# THE SATURATION NUMBER FOR THE LENGTH OF DEGREE MONOTONE PATHS 

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#### Abstract

A degree monotone path in a graph $G$ is a path $P$ such that the sequence of degrees of the vertices in the order in which they appear on $P$ is monotonic. The length (number of vertices) of the longest degree monotone path in $G$ is denoted by $m p(G)$. This parameter, inspired by the well-known ErdősSzekeres theorem, has been studied by the authors in two earlier papers. Here we consider a saturation problem for the parameter $\operatorname{mp}(G)$. We call $G$ saturated if, for every edge $e$ added to $G, m p(G+e)>m p(G)$, and we define $h(n, k)$ to be the least possible number of edges in a saturated graph $G$ on $n$ vertices with $m p(G)<k$, while $m p(G+e) \geq k$ for every new edge $e$.

We obtain linear lower and upper bounds for $h(n, k)$, we determine exactly the values of $h(n, k)$ for $k=3$ and 4 , and we present constructions of saturated graphs.


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## 1. Introduction

Given a graph $G$, a degree monotone path is a path $v_{1} v_{2} \cdots v_{k}$ such that $\operatorname{deg}\left(v_{1}\right) \leq$ $\operatorname{deg}\left(v_{2}\right) \leq \cdots \leq \operatorname{deg}\left(v_{k}\right)$ or $\operatorname{deg}\left(v_{1}\right) \geq \operatorname{deg}\left(v_{2}\right) \geq \cdots \geq \operatorname{deg}\left(v_{k}\right)$. This notion,
inspired by the well-known Erdős-Szekeres theorem [7, 9], was introduced in [6] under the name of uphill and downhill path in relation to domination problems, also studied in $[4,5,11]$. In [6], the study of the parameter $m p(G)$, which denotes the length of the longest degree monotone path in $G$, was specifically suggested. This parameter was studied by the authors in [2, 3], and among many results obtained, the parameter $f(n, k)=\max \{|E(G)|:|V(G)|=n, m p(G)<k\}$ was also defined. It was shown that this is closely related to the Turán numbers $t(n, k)=\max \left\{|E(G)|:|V(G)|=n, G\right.$ contains no copy of $\left.K_{k}\right\}$.

A general form of the Turán numbers with respect to a graph $H$ is $t(n, H)=$ $\max \{|E(G)|:|V(G)|=n, G$ contains no copy of $H\}$. The study of Turán numbers is undoubtedly considered as one of the fundamental problems in extremal graph and hypergraph theory [1].

The Turán number has a counter-part known as the saturation number with respect to a given graph $H$, defined as

$$
\begin{aligned}
\operatorname{sat}(n, H)= & \min \{|E(G)|:|V(G)|=n, G \text { contains no copy of } H, \\
& \text { but } G+e \text { contains } H \text { for any edge added to } G\}
\end{aligned}
$$

Tuza and Kászonyi in [12] launched a systematic study of $\operatorname{sat}(n, H)$ following an earlier result by Erdős, Hajnal and Moon [8] who proved that $\operatorname{sat}\left(n, K_{k}\right)=$ $\left(_{2}^{k-2}\right)+(k-2)(n-k+2)$ with a unique graph attaining this bound, namely $K_{k-2}+$ $\bar{K}_{n-k+2}^{2}$. For the current paper, it is worth noting that $\operatorname{sat}\left(n, P_{k}\right)(\operatorname{sat}(n, k)$ for short) is known [12] for every $k$ and $n$ sufficiently large with respect to $k$, and in particular for $n$ large enough, $\operatorname{sat}(n, k)=n(1-c(k))$, where $c(k)<1$ is a positive constant which depends only on $k$ (for the exact value we refer the interested reader to [12]). For a survey and recent information about saturation, see [10].

In this spirit, we call a graph $G$ saturated if $m p(G+e)>m p(G)$ for all new edges $e$ joining non-adjacent vertices in $G$. If it happens that $m p(G+e) \geq k$ for all new edges $e$ we sometimes refer to the saturated graph $G$ as $k$-saturated. By convention we say that $K_{m}$ is $k$-saturated for $m \leq k-1$. Then we define

$$
h(n, k)=\min \{|E(G)|:|V(G)|=n, G \text { is } k \text {-saturated }\}
$$

In Section 2, we prove linear lower and upper bounds for this parameter. In Section 3, we provide exact determination of $h(n, k)$ for $k=3$, 4. In Section 4 we present several open problems concerning $h(n, k)$ for $k \geq 5$ as well as several other problems and conjectures.

## 2. General Lower and Upper bounds

### 2.1. Lower bounds

We begin by showing that $\operatorname{sat}(n, k)$ is a lower bound for $h(n, k)$.

