then on one of these branches (including the vertex on the cycle) there must be a vertex of maximum degree, and on the other branch there must be a leaf not connected to this vertex of maximum degree, and hence by Lemma 2.2 G is not saturated. So there is precisely one vertex on the cycle with degree greater than two, and if the cycle has at least four vertices, then  $mp(G) \ge 4$ , a contradiction.

So it remains to consider the cycle  $K_3$  with exactly one vertex x with degree greater than two. Suppose the vertex x has p leaves and q branches with  $p, q \ge 0$ . We consider several cases.

Case 1. If  $p \ge 2$ , then we connect two leaves to get H with mp(H) = mp(G) = 3, and G is not saturated. Hence  $p \le 1$ .

Case 2. If p = 1 and  $q \ge 1$ , then either x is a vertex of maximum degree  $\Delta \ge 3$ , and there is a leaf not connected to x, so by Lemma 2.2 G is not saturated, or there is a vertex of maximum degree in one of these branch, so the leaf at x is not connected to the vertex of maximum degree and again by Lemma 2.2, G is not saturated.

Case 3. If p = 1 and q = 0, then G is  $K_3$  with a leaf attached and clearly it is not saturated.

Case 4. If p = 0 and  $q \ge 2$ , then either x is a vertex of maximum degree  $\Delta \ge 3$  and there is a leaf in the branch not connected to x, so by Lemma 2.2 G is not saturated, or there is a vertex of maximum degree in one of these branches, so the leaf at x is not connected to the vertex of maximum degree and again by Lemma 2.2, G is not saturated.

Case 5. If p = 0 and q = 1, then deg(x) = 3. Let z be the neighbour of x in this branch. If  $deg(z) \ge 3$ , then  $mp(G) \ge 4$ , a contradiction. Hence deg(z) = 2, and let w be the neighbour of z. If deg(w) = 1, then x has maximum degree, w

is not connected to x and by Lemma 2.2, G is not saturated. So  $deg(w) \ge 2$  and we consider two cases.

Case 5.1. deg(w) = 2. Let u be the neighbour of w. If  $deg(u) \le 2$ , then we have a degree monotone path uwzv of length four. So  $deg(u) \ge 3$ .

If deg(u) > 3, then if the edge xw is added, mp(G + xw) = 3 and G is not saturated. Hence deg(u) = 3. Let s and y be the neighbours of u. If either s or y have degree at least three, then we have degree montone paths of length four zwux or zwuy, a contradiction. So both s and y have degree at most two.

If either s or y is a leaf, say s, then either  $\Delta = 3$  and s is a leaf not connected to x, so by Lemma 2.2 G is not saturated, or  $\Delta \ge 4$  and is realized by a vertex r say on the branch at y. Again s is a leaf not adjacent to r, and by Lemma 2,2 G is not saturated.

So deg(s) = deg(y) = 2, and either the maximum degree  $\Delta = 3$  and there is a leaf not adjacent to x, so by Lemma 2.2 G is not saturated, or there is a vertex r of maximum degree  $\Delta \ge 4$ , which is on one of the branches starting at s or y, say s. But then there is a leaf on the branch starting at y not adjacent to the vertex r, and again by Lemma 2.2 G is saturated.

Case 5.2.  $deg(w) = t \ge 3$ . Let  $x_1, \ldots, x_t$  be the neighbors of w. If for some  $j, deg(x_j) = 1$ , then either  $\Delta = 3$  and  $x_j$  is not connected to x, so by Lemma 2.2 G is not saturated, or  $\Delta \ge 4$  and is realized by a vertex r on a branch at some  $x_i$ ,  $i \ne j$ . Then  $x_j$  is a leaf not adjacent to r, and by Lemma 2.2 G is not saturated.

So  $deg(x_j) \geq 2$  for j = 1, ..., t. Now if  $\Delta = 3$ , then a leaf on one these branches starting at  $x_1, ..., x_t$  is not connected to x, and by Lemma 2.2 G is not saturated. Otherwise,  $\Delta \geq 4$  and a vertex r of maximum degree appears on the branch starting at say  $x_j$ . Then a leaf on any other branch is not connected to r, and by Lemma 2.2 G is not saturated.

Hence  $G = K_3$  is the only saturated graph with  $|E(G)| \leq |V(G)|$  and mp(G) = 3.

**Theorem 3.3.** For  $n \equiv 0 \pmod{3}$ , h(n, 4) = n, while for  $n \equiv 1, 2 \pmod{3}$ , h(n, 4) = n + 1.

**Proof.** First we prove the upperbound for h(n, 4). Consider the following cases.

Case 1. Assume  $n \equiv 0 \pmod{3}$ . If G is made up of  $\frac{n}{3}$  copies of  $K_3$ , then clearly mp(G) = 3. Any edge we add gives a degree monotone path of length 4. So G is saturated and hence  $h(n, 4) \leq n$  for  $n \equiv 0 \pmod{3}$ .

