Discussiones Mathematicae
Graph Theory 35 (2015) 483–491
doi:10.7151/dmgt.1816

RECONSTRUCTING SURFACE TRIANGULATIONS BY THEIR INTERSECTION MATRICES[1]

Jorge L. Arocha, Javier Bracho

Natalia García-Colín

and

Isabel Hubard

Instituto de Matemáticas
Universidad Nacional Autónoma de México
CU, México D.F. 04510

e-mail: arocha@matem.unam.mx
jbracho@matem.unam.mx
garcia@matem.unam.mx
hubard@matem.unam.mx

Abstract

The intersection matrix of a simplicial complex has entries equal to the rank of the intersecction of its facets. We prove that this matrix is enough to define up to isomorphism a triangulation of a surface.

Keywords: triangulated surface, isomorphism, intersection matrix.

2010 Mathematics Subject Classification: 05C10, 57M15.

1. Introduction

There are several classes of simplicial complexes that are defined by the adjacency relation between maximal simplices. This is true for triangulations of the sphere due to the classical theorem of Whitney about the embeddings of 3-connected planar graphs. It is also true for simplicial convex polytopes of arbitrary dimension [1,2,4].

[1]Research was partially supported by grants CONACyT 166951 and DGAPA IN112511.

A triangulated surface is a simplicial complex whose underlying topological space is a connected 2-manifold (without boundary). It turns out that triangulated surfaces are not defined up to isomorphism by their dual graphs. There exist different triangulated surfaces having the same dual graph (see Goldberg snarks in [3], pages 890–892). However, it is not easy to build the appropriate examples. This strongly suggests that with little more information we should be able to define triangulated surfaces up to isomorphism.

If S is a triangulated surface, then we will denote by V, E and T the sets of its vertices, edges and triangles. We say that two triangulated surfaces S and S' have the *same intersection matrix* if there is a bijective map $f : T \longrightarrow T'$ such that for any two triangles t_1, t_2 the equality of cardinalities $|t_1 \cap t_2| = |f(t_1) \cap f(t_2)|$ holds. In this case we will say that f is an *intersection preserving map*. We shall prove the following.

Theorem 1. *If two triangulated surfaces have the same intersection matrix, then they are isomorphic.*

Now we shall state a more detailed result from which Theorem 1 follows. We say that a map $f : T \longrightarrow T'$ *extends to an isomorphism* if there is an isomorphism $g : V \longrightarrow V'$ which induces f. This means that for any triangle $t = \{v_1, v_2, v_3\}$ the equality $f(t) = \{g(v_1), g(v_2), g(v_3)\}$ holds.

It is not true that any intersection preserving map extends to an isomorphism. To see this, consider the two triangulations of the projective plane in Figure 1.

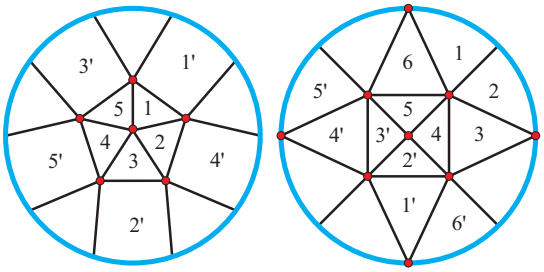

Figure 1. Two triangulations of the projective plane which have intersection preserving maps not extendable to automorphisms.

The triangulation on the left is the *half-icosahedron* and we will denote it by $\mathcal{I}/2$. The one on the right can be obtained by triangulating the 3 squares of the *half-cube* with 3 additional vertices. We will denote it by $\mathcal{TC}/2$.

It can be easily checked that in the half-icosahedron case, the permutation of the triangles whose descomposition in cycles is $(1, 1')(2, 2', 5, 5')(3, 3', 4, 4')$ is intersection preserving. In the $\mathcal{TC}/2$ case, the involution $i \leftrightarrow i'$ is intersection preserving. However, these two maps do not extend to automorphisms because disks are mapped to Möbius bands.

Theorem 1 is true because essentially there are no more such examples. More precisely, our main result is the following.

