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THE SATURATION NUMBER FOR THE LENGTH OF DEGREE MONOTONE PATHS

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Abstract

A degree monotone path in a graph G is a path P such that the sequence of degrees of the vertices in the order in which they appear on P is monotonic. The length (number of vertices) of the longest degree monotone path in G is denoted by $mp(G)$. This parameter, inspired by the well-known Erdős-Szekeres theorem, has been studied by the authors in two earlier papers. Here we consider a saturation problem for the parameter $mp(G)$. We call G saturated if, for every edge e added to G, $mp(G+e) > mp(G)$, and we define $h(n,k)$ to be the least possible number of edges in a saturated graph G on n vertices with $mp(G) < k$, while $mp(G+e) \geq k$ for every new edge e.

We obtain linear lower and upper bounds for $h(n,k)$, we determine exactly the values of $h(n,k)$ for $k = 3$ and 4, and we present constructions of saturated graphs.

Keywords: paths, degrees, saturation.

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1. Introduction

Given a graph G, a degree monotone path is a path $v_1v_2\cdots v_k$ such that $deg(v_1) \leq deg(v_2) \leq \cdots \leq deg(v_k)$ or $deg(v_1) \geq deg(v_2) \geq \cdots \geq deg(v_k)$. This notion, inspired by the well-known Erdős-Szekeres theorem [7, 9], was introduced in [6] under the name of uphill and downhill path in relation to domination problems, also studied in [4, 5, 11]. In [6], the study of the parameter $mp(G)$, which denotes the length of the longest degree monotone path in G, was specifically suggested. This parameter was studied by the authors in [2, 3], and among many results obtained, the parameter $f(n,k) = \max\{|E(G)| : |V(G)| = n, mp(G) < k\}$ was also defined. It was shown that this is closely related to the Turán numbers $t(n,k) = \max\{|E(G)| : |V(G)| = n, G \text{ contains no copy of } K_k\}$.

A general form of the Turán numbers with respect to a graph H is $t(n,H) = \max\{|E(G)| : |V(G)| = n, G \text{ contains no copy of } H\}$. The study of Turán numbers is undoubtedly considered as one of the fundamental problems in extremal graph and hypergraph theory [1].

The Turán number has a counter-part known as the saturation number with respect to a given graph H, defined as

$$sat(n,H) = \min\{|E(G)| : |V(G)| = n,\ G \text{ contains no copy of } H, \text{ but } G+e \text{ contains } H \text{ for any edge added to } G\}.$$

Tuza and Kászonyi in [12] launched a systematic study of $sat(n,H)$ following an earlier result by Erdős, Hajnal and Moon [8] who proved that $sat(n,K_k) = \binom{k-2}{2} + (k-2)(n-k+2)$ with a unique graph attaining this bound, namely $K_{k-2} + \overline{K}_{n-k+2}$. For the current paper, it is worth noting that $sat(n,P_k)$ ($sat(n,k)$ for short) is known [12] for every k and n sufficiently large with respect to k, and in particular for n large enough, $sat(n,k) = n(1-c(k))$, where $c(k) < 1$ is a positive constant which depends only on k (for the exact value we refer the interested reader to [12]). For a survey and recent information about saturation, see [10].

In this spirit, we call a graph G saturated if $mp(G+e) > mp(G)$ for all new edges e joining non-adjacent vertices in G. If it happens that $mp(G+e) \geq k$ for all new edges e we sometimes refer to the saturated graph G as k-saturated. By convention we say that K_m is k-saturated for $m \leq k-1$. Then we define

$$h(n,k) = \min\{|E(G)| : |V(G)| = n, G \text{ is } k\text{-saturated}\}.$$

In Section 2, we prove linear lower and upper bounds for this parameter. In Section 3, we provide exact determination of $h(n,k)$ for $k = 3,4$. In Section 4 we present several open problems concerning $h(n,k)$ for $k \geq 5$ as well as several other problems and conjectures.

2. General Lower and Upper bounds

2.1. Lower bounds

We begin by showing that $sat(n,k)$ is a lower bound for $h(n,k)$.