Case 2. Assume  $n \equiv 1 \pmod{3}$ . Let G be made up of  $\frac{n-4}{3}$  copies of  $K_3$  and a copy of  $K_4 - e$ ,  $e \in E(K_4)$ . Clearly mp(G) = 3 and it is easy to see that  $mp(G + e) \geq 4$ . So G is saturated and hence  $h(n, 4) \leq n + 1$  for  $n \equiv 1 \pmod{3}$ .

Case 3. Assume  $n \equiv 2 \pmod{3}$ . Let G be made up of  $\frac{n-5}{3}$  copies of  $K_3$  and two copies of  $K_3$  with a common vertex. Clearly mp(G) = 3 and it is easy to see that  $mp(G+e) \ge 4$ . So G is saturated and hence  $h(n,4) \le n+1$  for  $n \equiv 2 \pmod{3}$ .

Now to the lower bound. Suppose G is a graph on  $n \ge 3$  vertices realising h(n, 4). If G is connected, then by Lemma 3.2, either G is  $K_3$  or  $|E(G)| \ge n+1$ . Hence we may assume that G is not connected, and let  $G_1, G_2, \ldots, G_t$  be the connected components of G. Again, by Lemma 3.2, every component  $G_j$  on at least 3 vertices is either  $K_3$  or contains at least  $|V(G_j)| + 1$  edges.

If there are at least two components, say  $G_i$  and  $G_j$ , on at most two vertices each, then we can just add an edge between a vertex in  $G_i$  and one in  $G_j$  without creating a degree monotone path of length more than 3, contradicting the fact that G is saturated.

Lastly, if there is just one component  $G_j$  on at most two vertices, then if we connect a vertex in this component to a vertex v of maximum degree in another component of G, then clearly no degree monotone path of length 4 or more is created, once again contradicting that G is saturated.

Hence all components of G have at least 3 vertices. If there are at least two components which are not  $K_3$ , then  $|E(G)| \ge n + 2$ , and this is not optimal by the constructions above. If there is just one component which is not  $K_3$ , then  $|E(G)| \ge n + 1$  and so for  $n \equiv 1, 2 \pmod{3}$ ,  $h(n, 4) \ge n + 1$  proving the constructions above are optimal.

Finally, if all components are  $K_3$ , then |E(G)| = n, proving h(n, 4) = n for  $n \equiv 0 \pmod{3}$ .

# 4. Concluding Remarks and Open Problems

Several open problems have arised during our work on this paper. We list some of the more interesting ones.

• The major role played in this paper by Lemma 2.2 and its consequences suggest:

**Problem 1**: Find further structural conditions (along the lines indicated in Lemma 2.2) indicating that a graph G is not saturated.

• In Corollary 2.3, we characterise saturated trees. In a previous paper [2] we characterised saturated graphs with mp(G) = 2. This leads to the following:

**Problem 2**: Characterise *k*-saturated graphs for other families of graphs such as maximal outerplanar graphs, maximal planar graphs, regular graphs, etc.

**Problem 3**: Characterise saturated graphs with mp(G) = 3.

• The parameter mp(G) can be very sensitive to edge-addition and edgedeletion, as shown in [3]. Also Theorem 2.5 gives  $h(n,7) \leq \frac{5n}{3}$  for  $n \equiv 0$ (mod 9) while Theorem 2.7 gives  $h(n,6) \leq \frac{13n}{7}$  for  $n \equiv 0 \pmod{7}$ . These facts suggest the following monotonicity problem.

**Problem 4**: Is it true that, at least for *n* large enough, depending on *k*, and for  $k \ge 2$ ,  $h(n, k+1) \ge h(n, k)$ ?

If true, this will have the immediate implication that the construction for h(n, 6) is not optimal and that in fact  $h(n, 6) \leq \frac{5n(1+o(1))}{3}$  by the above upper bound for h(n, 7).

• The upper bound constructions given in Theorem 2.5 and Theorem 2.7 are probably not optimal.

**Problem 5**: Improve upon the upper bounds obtained in Theorems 2.5 and 2.7.

- The lower bound given in Theorem 2.4 proved to be sharp in the case k = 4.
  Problem 6: Improve upon the lower bound h(n, k) ≥ n for k ≥ 5.
- In Proposition 2.8 we have shown that  $h(n,5) \leq \frac{7n}{6} + c(n \pmod{6})$ .
- **Problem 7**: Determine h(n, 5) exactly. In particular, is it true that  $h(n, 5) = \frac{7n(1+o(1))}{6}$ ?
- Lastly, recall that sat(n, k) = n(1 c(k)) < n for every large k and n.

**Problem 8**: Is it true that  $h(n,k) \leq cn$  for some constant c independent of k?

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