Theorem 2. *Let S and S' be two triangulated surfaces and $f : T \longrightarrow T'$ be an intersection preserving map that does not extend to an isomorphism. Then, $S \simeq S' \simeq \mathcal{I}/2$ or $S \simeq S' \simeq \mathcal{TC}/2$.*

The next section is dedicated to introduce and classify cyclic shells: a tool that we use to prove Theorem 2. The proof of Theorem 2 is given in Section 3. In the last section we outline some directions of further research.

2. CYCLIC SHELLS

A *cyclic shell* is a simplicial complex of dimension two spanned by a set of $n \geq 3$ triangles $\{t_0, t_1, \ldots, t_{n-1}\}$ such that for all $0 \leq i < j \leq n-1$

$$|t_i \cap t_j| = \begin{cases} 2 & \text{if } i = 0 \text{ and } j = n-1, \\ 2 & \text{if } j = i+1, \\ 1 & \text{otherwise.} \end{cases} .$$

Of course, triangulations of disks known as "wheels" in graph theory (see Figure 2) are cyclic shells. The neighborhoods of all vertices in triangulated surfaces are wheels. The wheels will be denoted by CW_n. Cyclic shells are exactly those simplicial complexes whose intersection matrix is equal to the intersection matrix of a wheel.

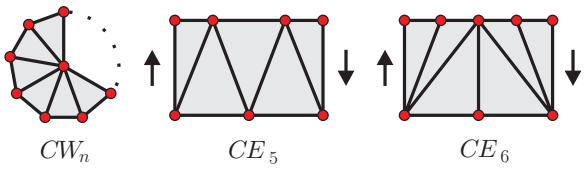

Figure 2. Cyclic shells.

More interesting cyclic shells are the two triangulations of the Möbius band showed in Figure 2. They will be denoted by CE_5 and CE_6 according to their number of triangles. We shall prove in this section that there are no more cyclic shells. This is a necessary step because any intersection preserving map transforms cyclic shells onto cyclic shells.

A *linear shell* is a simplicial complex of dimension two spanned by a set of $n \geq 2$ triangles $\{t_0, t_1, \ldots, t_{n-1}\}$ such that for all $0 \leq i < j \leq n-1$

$$|t_i \cap t_j| = \begin{cases} 2 & \text{if } j = i+1, \\ 1 & \text{otherwise.} \end{cases} .$$

An n-shell is a (linear or cyclic) shell with n triangles. It is clear that if $\{t_0, \ldots, t_{n-1}\}$ spans a cyclic or linear shell, then for every $0 \leq j < k < n$ the set $\{t_j, t_{j+1}, \ldots, t_k\}$ spans a linear shell. This allows us to classify linear shells starting with small ones and "gluing" new triangles one by one. The reader should be prepared to deal with a relative large number of cases but the arguments used in each case are very simple.

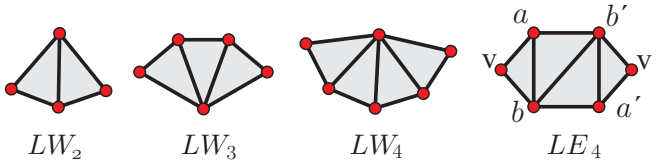

Figure 3. Small shells.

Easy arguments show that the only linear shells with less than 5 triangles are those in Figure 3. The first three are part of an infinite family obtained by deleting one triangle from wheels. They will be denoted by LW_n. The one on the right will be denoted by LE_4 and can be obtained from LW_3 by adding a triangle and identifying the two vertices labeled the letter "v".

Observe that the map $v \leftrightarrow v$, $a \leftrightarrow a'$, $b \leftrightarrow b'$ is an automorphism of LE_4 and therefore there are only two ways to glue a triangle to LE_4 in order to obtain a linear 5-shell. They are shown in Figure 4.

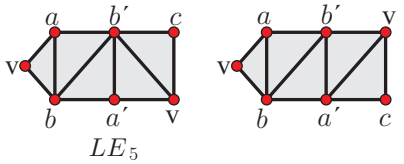

Figure 4. Adding a triangle to LE_4.

The one on the left is a linear 5-shell and will be denoted by LE_5. In the one on the right, the triangles $\{a, b', b\}$ and $\{v, a', c\}$ do not intersect and therefore we should identify the new vertex c with one in $\{a, b', b\}$.