Proposition 2.1. *For $k \geq 2$, $h(n,k) \geq sat(n,k)$.*

Proof. Clearly, if G is a graph realising $sat(n,P_k) = sat(n,k)$, this means that G does not contain a copy of P_k, and hence no degree monotone path of length k. But $G+e$ contains P_k, but not necessarily a degree monotone path of length k. Hence $h(n,k) \geq sat(n,k)$. ■

Recall that for fixed k and large n, $sat(n,k) = n(1-c(k)) < n$. We now strengthen Proposition 2.1 to show that for $k \geq 4$, $h(n,k) \geq n$. First we prove a lemma, and subsequently a corollary, which will then be used in the main proof.

Lemma 2.2. *Let G be a connected graph with a vertex u of degree 1 and a vertex v of maximum degree $\Delta \geq 2$ which are not adjacent. Then $mp(G+uv) \leq mp(G)$, namely G is not saturated.*

Proof. Let $H = G+uv$ and let P be a path in H which realizes $mp(H)$. Let u^* and v^* be the vertices u and v as they appear in H.

If $\Delta = 2$, then clearly G is a path on $k \geq 4$ vertices and $mp(G) = k-1$, and if we take u to be the first vertex of the path, and v to be the $(k-1)^{th}$ vertex, then $mp(H) = k-1 = mp(G)$.

So we may assume $\Delta \geq 3$. Now, if u^* and v^* are not on P, then P is degree monotone in G and hence $mp(H) \leq mp(G)$. If v^* is on P but u^* is not, then v^* must be the last vertex on P, and hence the path P with v^* replaced by v is also degree monotone in G and $mp(H) \leq mp(G)$. Similarly, if u^* is on P but v^* is not, then u^* must be the first vertex on P, since clearly u^* cannot be in the "middle" of the path as then the next vertex on P must be v^*, which is not on P. Then the path P in G with u^* replaced by u is also degree monotone in G and again $mp(H) \leq mp(G)$. If u^* is the last vertex on the path, then clearly P is not maximal as $P \cup \{v^*\}$ via the edge u^*v^* is a longer degree monotone path, contradicting maximality of P.

So the only remaining case to consider is when u^* and v^* are both on P. Then clearly v^* is the last vertex on P. If u^* is the first vertex, then either $P = u^*v^*$ and $mp(H) = 2 \leq mp(G)$, or the path P is degree monotone in G too. If u^* is not the first vertex, then the next vertex on P must be v^* which is the last vertex. Hence, in this case, all predecessors of u^* on P must have degree at most 2. But if the first vertex y in P has degree 1, then, in G, the path $y \cdots u$ is disconnected from the rest of G, which is impossible. Therefore $deg(y) = 2$ and y has a neighbour w which must have degree greater than 2 (note that w may be equal to v^* but cannot be any other vertex on P). But then, the path $u \cdots yw$ is degree monotone in G and is of the same length as P, and hence $mp(H) \leq mp(G)$. ■

Lemma 2.2 is best possible with respect to the adjacency condition between minimum degrees and maximum degrees because if the minimum degree is greater than 1, and a vertex u of minimum degree is not adjacent to vertex v, then $mp(G + uv)$ may be larger than $mp(G)$. As an example, consider a graph G_n made up of the cycle C_{2n}, $n \geq 3$, with vertices labelled $v_1, v_2, \ldots, v_{2n}$, and a vertex w connected to vertices $v_1, v_3, v_5, \ldots, v_{2n-1}$. Thus w has degree $\Delta = n$ and $\delta = 2$, and $mp(G_n) = 3$. The vertices of degree 2 are not connected to w, but connecting any such vertex to w by an edge e gives $mp(G_n + e) = 5$. In fact, these graphs are 5-saturated even though they have non-adjacent vertices of maximum degree $\Delta \geq 3$ and minimum degree $\delta = 2$.

Corollary 2.3. *Let T be a tree on $n \geq 3$ vertices. Then T is saturated for a degree monotone path if and only if $T = K_{1,n-1}$.*

Proof. Suppose first $mp(T) \geq 3$. Then clearly T is not a star, hence there is a leaf not connected to a vertex of maximum degree and by Lemma 2.2, T is not saturated.

So suppose $mp(T) = 2$. If not all leaves are adjacent to the same vertex of maximum degre, then again by Lemma 2.2, T is not saturated. Hence T must be a star $K_{1,n-1}$.