Proposition 2.1. For $k \geq 2, h(n, k) \geq \operatorname{sat}(n, k)$.
Proof. Clearly, if $G$ is a graph realising $\operatorname{sat}\left(n, P_{k}\right)=\operatorname{sat}(n, k)$, this means that $G$ does not contain a copy of $P_{k}$, and hence no degree monotone path of length $k$. But $G+e$ contains $P_{k}$, but not necessarily a degree monotone path of length $k$. Hence $h(n, k) \geq \operatorname{sat}(n, k)$.

Recall that for fixed $k$ and large $n$, $\operatorname{sat}(n, k)=n(1-c(k))<n$. We now strengthen Proposition 2.1 to show that for $k \geq 4, h(n, k) \geq n$. First we prove a lemma, and subsequently a corollary, which will then be used in the main proof.
Lemma 2.2. Let $G$ be a connected graph with a vertex u of degree 1 and a vertex $v$ of maximum degree $\Delta \geq 2$ which are not adjacent. Then $m p(G+u v) \leq m p(G)$, namely $G$ is not saturated.
Proof. Let $H=G+u v$ and let $P$ be a path in $H$ which realizes $m p(H)$. Let $u^{*}$ and $v^{*}$ be the vertices $u$ and $v$ as they appear in $H$.

If $\Delta=2$, then clearly $G$ is a path on $k \geq 4$ vertices and $m p(G)=k-1$, and if we take $u$ to be the first vertex of the path, and $v$ to be the $(k-1)^{t h}$ vertex, then $m p(H)=k-1=m p(G)$.

So we may assume $\Delta \geq 3$. Now, if $u^{*}$ and $v^{*}$ are not on $P$, then $P$ is degree monotone in $G$ and hence $m p(H) \leq m p(G)$. If $v^{*}$ is on $P$ but $u^{*}$ is not, then $v^{*}$ must be the last vertex on $P$, and hence the path $P$ with $v^{*}$ replaced by $v$ is also degree monotone in $G$ and $m p(H) \leq m p(G)$. Similarly, if $u^{*}$ is on $P$ but $v^{*}$ is not, then $u^{*}$ must be the first vertex on $P$, since clearly $u^{*}$ cannot be in the "middle" of the path as then the next vertex on $P$ must be $v^{*}$, which is not on $P$. Then the path $P$ in $G$ with $u^{*}$ replaced by $u$ is also degree monotone in $G$ and again $m p(H) \leq m p(G)$. If $u^{*}$ is the last vertex on the path, then clearly $P$ is not maximal as $P \cup\left\{v^{*}\right\}$ via the edge $u^{*} v^{*}$ is a longer degree monotone path, contradicting maximality of $P$.

So the only remaining case to consider is when $u^{*}$ and $v^{*}$ are both on $P$. Then clearly $v^{*}$ is the last vertex on $P$. If $u^{*}$ is the first vertex, then either $P=u^{*} v^{*}$ and $m p(H)=2 \leq m p(G)$, or the path $P$ is degree monotone in $G$ too. If $u^{*}$ is not the first vertex, then the next vertex on $P$ must be $v^{*}$ which is the last vertex. Hence, in this case, all predecessors of $u^{*}$ on $P$ must have degree at most 2. But if the first vertex $y$ in $P$ has degree 1 , then, in $G$, the path $y \cdots u$ is disconnected from the rest of $G$, which is impossible. Therefore $\operatorname{deg}(y)=2$ and $y$ has a neighbour $w$ which must have degree greater than 2 (note that $w$ may be equal to $v^{*}$ but cannot be any other vertex on $P$ ). But then, the path $u \cdots y w$ is degree monotone in $G$ and is of the same length as $P$, and hence $m p(H) \leq m p(G)$.

Lemma 2.2 is best possible with respect to the adjacency condition between minimum degrees and maximum degrees because if the minimum degree is greater
than 1 , and a vertex $u$ of minimum degree is not adjacent to vertex $v$, then $m p(G+u v)$ may be larger than $m p(G)$. As an example, consider a graph $G_{n}$ made up of the cycle $C_{2 n}, n \geq 3$, with vertices labelled $v_{1}, v_{2}, \ldots, v_{2 n}$, and a vertex $w$ connected to vertices $v_{1}, v_{3}, v_{5}, \ldots, v_{2 n-1}$. Thus $w$ has degree $\Delta=n$ and $\delta=2$, and $m p\left(G_{n}\right)=3$. The vertices of degree 2 are not connected to $w$, but connecting any such vertex to $w$ by an edge $e$ gives $m p\left(G_{n}+e\right)=5$. In fact, these graphs are 5 -saturated even though they have non-adjacent vertices of maximum degree $\Delta \geq 3$ and minimum degree $\delta=2$.

Corollary 2.3. Let $T$ be a tree on $n \geq 3$ vertices. Then $T$ is saturated for $a$ degree monotone path if and only if $T=K_{1, n-1}$.