Case	It is not a linear 5-shell because:
$c = a$	$\lvert\{v, a', c\} \cap \{a, b, v\}\rvert = 2$
$c = b$	$\lvert\{v, a', c\} \cap \{a', b, b'\}\rvert = 2$
$c = b'$	$\{v, a', c\} = \{v, a', b'\}$.

In the three cases we do not obtain a linear 5-shell and we must discard this possibility. Also observe that LE_5 contains LW_4 and therefore we proved that the only linear 5-shells are LE_5 and LW_5.

Lemma 3. *If a linear n-shell contains LW_5, then it is LW_n.*

Proof. If we add a new triangle to LW_5, then we obtain either LW_6 or the triangulation in Figure 5.

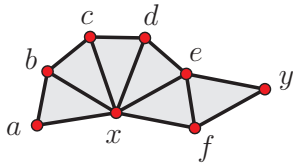

Figure 5. Adding a triangle to LW_5.

The new triangle $\{e, f, y\}$ does not intersect the triangles $\{a, b, x\}$, $\{b, c, x\}$ and $\{c, d, x\}$ and therefore we should identify the new vertex y with a non existing common vertex to these three triangles. So, we must discard this possibility. The same happens when we add a new triangle to LW_n for $n \geq 5$. ■

If we add a new triangle to LE_5, then we obtain one of the triangulations in Figure 6. The new triangle is always the one that contains the vertex d.

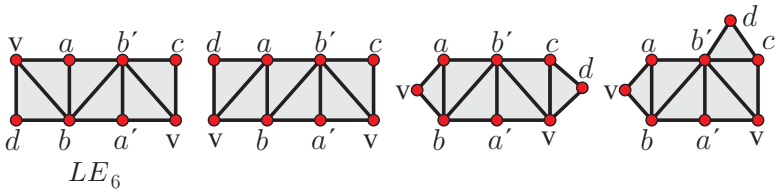

Figure 6. Adding a triangle to LE_5.

The first triangulation from the left is a linear 6-shell and will be denoted by LE_6. In the second one the new triangle does not intersect $\{a', b, b'\}$. If we identify d with one of these vertices, then a linear 6-shell is not obtained:

Case	It is not a linear 6-shell because:
$d = a'$	$\lvert\{v, a, d\} \cap \{v, a', b'\}\rvert = 2$
$d = b'$	$\lvert\{v, a, d\} \cap \{v, a', b'\}\rvert = 2$
$d = b$	$\{v, a, d\} = \{v, a, b\}$.

In the third one from the left, the new triangle does not intersect $\{a', b, b'\}$. If we identify d with one of these vertices, then a linear 6-shell is not obtained:

Case	It is not a linear 6-shell because:
$d = a'$	$\lvert\{v, c, d\} \cap \{v, b', a'\}\rvert = 2$
$d = b$	$\lvert\{v, c, d\} \cap \{v, a, b\}\rvert = 2$
$d = b'$	$\{v, c, d\} = \{v, c, b'\}$.

The last one contains LW_5 and by Lemma 3 we must discard it. So, we proved that the only linear 6-shells are LE_6 and LW_6.

The map $v \leftrightarrow v$, $a \leftrightarrow a'$, $b \leftrightarrow b'$, $c \leftrightarrow d$ is an automorphism of LE_6 and therefore there are only two ways to glue a triangle to LE_6 in order to obtain a linear 7-shell. They are showed in Figure 7.

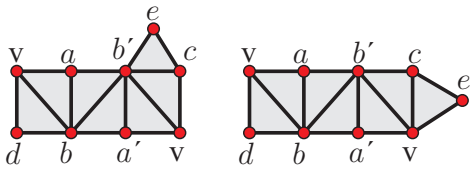

Figure 7. Adding a triangle to LE_6.

The one on the left contains LW_5 and by Lemma 3 we must discard it. In the one on the right, the new triangle does not intersect the triangles $\{a, b, b'\}$ and $\{a', b, b'\}$. If we identify e with b, then the new triangle has two vertices in common with $\{d, b, v\}$. If we identify e with b', then the triangulation has only 6 triangles. This shows that LW_7 is the only linear 7-shell. Therefore, we proved the following.

Proposition 4. The linear shells are LW_n for $n \geq 2$ and LE_m for $m \in \{4, 5, 6\}$.