Indeed, $K_{1,n-1}$ is saturated and $mp(K_{1,n-1}) = 2$ while $mp(K_{1,n-1} + e) = 3$ for every edge $e \notin E(K_{1,n-1})$. ■

Theorem 2.4. *For $n \geq 3$ and $k \geq 4$, $h(n, k) \geq n$.*

Proof. We may assume that $n \geq k$ for otherwise, trivially, K_n is saturated having $\binom{n}{2} \geq n$ edges for $n \geq 3$.

So let G be a graph on $n \geq k$ vertices realizing $h(n, k)$, $k \geq 4$. If G is connected, then by Corollary 2.3, G is not a tree, and hence $|E(G)| \geq n$ as required.

So we may assume that G is not connected, and let $G_1, G_2, \ldots, G_t$ be the connected components of G. Again, by Corollary 2.3, we infer that every component on at least three vertices is not a tree and hence must have at least $|V(G_j)|$ edges.

If there are two components G_i and G_j on at most two vertices, adding an edge joining these two components does not create a degree monotone path of length 4 or more, contradicting the fact that G is saturated.

If there is just one component on at most two vertices, then one can connect one vertex of this component to a vertex of maximum degree in another component, and again no degree monotone path of length four or more is created, contradicting the fact that G is saturated.

Hence
$$|E(G)| = \sum_{i=1} |E(G_i)| \geq \sum_{i=1} |V(G_i)| = n,$$
and therefore $h(n, k) \geq n$ for $n \geq 3$ and $k \geq 4$. ■

2.2. Upper bounds

We now give a linear upper bound for $h(n,k)$. We consider separately k odd and k even.

First, we recall the definition of the Cartesian product $G \square H$ for two graphs G and H. The vertex set of the product is $V(G) \times V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either u_1 and u_2 are adjacent in G and $v_1 = v_2$, or v_1, v_2 are adjacent in H and $u_1 = u_2$.

Theorem 2.5. *If $k \geq 3$ is an odd integer, then $h(n,k) \leq \frac{n(3k-1)}{12}$ for $n \equiv 0 \pmod{\frac{3(k-1)}{2}}$.*

Proof. Consider the graph $G = P_3 \square K_t$ for $k \geq 3$ odd and $t = \frac{k-1}{2}$. Clearly, $|V(G)| = \frac{3(k-1)}{2}$ and $|E(G)| = \frac{3(k-1)(k-3)}{8} + \frac{2(k-1)}{2} = \frac{(k-1)(3k-1)}{8}$. For $k = 3$ (so $t = 1$) this is simply P_3 and $mp(P_3) = 2$, while for $k = 5$ (so $t = 2$) this gives the graph $G = P_3 \square K_2$, which is C_6 plus one edge joining two antipodal vertices and clearly $mp(G) = 4$.

We now show that this graph, which has $mp(G) = k - 1$, is saturated. In $G = P_3 \square K_t$, let the top t vertices be $u_1, \ldots, u_t$, all having degree t, the middle vertices $v_1, \ldots, v_t$ all having degree $t+1$, and the bottom vertices $w_1, \ldots, w_t$ all having degree t. It is clear that $mp(G) = 2t = k - 1$, taking for example the path $u_1 \cdots u_t v_t \cdots v_1$. Because of the symmetry of G, we only need to check the addition of the edges $u_1 v_2$, $v_1 w_2$ and $u_1 w_1$.

- If the edge $u_1 v_2$ is added, then the path $w_1 \cdots w_t v_t \cdots v_3 v_1 u_1 v_2$ has exactly $t + t - 2 + 3 = 2t + 1 = k$ vertices.
- If the edge $v_1 w_2$ is added, then the path $u_1 \cdots u_t v_t \cdots v_2 w_2 v_1$ has exactly $t + t - 1 + 2 = 2t + 1 = k$ vertices.
- If the edge $u_1 w_1$ is added, then the path $u_2 \cdots u_t v_t \cdots v_1 w_1 u_1$ has exactly $t - 1 + t + 2 = 2t + 1 = k$ vertices.

Hence G is saturated with $mp(G) = k - 1$.

