# WORM COLORINGS 

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#### Abstract

Given a coloring of the vertices, we say subgraph $H$ is monochromatic if every vertex of $H$ is assigned the same color, and rainbow if no pair of vertices of $H$ are assigned the same color. Given a graph $G$ and a graph $F$, we define an $F$-WORM coloring of $G$ as a coloring of the vertices of $G$ without a rainbow or monochromatic subgraph $H$ isomorphic to $F$. We present some results on this concept especially as regards to the existence, complexity, and optimization within certain graph classes. The focus is on the case that $F$ is the path on three vertices.


Keywords: coloring, rainbow, monochromatic, forbidden, path.
2010 Mathematics Subject Classification: 05C15.

## 1. Introduction

Let $F$ be a graph. Consider a coloring of the vertices of $G$. We say that a copy of $F$ (as a subgraph) is rainbow if all its vertices receive different colors. We say that a copy of $F$ (as a subgraph) is monochromatic if all its vertices receive the same color. It is easy to avoid monochromatic copies: color every vertex in $G$ a different color. It is also easy to avoid rainbow copies: color every vertex the same color. But things are more challenging if one tries to avoid both simultaneously.

For example, if $G=K_{5}$, then any coloring of $G$ yields either a monochromatic or a rainbow $P_{3}$. On the other hand, if we color $G=K_{4}$ giving two vertices red and two vertices blue, we avoid both a rainbow and a monochromatic $P_{3}$.

So we define an $F$-WORM coloring of $G$ as a coloring of the vertices of $G$ without a rainbow or monochromatic subgraph $H$ isomorphic to $F$. We assume the graph $F$ has at least 3 vertices, since any subgraph on 1 or 2 vertices is automatically rainbow or monochromatic. For example, if $G$ is bipartite (and $F$ is not empty), then the bipartition is automatically an $F$-WORM coloring. Indeed, if $G$ is $k$-colorable with $k$ less than the order of $F$ (and $F$ is nonempty), then a proper $k$-coloring of $G$ is an $F$-WORM coloring.

In this paper we explore the concept and establish some basic properties. We also consider, for a graph that has such a coloring, what the range of colors is. To this end, we define $W^{+}(G, F)$ as the maximum number of colors and $W^{-}(G, F)$ as the minimum number of colors in an $F$-WORM coloring of graph $G$. In this paper we focus on the fundamental results and the case that $F$ is the path $P_{3}$. Some further results where a cycle or clique is forbidden are given in [6].

Vertex colorings with various local constraints, especially avoiding monochromatic subgraphs, have been studied extensively; see for example [10]. Edgecolorings that avoid rainbow subgraphs have been studied under the term "antiRamsey numbers"; see for example [2]. There are also a few papers on edgecolorings that avoid some monochromatic and some rainbow subgraph; see for example [1].

More recently, vertex colorings that avoid rainbow subgraphs have been considered by Bujtás et al. [4], whose 3-consecutive $C$-coloring is equivalent to a coloring without a rainbow $P_{3}$, and by Bujtás et al. [3] who defined the star- $[k]$ upper chromatic number as the maximum number of colors in a coloring of the vertices without a rainbow $K_{1, k}$. It follows that $W^{+}\left(G, P_{3}\right)$ is at most the star-[2] upper chromatic number. However, our parameter is not equal to theirs (even in graphs where $W^{+}\left(G, P_{3}\right)$ exists). For example, take the tree $S$ obtained from the star on three edges by subdividing each edge once. Then one can color $S$ with 4 colors while avoiding a rainbow $P_{3}$ (color the center and all its neighbors the same color, and color each leaf with a different color). On the other hand, one can easily check that any $P_{3}$-WORM coloring of $S$ uses at most three colors (or see Theorem 17 below).

We proceed as follows. In Section 2 we show that if a graph has a $P_{3}{ }^{-}$ WORM coloring then it has one using two colors, from which it follows that the decision problem is NP-hard. In Sections 3 and 4 we consider the existence and range of $P_{3}$-WORM colorings for several graph families including bipartite graphs, Cartesian products, cubic graphs, outerplanar graphs, and trees. Finally, we consider some related complexity results in Section 5 and an extremal question in Section 6.