Proposition 5. The cyclic shells are CW_n for $n \geq 3$ and CE_m for $m \in \{5, 6\}$.

Proof. We will add a triangle to a linear shell to obtain a cyclic shell. It is easy to see that the only cyclic shell than can be obtained from LW_n is CW_n.

Recall that LE_4 is on the right of Figure 3, LE_5 is on the left of Figure 4 and LE_6 is on the left of Figure 6. We will use the names of vertices given in those figures. We have that $CE_5 = LE_4 + \{v, a, a'\}$ and $CE_6 = LE_5 + \{v, c, b\}$.

To prove that there are no more cyclic shells, observe that the new triangle has to contain an edge from the first triangle and another edge from the last triangle. Hence, it contains the common vertex of these two triangles and no new vertices. So, we only need to check the following four cases:

Triangulation	It is not a cyclic shell because:
$LE_4 + \{v, b', a\}$	$\lvert\{v, b', a\} \cap \{a, b, b'\}\rvert = 2$
$LE_4 + \{v, b', b\}$	$\lvert\{v, b', b\} \cap \{a, b, b'\}\rvert = 2$
$LE_5 + \{v, c, a\}$	$\lvert\{v, c, a\} \cap \{a', b, b'\}\rvert = 0$
$LE_6 + \{v, c, d\}$	$\lvert\{v, c, d\} \cap \{a', b, b'\}\rvert = 0$.

■

3. THE PROOF OF THEOREM 2

Proof. Now we are ready to prove Theorem 2. Let $f : T \longrightarrow T'$ be an intersection preserving map between the sets of triangles of two triangulated surfaces S and S'. Since cyclic shells are defined in terms of the cardinalities of the intersections, then f must map cyclic shells onto cyclic shells.

Let v be a vertex of S. Its neighborhood is some wheel CW_n. If the image by f is also a wheel, then there is a vertex w in S' which is the center of this wheel. If this happens for every vertex in S, then we can define in this way the function $g : V \ni v \mapsto w \in V'$. It is not hard to prove that the action of g on triangles is f. Moreover, g is an isomorphism which extends f.

So, the only interesting cases arise when some wheel is mapped by f to a cyclic shell which is not a wheel. By the classification of cyclic shells, the image must be CE_5 or CE_6.

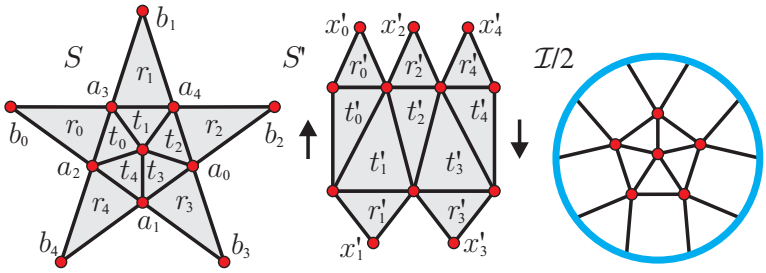

Figure 8. Mapping CW_5 to CE_5.

Case CE_5. Let $\{t_0, t_1, \ldots, t_4\}$ be a set of triangles on S which spans a wheel. Denote $t'_i = f(t_i)$. In this case we suppose that $\{t'_0, t'_1, \ldots, t'_4\}$ spans a CE_5. Since S is a surface without boundary, there must exist five triangles $r_0, r_1, \ldots, r_4$ such that $|t_i \cap r_i| = 2$. Denote $r'_i = f(r_i)$. By the properties of f we have $|t'_i \cap r'_i| = 2$. This situation is showed in Figure 8. We will use the names of the vertices in this figure. The drawing on the right is our goal: we must prove that $S \simeq \mathcal{I}/2 \simeq S'$.

Observe that r'_2 has non empty intersection with t'_0 and t'_4. By the properties of f, the triangle r_2 has non empty intersection with t_0 and t_4. Since S is a surface, the vertices a_2 and b_2 are the same. By dihedral symmetry we also have $a_i = b_i$ and therefore $S \simeq \mathcal{I}/2$.