We now consider two disjoint copies of G, G_1 and G_2. We label this graph $2G$ and show that this graph is also saturated. Again labelling the vertices of G as above, by the symmetry of G we only need to consider the addition of the edges joining u_t in G_1 to u_1 in G_2, u_t in G_1 to v_1 in G_2, and v_t in G_1 to v_1 in G_2.

- If the edge joining u_t in G_1 to u_1 in G_2 is added, then the path $u_1 \cdots u_t$ in G_1 followed by $u_1 v_1 \cdots v_t$ in G_2 has exactly $t + t + 1 = 2t + 1 = k$ vertices.
- If the edge joining u_t in G_1 to v_1 in G_2 is added, then the path $v_1 \cdots v_t u_t$ in G_1 followed by $v_1 \cdots v_t$ in G_2 has exactly $t + 1 + t = 2t + 1 = k$ vertices.

- If the edge joining v_t in G_1 to v_1 in G_2 is added, then the path $u_t \cdots u_1 v_1 \cdots v_t$ in G_1 followed by v_1 in G_2 has exactly $2t+1=k$ vertices.

Hence $2G$ is saturated, and clearly this also applies to $p \geq 3$ disjoint copies of G, pG. Now pG has $n = p\frac{3(k-1)}{2}$ vertices and $p\frac{(k-1)(3k-1)}{8}$ edges. Hence, for $n \equiv 0 \pmod{\frac{3(k-1)}{2}}$, the number of edges is $\frac{n(3k-1)}{12}$, as stated. ∎

Lemma 2.6. *Let G be a saturated graph with $mp(G)=k$. Consider the graph $H = G+v$, where v is a new vertex connected to all the vertices of G. Then $mp(H)=k+1$, and H is saturated.*

Proof. Consider the graph H. Then $deg(v) = |V(G)|$ and v has maximum degree. So any degree monotone path in G can be extended in H by including vertex v, and hence $mp(H) = mp(G)+1 = k+1$.

Now, since G is saturated, adding any edge e creates a degree monotone path of length $k+1$, and hence, adding the same edge in H creates a path of length $k+2$. The only edges which can be added in H are those that can be added in G, and hence H is saturated with $mp(H)=k+1$, as required. ∎

This lemma together with Theorem 2.5 leads to the following result.

Theorem 2.7. *For even k, $k \geq 4$, $h(n,k) \leq \frac{n(3k+8)(k-2)}{4(3k-4)}$ for $n \equiv 0 \pmod{\frac{3k-4}{2}}$.*

Proof. In Theorem 2.5 we proved that $G = P_3 \square K_t$, where $t = \frac{j-1}{2}$, has $mp(G) = j-1$, and G is saturated for $j \geq 3$ and j odd. Now by Lemma 2.6, $H = G+v$ has $mp(H) = j+1$ (even) and is saturated. Then H has $\frac{3(j-1)}{2}+1 = \frac{3j-1}{2}$ vertices and $\frac{(j-1)(3j-1)}{8} + \frac{3(j-1)}{2} = \frac{(3j+11)(j-1)}{8}$ edges. Now let $k = j+1$, and hence we have $\frac{3k-4}{2}$ vertices and $\frac{(3k+8)(k-2)}{2}$ edges.

We now consider two disjoint copies of H, H_1 and H_2 and call this graph $2H$. We need only consider edges which involve the new vertex of degree $\frac{3(k-2)}{2}$, which has the largest degree, as other edges have the same effect as they have in $2G$. If we connect the vertex of degree $\frac{3(k-2)}{2}$ in H_1 to that of the same degree in H_2, then we can take a path of length $k-1$ in H_1 ending with the vertex of maximum degree, and then move to the vertex in H_2, giving a path of length k. If we connect the vertex of degree $\frac{3(k-2)}{2}$ in H_1 to one of degree $\frac{k}{2}$ in H_2, then we take a path of length $k-1$ in H_2 ending with the vertex connected to the vertex in H_1, and then move to this vertex in H_1 to give a degree monotone path of length k. Finally, if we connect the vertex of degree $\frac{3(k-2)}{2}$ in H_1 to one of degree $\frac{k+2}{2}$ in H_2, then we can take a degree monotone path in H_2 of length $k-1$ ending with the vertex connected to H_2, and then the vertex in H_2 to give a degree monotone path of length k in $2H$.