## 2. Basics

In this section we consider the maximum and minimum number of colors in a $P_{3}$-WORM coloring. In particular, we show that if a graph has a $P_{3}$-WORM coloring then it has such a coloring with only two colors. Note that we consider only connected graphs $G$, since if $G$ is disconnected then the existence and range of colors for $G$ is determined by the existence and range of colors for the components. For example, $W^{+}\left(G, P_{3}\right)$ is the sum of $W^{+}\left(G_{i}, P_{3}\right)$ over all components $G_{i}$ of $G$.

We consider first the maximum number of colors that a $P_{3}$-WORM coloring may use.
Theorem 1. If a graph $G$ on $n$ vertices has a $P_{3}$-WORM coloring, then

$$
W^{+}\left(G, P_{3}\right) \leq n / 2+1
$$

Proof. Note that if we add edges to the graph, then the constraints increase, and so $W^{+}$can only decrease. So it suffices to prove the result for $G$ a tree. The result is by induction. The base case of $n=1$ is trivial. Further, if $G$ is a star then $W^{+}\left(G, P_{3}\right)=2$; so we may assume that $G$ is not a star. Let $v$ be a non-leaf vertex that has at most one non-leaf neighbor $w$. Let $G^{\prime}=G-\{v\}-L_{v}$, where $L_{v}$ is the set of leaf-neighbors of $v$. Then $G^{\prime}$ is connected. Further, $N_{G}[v]$ receives at most two colors, and if exactly two colors, then one of those colors is the same color as $w$. It follows that the number of colors in $G$ is at most one more than the number of colors in $G^{\prime}$, and the bound follows.

One example of equality in Theorem 1 is the case that $G$ is a path.
Observation 2. For the path on $n$ vertices, $W^{+}\left(P_{n}, P_{3}\right)=\lfloor n / 2\rfloor+1$.
Proof. Say the vertices of the path are $v_{1} v_{2} \cdots v_{n}$. Color $v_{1}$ with color 1 , color $v_{2}$ and $v_{3}$ with color 2 , color $v_{4}$ and $v_{5}$ with color 3 , and so on. This coloring has neither a rainbow nor a monochromatic $P_{3}$ and uses $\lfloor n / 2\rfloor+1$ colors.

The above theorem can also be deduced from a result of Bujt'as et al. [4]. They showed that their 3 -consecutive $C$-coloring number of a connected graph is at most one more than the vertex cover number, which we denote by $\beta(G)$. By definition, a $P_{3}$-WORM coloring is a 3 -consecutive $C$-coloring. Thus:

Observation 3. If a connected graph $G$ has a $P_{3}$-WORM coloring, then

$$
W^{+}\left(G, P_{3}\right) \leq \beta(G)+1
$$

We next consider the minimum number of colors that a $P_{3}$-WORM coloring may use.

Theorem 4. A graph $G$ has a $P_{3}$-WORM coloring if and only if $G$ has a $P_{3}$ WORM coloring using only two colors.

Proof. Consider a $P_{3}$-WORM coloring of the graph $G$. Say an edge is monochromatic if its two ends have the same color. By the lack of monochromatic $P_{3}$ 's, the monochromatic edges form a matching. Let $H$ be the spanning subgraph of $G$ with the monochromatic edges removed. Consider any edge $u v$ in $H$; say $u$ is color $i$ and $v$ is color $j$. Then by the lack of rainbow $P_{3}$ 's, every neighbor of $u$ is color $j$ and every neighbor of $v$ is color $i$. It follows that $H$ is bipartite. If we 2 -color $G$ by the bipartition of $H$, the monochromatic edges still form a matching, and so this is a $P_{3}$-WORM coloring.

Recall that a 1-defective 2-coloring of a graph $G$ is a 2-coloring such that each vertex has at most one neighbor of its color. It follows that:

A 1-defective 2-coloring is equivalent to a $P_{3}$-WORM 2 -coloring.
For example, Cowen [5] proved that determining whether a graph has a 1-defective 2 -coloring is NP-complete. It follows that determining whether a graph has a $P_{3}$ WORM coloring is NP-complete.

It is not true, however, that if a graph has a a $P_{3}$-WORM coloring using $k$ colors, then it has one using $j$ colors for every $2<j<k$. Indeed, we now construct a graph $H_{k}$ that has a $P_{3}$-WORM coloring using $k$ colors and one using 2 colors, but for no other number of colors.