Proof. Suppose first $m p(T) \geq 3$. Then clearly $T$ is not a star, hence there is a leaf not connected to a vertex of maximum degree and by Lemma $2.2, T$ is not saturated.

So suppose $m p(T)=2$. If not all leaves are adjacent to the same vertex of maximum degre, then again by Lemma $2.2, T$ is not saturated. Hence $T$ must be a star $K_{1, n-1}$.

Indeed, $K_{1, n-1}$ is saturated and $m p\left(K_{1, n-1}\right)=2$ while $m p\left(K_{1, n-1}+e\right)=3$ for every edge $e \notin E\left(K_{1, n-1}\right)$.

Theorem 2.4. For $n \geq 3$ and $k \geq 4, h(n, k) \geq n$.
Proof. We may assume that $n \geq k$ for otherwise, trivially, $K_{n}$ is saturated having $\binom{n}{2} \geq n$ edges for $n \geq 3$.

So let $G$ be a graph on $n \geq k$ vertices realizing $h(n, k), k \geq 4$. If $G$ is connected, then by Corollary 2.3, $G$ is not a tree, and hence $|E(G)| \geq n$ as required.

So we may assume that $G$ is not connected, and let $G_{1}, G_{2}, \ldots, G_{t}$ be the connected components of $G$. Again, by Corollary 2.3, we infer that every component on at least three vertices is not a tree and hence must have at least $\left|V\left(G_{j}\right)\right|$ edges.

If there are two components $G_{i}$ and $G_{j}$ on at most two vertices, adding an edge joining these two components does not create a degree monotone path of length 4 or more, contradicting the fact that $G$ is saturated.

If there is just one component on at most two vertices, then one can connect one vertex of this component to a vertex of maximum degree in another component, and again no degree monotone path of length four or more is created, contradicting the fact that $G$ is saturated.

Hence

$$
|E(G)|=\sum_{i=1}\left|E\left(G_{i}\right)\right| \geq \sum_{i=1}\left|V\left(G_{i}\right)\right|=n,
$$

and therefore $h(n, k) \geq n$ for $n \geq 3$ and $k \geq 4$.

### 2.2. Upper bounds

We now give a linear upper bound for $h(n, k)$. We consider separately $k$ odd and $k$ even.

First, we recall the definition of the Cartesian product $G \square H$ for two graphs $G$ and $H$. The vertex set of the product is $V(G) \times V(H)$. Two vertices $\left(u_{1}, v_{1}\right)$ and $\left(u_{2}, v_{2}\right)$ are adjacent if either $u_{1}$ and $u_{2}$ are adjacent in $G$ and $v_{1}=v_{2}$, or $v_{1}, v_{2}$ are adjacent in $H$ and $u_{1}=u_{2}$.
Theorem 2.5. If $k \geq 3$ is an odd integer, then $h(n, k) \leq \frac{n(3 k-1)}{12}$ for $n \equiv 0$ $\left(\bmod \frac{3(k-1)}{2}\right)$.
Proof. Consider the graph $G=P_{3} \square K_{t}$ for $k \geq 3$ odd and $t=\frac{k-1}{2}$. Clearly, $|V(G)|=\frac{3(k-1)}{2}$ and $|E(G)|=\frac{3(k-1)(k-3)}{8}+\frac{2(k-1)}{2}=\frac{(k-1)(3 k-1)}{8}$. For $k=3$ (so $t=1$ ) this is simply $P_{3}$ and $m p\left(P_{3}\right)=2$, while for $k=5$ (so $t=2$ ) this gives the graph $G=P_{3} \square K_{2}$, which is $C_{6}$ plus one edge joining two antipodal vertices and clearly $m p(G)=4$.

We now show that this graph, which has $m p(G)=k-1$, is saturated. In $G=P_{3} \square K_{t}$, let the top $t$ vertices be $u_{1}, \ldots, u_{t}$, all having degree $t$, the middle vertices $v_{1}, \ldots, v_{t}$ all having degree $t+1$, and the bottom vertices $w_{1}, \ldots, w_{t}$ all having degree $t$. It is clear that $m p(G)=2 t=k-1$, taking for example the path $u_{1} \cdots u_{t} v_{t} \cdots v_{1}$. Because of the symmetry of $G$, we only need to check the addition of the edges $u_{1} v_{2}, v_{1} w_{2}$ and $u_{1} w_{1}$.

- If the edge $u_{1} v_{2}$ is added, then the path $w_{1} \cdots w_{t} v_{t} \cdots v_{3} v_{1} u_{1} v_{2}$ has exactly $t+t-2+3=2 t+1=k$ vertices.
- If the edge $v_{1} w_{2}$ is added, then the path $u_{1} \cdots u_{t} v_{t} \cdots v_{2} w_{2} v_{1}$ has exactly $t+t-1+2=2 t+1=k$ vertices.
- If the edge $u_{1} w_{1}$ is added, then the path $u_{2} \cdots u_{t} v_{t} \cdots v_{1} w_{1} u_{1}$ has exactly $t-1+t+2=2 t+1=k$ vertices.