Hence $2H$ is saturated, and this also applies to $p \geq 3$ disjoint copies of H, pH. This graph has $n = p\frac{3k-4}{2}$ vertices and $p\frac{(3k+8)(k-2)}{8}$ edges. Hence for $n \equiv 0$ (mod $\frac{3k-4}{2}$), the number of edges is $\frac{n(3k+8)(k-2)}{4(3k-4)}$, as stated. ■

Next We show, as an example, how to extend the results given in Theorems 2.5 and 2.7 , to the case where $n \not\equiv 0 \pmod{f(k)}$, where $f(k)$ is the modulus given in these theorems. We will demonstrate it in the case $k = 5$.

Proposition 2.8. *For $n \geq 8$, $h(n, 5) \leq \frac{7n+c(n \pmod 6)}{6}$, where $c(n \pmod 6)) = \{0, 35, 16, 27, 8, 28\}$ for $n \equiv 0, 1, 2, 3, 4, 5 \pmod 6$, respectively.*

Proof. Consider the graphs $G = P_3 \square K_2$, $H = K_5 - e$ for $e \in E(K_5)$ and K_4, which are sturated for $k = 5$ and clearly $mp(G) = mp(H) = mp(K_4) = 4$. Every integer $n \geq 8$ can be represented in the form $6x + 5y + 4z$ with x, y, z non-negative integers. Hence x copies of G, y copies of H and z copies of K_4 produce graphs for every $n \geq 8$. It is easy to check that any graph made up of two vertex disjoint copies of any combination of G, H and K_4 is also saturated, and hence any combination of vertex disjoint copies of these graphs is saturated.

Hence any graph made up of a disjoint combination of any number of these three graphs is saturated.

For $n \equiv 0 \pmod 6$, the result follows immediately by substituting $k = 5$ in Theorem 2.5.

For $n \equiv 1 \pmod 6$, we take the graph made up of $\frac{n-13}{6}$ copies of G, two copies K_4 and one copy of H. The graph thus obtained is saturated and has $\frac{7(n-13)}{6} + 12 + 9 = \frac{7n+35}{6}$ edges.

For $n \equiv 2 \pmod 6$, we take the graph made up of $\frac{n-8}{6}$ copies of G and two copies K_4. The graph thus obtained is saturated and has $\frac{7(n-8)}{6} + 12 = \frac{7n+16}{6}$ edges.

For $n \equiv 3 \pmod 6$, we take the graph made up of $\frac{n-9}{6}$ copies of G, one copy of K_4 and one copy of H. The graph thus obtained is saturated and has $\frac{7(n-9)}{6} + 6 + 9 = \frac{7n+27}{6}$ edges.

For $n \equiv 4 \pmod 6$, we take the graph made up of $\frac{n-4}{6}$ copies of G and one copy of K_4 . The graph thus obtained is saturated and has $\frac{7(n-4)}{6} + 6 = \frac{7n+8}{6}$ edges.

For $n \equiv 5 \pmod 6$, we take the graph made up of $\frac{n-5}{6}$ copies of G and one copy of H. The graph thus obtained is saturated and has $\frac{7(n-5)}{6} + 9 = \frac{7n+28}{6}$ edges. ■

Note: Applying the technique demonstrated in Proposition 2.8, we can extend Theorems 2.5 and 2.7 to cover all $n \geq (k-1)(k-2)$, and we state it rather crudely as follows.

1. For odd k, $k \geq 3$, and $n \geq (k-1)(k-2)$, $h(n, k) \leq \frac{n(3k-1)}{12} + O(k^2)$.

2. For even k, $k \geq 4$, and $n \geq (k-1)(k-2)$, $h(n,k) \leq \frac{n(3k+8)(k-2)}{4(3k-4)} + O(k^2)$.

3. Determination of $h(n,k)$ for $k = 2,3,4$.

First we determine the exact value of $h(n,2)$ and $h(n,3)$.

Proposition 3.1. (1) $h(n,2) = 0$.
(2) $h(n,3) = \frac{n}{2}$ *for n even, while* $h(n,3) = \frac{n+1}{2}$ *for n odd.*

Proof. 1. $mp(G) = 1$ if and only if G is a graph with no edges, and any edge we add gives $mp(G+e) = 2$.