Since $r_0 \cap r_2 = \{a_2, a_0\}$, we have $|r'_0 \cap r'_2| = 2$ and therefore $x'_0 = x'_2$, By dihedral symmetry we also have $x'_2 = x'_4 = x'_1 = x'_3$. Therefore S' can be obtained by gluing CW_5 and CE_5 by their boundaries. This implies that $S' \simeq \mathcal{I}/2$.

Case CE_6. Let $\{t_0, t_1, \ldots, t_5\}$ be a set of triangles on S which spans a wheel. Denote $t'_i = f(t_i)$. We suppose that $\{t'_0, t'_1, \ldots, t'_5\}$ spans a CE_6. Since S is a surface without boundary, there must exist six triangles $r_0, r_1, \ldots, r_6$ such that

$|t_i \cap r_i| = 2$. Denote $r'_i = f(r_i)$. By the properties of f we have $|t'_i \cap r'_i| = 2$. This situation is showed in Figure 9. We will use the names of vertices in this figure. The drawing on the right is our goal: we must prove that $S \simeq \mathcal{TC}/2 \simeq S'$. This drawing is different from the one on Figure 1 but the triangulation is the same, the line at the infinity has changed.

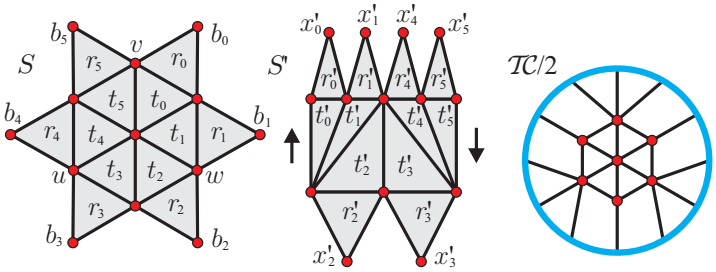

Figure 9. Mapping CW_6 to CE_6.

Since the triangle r'_0 intersects t'_3 and t'_4, we see that r_0 must intersects t_3 and t_4. Since S is a surface, $b_0 = u$. Since r'_1 intersects t'_3 and t'_4, we have $b_1 = u$. By triangular symmetry we also have that $b_5 = b_4 = w$ and $b_2 = b_3 = v$. We proved that S is isomorphic to $\mathcal{TC}/2$.

Since $|r_0 \cap r_1| = |r_1 \cap r_4| = |r_4 \cap r_5| = |r_5 \cap r_2| = |r_2 \cap r_3| = 2$ in S, we have $x'_0 = x'_1 = x'_4 = x'_5 = x'_2 = x'_3$ in S'. From this we conclude that S' can be obtained by gluing CW_6 and CE_6 by their boundaries. This implies that $S' \simeq \mathcal{TC}/2$. ∎

4. Conclusion

The reader might wonder if the intersection matrix characterizes triangulations of connected surfaces with boundaries. The answer to this question is negative and some examples of this are provided by the objects in Section 2.

However, some of the surfaces in Section 2 are non orientable, thus one may further ask if the intersection matrix of a triangulation of a connected orientable surface with boundary defines it. Here, we conjecture that the answer to this question is positive.

Further, one might wonder on the generalizations of Theorem 1 to simplicial complexes of higher dimension. We have strong evidence to suggest that the statement of Theorem 1 can indeed be extended to some classes of simplicial complexes of higher dimension. For example, triangulations of 3-dimensional balls.

Another interesting question is the algorithmic one. How fast can we build the lists of vertices and triangles of a triangulated closed surface from its intersection matrix?

References

[1] R. Blind and P. Mani-Levitska, *Puzzles and polytope isomorphisms*, Aequationes Math. **34** (1987) 287–297.
doi:10.1007/BF01830678

[2] G. Kalai, *A simple way to tell a simple polytope from its graph*, J. Combin. Theory Ser. A **49** (1988) 381–383.
doi:10.1016/0097-3165(88)90064-7

[3] B. Mohar and A. Vodopivec, *On polyhedral embeddings of cubic graphs*, Combin. Probab. Comput. **15** (2006) 877–893.
doi:10.1017/S0963548306007607

[4] G.M. Ziegler, Lectures on Polytopes, Graduate Texts in Mathematics, Springer, (1995).
doi:10.1007/978-1-4613-8431-1

Received 8 January 2014
Revised 26 September 2014
Accepted 22 October 2014