Hence $G$ is saturated with $m p(G)=k-1$.
We now consider two disjoint copies of $G, G_{1}$ and $G_{2}$. We label this graph $2 G$ and show that this graph is also saturated. Again labelling the vertices of $G$ as above, by the symmetry of $G$ we only need to consider the addition of the edges joining $u_{t}$ in $G_{1}$ to $u_{1}$ in $G_{2}, u_{t}$ in $G_{1}$ to $v_{1}$ in $G_{2}$, and $v_{t}$ in $G_{1}$ to $v_{1}$ in $G_{2}$.

- If the edge joining $u_{t}$ in $G_{1}$ to $u_{1}$ in $G_{2}$ is added, then the path $u_{1} \cdots u_{t}$ in $G_{1}$ followed by $u_{1} v_{1} \cdots v_{t}$ in $G_{2}$ has exactly $t+t+1=2 t+1=k$ vertices.
- If the edge joining $u_{t}$ in $G_{1}$ to $v_{1}$ in $G_{2}$ is added, then the path $v_{1} \cdots v_{t} u_{t}$ in $G_{1}$ followed by $v_{1} \cdots v_{t}$ in $G_{2}$ has exactly $t+1+t=2 t+1=k$ vertices.
- If the edge joining $v_{t}$ in $G_{1}$ to $v_{1}$ in $G_{2}$ is added, then the path $u_{t} \cdots u_{1} v_{1} \cdots$ $v_{t}$ in $G_{1}$ followed by $v_{1}$ in $G_{2}$ has exactly $2 t+1=k$ vertices.

Hence $2 G$ is saturated, and clearly this also applies to $p \geq 3$ disjoint copies of $G, p G$. Now $p G$ has $n=p \frac{3(k-1)}{2}$ vertices and $p \frac{(k-1)(3 k-1)}{8}$ edges. Hence, for $n \equiv 0\left(\bmod \frac{3(k-1)}{2}\right)$, the number of edges is $\frac{n(3 k-1)}{12}$, as stated.

Lemma 2.6. Let $G$ be a saturated graph with $m p(G)=k$. Consider the graph $H=G+v$, where $v$ is a new vertex connected to all the vertices of $G$. Then $m p(H)=k+1$, and $H$ is saturated.

Proof. Consider the graph $H$. Then $\operatorname{deg}(v)=|V(G)|$ and $v$ has maximum degree. So any degree monotone path in $G$ can be extended in $H$ by including vertex $v$, and hence $m p(H)=m p(G)+1=k+1$.

Now, since $G$ is saturated, adding any edge $e$ creates a degree monotone path of length $k+1$, and hence, adding the same edge in $H$ creates a path of length $k+2$. The only edges which can be added in $H$ are those that can be added in $G$, and hence $H$ is saturated with $m p(H)=k+1$, as required.

This lemma together with Theorem 2.5 leads to the following result.
Theorem 2.7. For even $k, k \geq 4, h(n, k) \leq \frac{n(3 k+8)(k-2)}{4(3 k-4)}$ for $n \equiv 0\left(\bmod \frac{3 k-4}{2}\right)$.
Proof. In Theorem 2.5 we proved that $G=P_{3} \square K_{t}$, where $t=\frac{j-1}{2}$, has $m p(G)=$ $j-1$, and $G$ is saturated for $j \geq 3$ and $j$ odd. Now by Lemma $2.6, H=G+v$ has $m p(H)=j+1$ (even) and is saturated. Then $H$ has $\frac{3(j-1)}{2}+1=\frac{3 j-1}{2}$ vertices and $\frac{(j-1)(3 j-1)}{8}+\frac{3(j-1)}{2}=\frac{(3 j+11)(j-1)}{8}$ edges. Now let $k=j+1$, and hence we have $\frac{3 k-4}{2}$ vertices and $\frac{(3 k+8)(k-2)}{2}$ edges.

We now consider two disjoint copies of $H, H_{1}$ and $H_{2}$ and call this graph $2 H$. We need only consider edges which involve the new vertex of degree $\frac{3(k-2)}{2}$, which has the largest degree, as other edges have the same effect as they have in $2 G$. If we connect the vertex of degree $\frac{3(k-2)}{2}$ in $H_{1}$ to that of the same degree in $H_{2}$, then we can take a path of length $k-1$ in $H_{1}$ ending with the vertex of maximum degree, and then move to the vertex in $H_{2}$, giving a path of length $k$. If we connect the vertex of degree $\frac{3(k-2)}{2}$ in $H_{1}$ to one of degree $\frac{k}{2}$ in $H_{2}$, then we take a path of length $k-1$ in $H_{2}$ ending with the vertex connected to the vertex in $H_{1}$, and then move to this vertex in $H_{1}$ to give a degree monotone path of length $k$. Finally, if we connect the vertex of degree $\frac{3(k-2)}{2}$ in $H_{1}$ to one of degree $\frac{k+2}{2}$ in $H_{2}$, then we can take a degree monotone path in $H_{2}$ of length $k-1$ ending with the vertex connected to $H_{2}$, and then the vertex in $H_{2}$ to give a degree monotone path of length $k$ in $2 H$.