2. By Proposition 2.1, $h(n,3) \geq sat(n,3) = \lfloor \frac{n}{2} \rfloor$. Consider even. Let G be made up of $\frac{n}{2}$ copies of K_2. This is the only graph which achieves $sat(n,3)$. Clearly $mp(G) = 2$, and adding any edge will create a copy of P_4 so $mp(G+e) = 3$.

Now if n is odd, then the graph G made up of $\lfloor \frac{n}{2} \rfloor$ copies of K_2, and one copy of K_1 achieves $sat(n,3)$, and is the only such graph. Again $mp(G) = 2$. If we add an edge joining two vertices from disjoint copies of K_2, then we get a copy of P_4 and $mp(G+e) = 3$. However, if we add a vertex joining a vertex from K_2 to the vertex in K_1, then this gives a copy of P_3, and $mp(G+e) = 2$, hence $h(n,3) \geq sat(n,3) + 1$.

Consider the graph G made up of $\frac{n-3}{2}$ copies of K_2, and a single copy of P_3. Again it is clear that $mp(G) = 2$. Adding an edge joining two vertices from disjoint copies of K_2 then we get a copy of P_4 and $mp(G+e) = 3$, while adding an edge joining a vertex from K_2 to one in P_3 gives $mp(G+e) = 4$. The number of edges in this graph is $\frac{n+1}{2} = sat(n,3) + 1$, as stated. ■

We now determine the exact value of $h(n,4)$. For this we need another lemma.

Lemma 3.2. *Let G be a saturated connected graph with $|E(G)| \leq |V(G)|$ and $2 \leq mp(G) \leq 3$. Then*
(1) *If $mp(G) = 2$, then $G = K_{1,\Delta}$, and for $\Delta \geq 2$, $mp(G+e) = 3$, for every $e \notin E(G)$.*
(2) *If $mp(G) = 3$, then $G = K_3$, which is saturated by definition.*

Proof. Let G be such a graph. Then since $|E(G)| \leq |V(G)|$, G is either a tree or is unicyclic.

If G is a tree such that all leaves are adjacent to the same vertex which has maximum degree, that is $G = K_{1,\Delta}$, then $mp(G) = 2$ and, in case $\Delta \geq 2$, adding any edge between two leaves u and v gives $mp(G+uv) = 3$. If G is a tree but not $K_{1,\Delta}$, then there is a leaf u and a vertex v of maximum degree which are not adjacent, and hence by Lemma 2.2, G is not saturated.

So suppose G is unicyclic. Then it cannot be a simple cycle C_n on $n \geq 4$ vertices, since otherwise $mp(C_n) = n \geq 4$. Observe that $C_3 = K_3$ is saturated by definition. So G is unicyclic with at least one leaf if the cycle has at least four vertices.

Suppose $mp(G) = 2$. If there are at least two vertices on the cycle which have branches attached, then on one of these branches (including the vertex on the cycle) there must be a vertex of maximum degree, and on the other branch there must be a leaf not connected to this vertex of maximum degree, and hence by Lemma 2.2 G is not saturated. So there is precisely one vertex on the cycle with degree greater than two, which means that $mp(G) > 2$, a contradiction.

So now suppose $mp(G) = 3$. If there are at least two vertices on the cycle which have branches attached, then on one of these branches (including the vertex on the cycle) there must be a vertex of maximum degree, and on the other branch there must be a leaf not connected to this vertex of maximum degree, and hence by Lemma 2.2 G is not saturated. So there is precisely one vertex on the cycle with degree greater than two, and if the cycle has at least four vertices, then $mp(G) \geq 4$, a contradiction.

So it remains to consider the cycle K_3 with exactly one vertex x with degree greater than two. Suppose the vertex x has p leaves and q branches with $p, q \geq 0$. We consider several cases.

Case 1. If $p \geq 2$, then we connect two leaves to get H with $mp(H) = mp(G) = 3$, and G is not saturated. Hence $p \leq 1$.

Case 2. If $p = 1$ and $q \geq 1$, then either x is a vertex of maximum degree $\Delta \geq 3$, and there is a leaf not connected to x, so by Lemma 2.2 G is not saturated, or there is a vertex of maximum degree in one of these branch, so the leaf at x is not connected to the vertex of maximum degree and again by Lemma 2.2, G is not saturated.

Case 3. If $p = 1$ and $q = 0$, then G is K_3 with a leaf attached and clearly it is not saturated.

Case 4. If $p = 0$ and $q \geq 2$, then either x is a vertex of maximum degree $\Delta \geq 3$ and there is a leaf in the branch not connected to x, so by Lemma 2.2 G is not saturated, or there is a vertex of maximum degree in one of these branches, so the leaf at x is not connected to the vertex of maximum degree and again by Lemma 2.2, G is not saturated.

Case 5. If $p = 0$ and $q = 1$, then $deg(x) = 3$. Let z be the neighbour of x in this branch. If $deg(z) \geq 3$, then $mp(G) \geq 4$, a contradiction. Hence $deg(z) = 2$, and let w be the neighbour of z. If $deg(w) = 1$, then x has maximum degree, w

is not connected to x and by Lemma 2.2, G is not saturated. So $deg(w) \geq 2$ and we consider two cases.

Case 5.1. $deg(w) = 2$. Let u be the neighbour of w. If $deg(u) \leq 2$, then we have a degree monotone path $uwzv$ of length four. So $deg(u) \geq 3$.

If $deg(u) > 3$, then if the edge xw is added, $mp(G + xw) = 3$ and G is not saturated. Hence $deg(u) = 3$. Let s and y be the neighbours of u. If either s or y have degree at least three, then we have degree montone paths of length four $zwux$ or $zwuy$, a contradiction. So both s and y have degree at most two.

If either s or y is a leaf, say s, then either $\Delta = 3$ and s is a leaf not connected to x, so by Lemma 2.2 G is not saturated, or $\Delta \geq 4$ and is realized by a vertex r say on the branch at y. Again s is a leaf not adjacent to r, and by Lemma 2,2 G is not saturated.

So $deg(s) = deg(y) = 2$, and either the maximum degree $\Delta = 3$ and there is a leaf not adjacent to x, so by Lemma 2.2 G is not saturated, or there is a vertex r of maximum degree $\Delta \geq 4$, which is on one of the branches starting at s or y, say s. But then there is a leaf on the branch starting at y not adjacent to the vertex r, and again by Lemma 2.2 G is saturated.

Case 5.2. $deg(w) = t \geq 3$. Let $x_1, \ldots, x_t$ be the neighbors of w. If for some j, $deg(x_j) = 1$, then either $\Delta = 3$ and x_j is not connected to x, so by Lemma 2.2 G is not saturated, or $\Delta \geq 4$ and is realized by a vertex r on a branch at some x_i, $i \neq j$. Then x_j is a leaf not adjacent to r, and by Lemma 2.2 G is not saturated.

So $deg(x_j) \geq 2$ for $j = 1, \ldots, t$. Now if $\Delta = 3$, then a leaf on one these branches starting at $x_1, \ldots, x_t$ is not connected to x, and by Lemma 2.2 G is not saturated. Otherwise, $\Delta \geq 4$ and a vertex r of maximum degree appears on the branch starting at say x_j. Then a leaf on any other branch is not connected to r, and by Lemma 2.2 G is not saturated.

Hence $G = K_3$ is the only saturated graph with $|E(G)| \leq |V(G)|$ and $mp(G) = 3$. ∎

Theorem 3.3. *For* $n \equiv 0 \pmod 3$, $h(n, 4) = n$, *while for* $n \equiv 1, 2 \pmod 3$, $h(n, 4) = n + 1$.

Proof. First we prove the upperbound for $h(n, 4)$. Consider the following cases.

Case 1. Assume $n \equiv 0 \pmod 3$. If G is made up of $\frac{n}{3}$ copies of K_3, then clearly $mp(G) = 3$. Any edge we add gives a degree monotone path of length 4. So G is saturated and hence $h(n, 4) \leq n$ for $n \equiv 0 \pmod 3$.

Case 2. Assume $n \equiv 1 \pmod 3$. Let G be made up of $\frac{n-4}{3}$ copies of K_3 and a copy of $K_4 - e$, $e \in E(K_4)$. Clearly $mp(G) = 3$ and it is easy to see that $mp(G + e) \geq 4$. So G is saturated and hence $h(n, 4) \leq n + 1$ for $n \equiv 1 \pmod 3$.