Hence $2 H$ is saturated, and this also applies to $p \geq 3$ disjoint copies of $H$, $p H$. This graph has $n=p \frac{3 k-4}{2}$ vertices and $p \frac{(3 k+8)(k-2)}{8}$ edges. Hence for $n \equiv 0$ $\left(\bmod \frac{3 k-4}{2}\right)$, the number of edges is $\frac{n(3 k+8)(k-2)}{4(3 k-4)}$, as stated.

Next We show, as an example, how to extend the results given in Theorems 2.5 and 2.7 , to the case where $n \not \equiv 0(\bmod f(k))$, where $f(k)$ is the modulus given in these theorems. We will demonstrate it in the case $k=5$.
Proposition 2.8. For $n \geq 8, h(n, 5) \leq \frac{7 n+c(n(\bmod 6))}{6}$, where $c(n(\bmod 6))=$ $\{0,35,16,27,8,28\}$ for $n \equiv 0,1,2,3,4,5(\bmod 6)$, respectively.

Proof. Consider the graphs $G=P_{3} \square K_{2}, H=K_{5}-e$ for $e \in E\left(K_{5}\right)$ and $K_{4}$, which are sturated for $k=5$ and clearly $m p(G)=m p(H)=m p\left(K_{4}\right)=4$. Every integer $n \geq 8$ can be represented in the form $6 x+5 y+4 z$ with $x, y, z$ non-negative integers. Hence $x$ copies of $G, y$ copies of $H$ and $z$ copies of $K_{4}$ produce graphs for every $n \geq 8$. It is easy to check that any graph made up of two vertex disjoint copies of any combination of $G, H$ and $K_{4}$ is also saturated, and hence any combination of vertex disjoint copies of these graphs is saturated.

Hence any graph made up of a disjoint combination of any number of these three graphs is saturated.

For $n \equiv 0(\bmod 6)$, the result follows immediately by substituting $k=5$ in Theorem 2.5.

For $n \equiv 1(\bmod 6)$, we take the graph made up of $\frac{n-13}{6}$ copies of $G$, two copies $K_{4}$ and one copy of $H$. The graph thus obtained is saturated and has $\frac{7(n-13)}{6}+12+9=\frac{7 n+35}{6}$ edges.

For $n \equiv 2(\bmod 6)$, we take the graph made up of $\frac{n-8}{6}$ copies of $G$ and two copies $K_{4}$. The graph thus obtained is saturated and has $\frac{7(n-8)}{6}+12=\frac{7 n+16}{6}$ edges.

For $n \equiv 3(\bmod 6)$, we take the graph made up of $\frac{n-9}{6}$ copies of $G$, one copy of $K_{4}$ and one copy of $H$. The graph thus obtained is saturated and has $\frac{7(n-9)}{6}+6+9=\frac{7 n+27}{6}$ edges.

For $n \equiv 4(\bmod 6)$, we take the graph made up of $\frac{n-4}{6}$ copies of $G$ and one copy of $K_{4}$. The graph thus obtained is saturated and has $\frac{7(n-4)}{6}+6=\frac{7 n+8}{6}$ edges.

For $n \equiv 5(\bmod 6)$, we take the graph made up of $\frac{n-5}{6}$ copies of $G$ and one copy of $H$. The graph thus obtained is saturated and has $\frac{7(n-5)}{6}+9=\frac{7 n+28}{6}$ edges.

Note: Applying the technique demonstrated in Proposition 2.8, we can extend Theorems 2.5 and 2.7 to cover all $n \geq(k-1)(k-2)$, and we state it rather crudely as follows.

1. For odd $k, k \geq 3$, and $n \geq(k-1)(k-2), h(n, k) \leq \frac{n(3 k-1)}{12}+O\left(k^{2}\right)$.
2. For even $k, k \geq 4$, and $n \geq(k-1)(k-2), h(n, k) \leq \frac{n(3 k+8)(k-2)}{4(3 k-4)}+O\left(k^{2}\right)$.

## 3. Determination of $h(n, k)$ for $k=2,3,4$.

First we determine the exact value of $h(n, 2)$ and $h(n, 3)$.
Proposition 3.1. (1) $h(n, 2)=0$.
(2) $h(n, 3)=\frac{n}{2}$ for $n$ even, while $h(n, 3)=\frac{n+1}{2}$ for $n$ odd.

Proof. 1. $m p(G)=1$ if and only if $G$ is a graph with no edges, and any edge we add gives $m p(G+e)=2$.
2. By Proposition 2.1, $h(n, 3) \geq \operatorname{sat}(n, 3)=\left\lfloor\frac{n}{2}\right\rfloor$. Consider even. Let $G$ be made up of $\frac{n}{2}$ copies of $K_{2}$. This is the only graph which achieves $\operatorname{sat}(n, 3)$. Clearly $m p(G)=2$, and adding any edge will create a copy of $P_{4}$ so $m p(G+e)=3$.

Now if $n$ is odd, then the graph $G$ made up of $\left\lfloor\frac{n}{2}\right\rfloor$ copies of $K_{2}$, and one copy of $K_{1}$ achieves $\operatorname{sat}(n, 3)$, and is the only such graph. Again $m p(G)=2$. If we add an edge joining two vertices from disjoint copies of $K_{2}$, then we get a copy of $P_{4}$ and $m p(G+e)=3$. However, if we add a vertex joining a vertex from $K_{2}$ to the vertex in $K_{1}$, then this gives a copy of $P_{3}$, and $m p(G+e)=2$, hence $h(n, 3) \geq \operatorname{sat}(n, 3)+1$.

Consider the graph $G$ made up of $\frac{n-3}{2}$ copies of $K_{2}$, and a single copy of $P_{3}$. Again it is clear that $m p(G)=2$. Adding an edge joining two vertices from disjoint copies of $K_{2}$ then we get a copy of $P_{4}$ and $m p(G+e)=3$, while adding an edge joining a vertex from $K_{2}$ to one in $P_{3}$ gives $m p(G+e)=4$. The number of edges in this graph is $\frac{n+1}{2}=\operatorname{sat}(n, 3)+1$, as stated.

We now determine the exact value of $h(n, 4)$. For this we need another lemma.

Lemma 3.2. Let $G$ be a saturated connected graph with $|E(G)| \leq|V(G)|$ and $2 \leq m p(G) \leq 3$. Then
(1) If $m p(G)=2$, then $G=K_{1, \Delta}$, and for $\Delta \geq 2, m p(G+e)=3$, for every $e \notin E(G)$.
(2) If $m p(G)=3$, then $G=K_{3}$, which is saturated by definition.

Proof. Let $G$ be such a graph. Then since $|E(G)| \leq|V(G)|, G$ is either a tree or is unicyclic.

If G is a tree such that all leaves are adjacent to the same vertex which has maximum degree, that is $G=K_{1, \Delta}$, then $m p(G)=2$ and, in case $\Delta \geq 2$, adding any edge between two leaves $u$ and $v$ gives $m p(G+u v)=3$. If $G$ is a tree but
not $K_{1, \Delta}$, then there is a leaf $u$ and a vertex $v$ of maximum degree which are not adjacent, and hence by Lemma 2.2, $G$ is not saturated.

So suppose $G$ is unicyclic. Then it cannot be a simple cycle $C_{n}$ on $n \geq 4$ vertices, since otherwise $m p\left(C_{n}\right)=n \geq 4$. Observe that $C_{3}=K_{3}$ is saturated by definition. So $G$ is unicyclic with at least one leaf if the cycle has at least four vertices.

Suppose $\operatorname{mp}(G)=2$. If there are at least two vertices on the cycle which have branches attached, then on one of these branches (including the vertex on the cycle) there must be a vertex of maximum degree, and on the other branch there must be a leaf not connected to this vertex of maximum degree, and hence by Lemma $2.2 G$ is not saturated. So there is precisely one vertex on the cycle with degree greater than two, which means that $m p(G)>2$, a contradiction.

So now suppose $m p(G)=3$. If there are at least two vertices on the cycle which have branches attached, then on one of these branches (including the vertex on the cycle) there must be a vertex of maximum degree, and on the other branch there must be a leaf not connected to this vertex of maximum degree, and hence by Lemma $2.2 G$ is not saturated. So there is precisely one vertex on the cycle with degree greater than two, and if the cycle has at least four vertices, then $m p(G) \geq 4$, a contradiction.

So it remains to consider the cycle $K_{3}$ with exactly one vertex $x$ with degree greater than two. Suppose the vertex $x$ has $p$ leaves and $q$ branches with $p, q \geq 0$. We consider several cases.

Case 1. If $p \geq 2$, then we connect two leaves to get $H$ with $m p(H)=$ $m p(G)=3$, and $G$ is not saturated. Hence $p \leq 1$.