Case 3. Assume $n \equiv 2\,(\text{mod } 3)$. Let G be made up of $\frac{n-5}{3}$ copies of K_3 and two copies of K_3 with a common vertex. Clearly $mp(G) = 3$ and it is easy to see that $mp(G+e) \geq 4$. So G is saturated and hence $h(n,4) \leq n+1$ for $n \equiv 2$ (mod 3).

Now to the lower bound. Suppose G is a graph on $n \geq 3$ vertices realising $h(n,4)$. If G is connected, then by Lemma 3.2, either G is K_3 or $|E(G)| \geq n+1$. Hence we may assume that G is not connected, and let $G_1, G_2, \ldots, G_t$ be the connected components of G. Again, by Lemma 3.2, every component G_j on at least 3 vertices is either K_3 or contains at least $|V(G_j)|+1$ edges.

If there are at least two components, say G_i and G_j, on at most two vertices each, then we can just add an edge between a vertex in G_i and one in G_j without creating a degree monotone path of length more than 3, contradicting the fact that G is saturated.

Lastly, if there is just one component G_j on at most two vertices, then if we connect a vertex in this component to a vertex v of maximum degree in another component of G, then clearly no degree monotone path of length 4 or more is created, once again contradicting that G is saturated.

Hence all components of G have at least 3 vertices. If there are at least two components which are not K_3, then $|E(G)| \geq n+2$, and this is not optimal by the constructions above. If there is just one component which is not K_3, then $|E(G)| \geq n+1$ and so for $n \equiv 1, 2\,(\text{mod } 3)$, $h(n,4) \geq n+1$ proving the constructions above are optimal .

Finally, if all components are K_3, then $|E(G)| = n$, proving $h(n,4) = n$ for $n \equiv 0\,(\text{mod } 3)$. ∎

4. Concluding Remarks and Open Problems

Several open problems have arised during our work on this paper. We list some of the more interesting ones.

- The major role played in this paper by Lemma 2.2 and its consequences suggest:

  Problem 1: Find further structural conditions (along the lines indicated in Lemma 2.2) indicating that a graph G is not saturated.

- In Corollary 2.3, we characterise saturated trees. In a previous paper [2] we characterised saturated graphs with $mp(G) = 2$. This leads to the following:

  Problem 2: Characterise k-saturated graphs for other families of graphs such as maximal outerplanar graphs, maximal planar graphs, regular graphs, etc.

Problem 3: Characterise saturated graphs with $mp(G) = 3$.

- The parameter $mp(G)$ can be very sensitive to edge-addition and edge-deletion, as shown in [3]. Also Theorem 2.5 gives $h(n, 7) \leq \frac{5n}{3}$ for $n \equiv 0 \pmod 9$ while Theorem 2.7 gives $h(n, 6) \leq \frac{13n}{7}$ for $n \equiv 0 \pmod 7$. These facts suggest the following monotonicity problem.

  Problem 4: Is it true that, at least for n large enough, depending on k, and for $k \geq 2$, $h(n, k + 1) \geq h(n, k)$?

  If true, this will have the immediate implication that the construction for $h(n, 6)$ is not optimal and that in fact $h(n, 6) \leq \frac{5n(1+o(1))}{3}$ by the above upper bound for $h(n, 7)$.

- The upper bound constructions given in Theorem 2.5 and Theorem 2.7 are probably not optimal.

  Problem 5: Improve upon the upper bounds obtained in Theorems 2.5 and 2.7.

- The lower bound given in Theorem 2.4 proved to be sharp in the case $k = 4$.

  Problem 6: Improve upon the lower bound $h(n, k) \geq n$ for $k \geq 5$.

- In Proposition 2.8 we have shown that $h(n, 5) \leq \frac{7n}{6} + c(n \pmod 6)$.

  Problem 7: Determine $h(n, 5)$ exactly. In particular, is it true that $h(n, 5) = \frac{7n(1+o(1))}{6}$?

- Lastly, recall that $sat(n, k) = n(1 - c(k)) < n$ for every large k and n.

  Problem 8: Is it true that $h(n, k) \leq cn$ for some constant c independent of k?

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