Case 2. If $p=1$ and $q \geq 1$, then either $x$ is a vertex of maximum degree $\Delta \geq 3$, and there is a leaf not connected to $x$, so by Lemma $2.2 G$ is not saturated, or there is a vertex of maximum degree in one of these branch, so the leaf at $x$ is not connected to the vertex of maximum degree and again by Lemma 2.2, $G$ is not saturated.

Case 3. If $p=1$ and $q=0$, then $G$ is $K_{3}$ with a leaf attached and clearly it is not saturated.

Case 4. If $p=0$ and $q \geq 2$, then either $x$ is a vertex of maximum degree $\Delta \geq 3$ and there is a leaf in the branch not connected to $x$, so by Lemma $2.2 G$ is not saturated, or there is a vertex of maximum degree in one of these branches, so the leaf at $x$ is not connected to the vertex of maximum degree and again by Lemma 2.2, $G$ is not saturated.

Case 5. If $p=0$ and $q=1$, then $\operatorname{deg}(x)=3$. Let $z$ be the neighbour of $x$ in this branch. If $\operatorname{deg}(z) \geq 3$, then $m p(G) \geq 4$, a contradiction. Hence $\operatorname{deg}(z)=2$, and let $w$ be the neighbour of $z$. If $\operatorname{deg}(w)=1$, then $x$ has maximum degree, $w$
is not connected to $x$ and by Lemma $2.2, G$ is not saturated. So $\operatorname{deg}(w) \geq 2$ and we consider two cases.

Case 5.1. $\operatorname{deg}(w)=2$. Let $u$ be the neighbour of $w$. If $\operatorname{deg}(u) \leq 2$, then we have a degree monotone path $u w z v$ of length four. So $\operatorname{deg}(u) \geq 3$.

If $\operatorname{deg}(u)>3$, then if the edge $x w$ is added, $m p(G+x w)=3$ and $G$ is not saturated. Hence $\operatorname{deg}(u)=3$. Let $s$ and $y$ be the neighbours of $u$. If either $s$ or $y$ have degree at least three, then we have degree montone paths of length four $z w u x$ or $z w u y$, a contradiction. So both $s$ and $y$ have degree at most two.

If either $s$ or $y$ is a leaf, say $s$, then either $\Delta=3$ and $s$ is a leaf not connected to $x$, so by Lemma $2.2 G$ is not saturated, or $\Delta \geq 4$ and is realized by a vertex $r$ say on the branch at $y$. Again $s$ is a leaf not adjacent to $r$, and by Lemma 2,2 $G$ is not saturated.

So $\operatorname{deg}(s)=\operatorname{deg}(y)=2$, and either the maximum degree $\Delta=3$ and there is a leaf not adjacent to $x$, so by Lemma 2.2 $G$ is not saturated, or there is a vertex $r$ of maximum degree $\Delta \geq 4$, which is on one of the branches starting at $s$ or $y$, say $s$. But then there is a leaf on the branch starting at $y$ not adjacent to the vertex $r$, and again by Lemma $2.2 G$ is saturated.

Case 5.2. $\operatorname{deg}(w)=t \geq 3$. Let $x_{1}, \ldots, x_{t}$ be the neighbors of $w$. If for some $j$, $\operatorname{deg}\left(x_{j}\right)=1$, then either $\Delta=3$ and $x_{j}$ is not connected to $x$, so by Lemma 2.2 $G$ is not saturated, or $\Delta \geq 4$ and is realized by a vertex $r$ on a branch at some $x_{i}$, $i \neq j$. Then $x_{j}$ is a leaf not adjacent to $r$, and by Lemma $2.2 G$ is not saturated.

So $\operatorname{deg}\left(x_{j}\right) \geq 2$ for $j=1, \ldots, t$. Now if $\Delta=3$, then a leaf on one these branches starting at $x_{1}, \ldots, x_{t}$ is not connected to $x$, and by Lemma 2.2 $G$ is not saturated. Otherwise, $\Delta \geq 4$ and a vertex $r$ of maximum degree appears on the branch starting at say $x_{j}$. Then a leaf on any other branch is not connected to $r$, and by Lemma $2.2 G$ is not saturated.

Hence $G=K_{3}$ is the only saturated graph with $|E(G)| \leq|V(G)|$ and $m p(G)=3$.

Theorem 3.3. For $n \equiv 0(\bmod 3), h(n, 4)=n$, while for $n \equiv 1,2(\bmod 3)$, $h(n, 4)=n+1$.

Proof. First we prove the upperbound for $h(n, 4)$. Consider the following cases.
Case 1. Assume $n \equiv 0(\bmod 3)$. If $G$ is made up of $\frac{n}{3}$ copies of $K_{3}$, then clearly $m p(G)=3$. Any edge we add gives a degree monotone path of length 4 . So $G$ is saturated and hence $h(n, 4) \leq n$ for $n \equiv 0(\bmod 3)$.

Case 2. Assume $n \equiv 1(\bmod 3)$. Let $G$ be made up of $\frac{n-4}{3}$ copies of $K_{3}$ and a copy of $K_{4}-e, e \in E\left(K_{4}\right)$. Clearly $m p(G)=3$ and it is easy to see that $m p(G+e) \geq 4$. So $G$ is saturated and hence $h(n, 4) \leq n+1$ for $n \equiv 1(\bmod 3)$.

Case 3. Assume $n \equiv 2(\bmod 3)$. Let $G$ be made up of $\frac{n-5}{3}$ copies of $K_{3}$ and two copies of $K_{3}$ with a common vertex. Clearly $m p(G)=3$ and it is easy to see that $m p(G+e) \geq 4$. So $G$ is saturated and hence $h(n, 4) \leq n+1$ for $n \equiv 2$ $(\bmod 3)$.

Now to the lower bound. Suppose $G$ is a graph on $n \geq 3$ vertices realising $h(n, 4)$. If $G$ is connected, then by Lemma 3.2, either $G$ is $K_{3}$ or $|E(G)| \geq n+1$. Hence we may assume that $G$ is not connected, and let $G_{1}, G_{2}, \ldots, G_{t}$ be the connected components of $G$. Again, by Lemma 3.2, every component $G_{j}$ on at least 3 vertices is either $K_{3}$ or contains at least $\left|V\left(G_{j}\right)\right|+1$ edges.

If there are at least two components, say $G_{i}$ and $G_{j}$, on at most two vertices each, then we can just add an edge between a vertex in $G_{i}$ and one in $G_{j}$ without creating a degree monotone path of length more than 3 , contradicting the fact that $G$ is saturated.

Lastly, if there is just one component $G_{j}$ on at most two vertices, then if we connect a vertex in this component to a vertex $v$ of maximum degree in another component of $G$, then clearly no degree monotone path of length 4 or more is created, once again contradicting that $G$ is saturated.

Hence all components of $G$ have at least 3 vertices. If there are at least two components which are not $K_{3}$, then $|E(G)| \geq n+2$, and this is not optimal by the constructions above. If there is just one component which is not $K_{3}$, then $|E(G)| \geq n+1$ and so for $n \equiv 1,2(\bmod 3), h(n, 4) \geq n+1$ proving the constructions above are optimal.

Finally, if all components are $K_{3}$, then $|E(G)|=n$, proving $h(n, 4)=n$ for $n \equiv 0(\bmod 3)$.

## 4. Concluding Remarks and Open Problems

Several open problems have arised during our work on this paper. We list some of the more interesting ones.

- The major role played in this paper by Lemma 2.2 and its consequences suggest:
Problem 1: Find further structural conditions (along the lines indicated in Lemma 2.2) indicating that a graph $G$ is not saturated.
- In Corollary 2.3, we characterise saturated trees. In a previous paper [2] we characterised saturated graphs with $m p(G)=2$. This leads to the following:
Problem 2: Characterise $k$-saturated graphs for other families of graphs such as maximal outerplanar graphs, maximal planar graphs, regular graphs, etc.

Problem 3: Characterise saturated graphs with $m p(G)=3$.

- The parameter $m p(G)$ can be very sensitive to edge-addition and edgedeletion, as shown in [3]. Also Theorem 2.5 gives $h(n, 7) \leq \frac{5 n}{3}$ for $n \equiv 0$ $(\bmod 9)$ while Theorem 2.7 gives $h(n, 6) \leq \frac{13 n}{7}$ for $n \equiv 0(\bmod 7)$. These facts suggest the following monotonicity problem.
Problem 4: Is it true that, at least for $n$ large enough, depending on $k$, and for $k \geq 2, h(n, k+1) \geq h(n, k)$ ?
If true, this will have the immediate implication that the construction for $h(n, 6)$ is not optimal and that in fact $h(n, 6) \leq \frac{5 n(1+o(1))}{3}$ by the above upper bound for $h(n, 7)$.
- The upper bound constructions given in Theorem 2.5 and Theorem 2.7 are probably not optimal.
Problem 5: Improve upon the upper bounds obtained in Theorems 2.5 and 2.7.
- The lower bound given in Theorem 2.4 proved to be sharp in the case $k=4$.

Problem 6: Improve upon the lower bound $h(n, k) \geq n$ for $k \geq 5$.

- In Proposition 2.8 we have shown that $h(n, 5) \leq \frac{7 n}{6}+c(n(\bmod 6))$.

Problem 7: Determine $h(n, 5)$ exactly. In particular, is it true that $h(n, 5)=\frac{7 n(1+o(1))}{6} ?$

- Lastly, recall that $\operatorname{sat}(n, k)=n(1-c(k))<n$ for every large $k$ and $n$.

Problem 8: Is it true that $h(n, k) \leq c n$ for some constant $c$ independent of $k$ ?

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