For $k \geq 3$ we construct graph $H_{k}$ as follows. Let $s=\max (3, k-2)$. For every ordered pair of distinct $i$ and $j$, with $i, j \in\{1, \ldots, k\}$, create disjoint sets $B_{i}^{j}$ of $s$ vertices. For each $i$ define $C_{i}=\bigcup_{j \neq i} B_{i}^{j}$. Then add all $s^{2}$ possible edges between sets $B_{i}^{j}$ and $B_{j}^{i}$ for all $i \neq j$. For each triple of distinct integers $i, j, j^{\prime}$, add exactly one edge between $B_{i}^{j}$ and $B_{i}^{j^{\prime}}$ such that for each $i$ the subgraph induced by $C_{i}$ has maximum degree 1. One possibility for the graph $H_{4}$ is shown in Figure 1.

Observation 5. For $k \geq 3$, every $P_{3}$-WORM coloring of $H_{k}$ uses either 2 or $k$ colors.

Proof. Consider a $P_{3}$-WORM coloring of $H_{k}$. Note that the subgraph induced by $B_{i}^{j} \cup B_{j}^{i}$ is $K_{s, s}$. It is easy to show that for $s \geq 3$ the only $P_{3}$-WORM coloring of $K_{s, s}$ is the bipartition. It follows that for each $i$ and $j$, all $s$ vertices in $B_{i}^{j}$ receive the same color; further, the color of $B_{i}^{j}$ is different from the color of $B_{j}^{i}$. Because there is a $P_{3}$ that goes from $B_{j}^{i}$ to $B_{i}^{j}$ to $B_{i}^{j^{\prime}}$, it must be that $B_{i}^{j^{\prime}}$ receives either the color of $B_{i}^{j}$ or the color of $B_{j}^{i}$. That is, there are precisely two colors on all of $C_{i} \cup C_{j}$.

So suppose some $C_{i}$ receives two colors. Then every other $C_{j}$ is colored with a subset of these two colors. Otherwise, assume every $C_{i}$ is monochromatic. It follows that every $C_{i}$ is a different color, and thus we use $k$ colors.


Figure 1. The graph $H_{4}$ whose $P_{3}$-WORM colorings use either 2 or 4 colors.

## 3. Some Calculations

Now we consider $P_{3}$-WORM colorings for some specific families of graphs.

### 3.1. Bipartite graphs

As observed earlier, the bipartite coloring of a bipartite graph is automatically a $P_{3}$-WORM coloring. So we focus on the maximum number of colors a WORMcoloring may use. We observed above that every $P_{3}$-WORM coloring of $K_{m, m}$ uses two colors. Indeed, we now observe the following slightly more general result:

Observation 6. For $n \geq m \geq 2, W^{+}\left(K_{n, n}, K_{1, m}\right)=2 m-2$.
Proof. One can achieve $2 m-2$ colors by using disjoint sets of $m-1$ different colors on each partite set. So we need to prove the upper bound.

Let $A$ and $B$ denote the partite sets of $K_{n, n}$ and consider a $K_{1, m}$-WORM coloring. Suppose that one partite set receives at least $m$ different colors, say $A$. Then starting with these $m$ vertices, it follows that every vertex $v$ in $B$ must be one of these $m$ colors. Furthermore, $v$ cannot see $m$ distinct colors different from it. That is, the coloring uses exactly $m$ colors. On the other hand, if every partite set has at most $m-1$ colors, the total number of colors is at most $2(m-1)$. It follows that $W^{+}\left(K_{n, n}, K_{1, m}\right) \leq \max (m, 2 m-2)=2 m-2$.

We saw earlier that $W^{+}\left(P_{n}, P_{3}\right)=\lfloor n / 2\rfloor+1$. This result can be generalized slightly to other forbidden paths:

Theorem 7. For $n \geq m \geq 3, W^{+}\left(P_{n}, P_{m}\right)=\left\lfloor\frac{(m-2) n}{m-1}\right\rfloor+1$.

Proof. Let the path $P_{n}$ be $v_{1} v_{2} \cdots v_{n}$. Give every vertex a different color except that $v_{a(m-1)}$ and $v_{a(m-1)+1}$ receive the same color for $1 \leq a \leq\lfloor(n-1) /(m-1)\rfloor$. For example, if $m=4$, then the coloring of $P_{n}$ starts $1,2,3,3,4,5,5,6,7,7,8,9, \ldots$ Thus, the total number of colors is $n-\lfloor(n-1) /(m-1)\rfloor$, which equals the claimed formula. It is easily checked that the coloring has neither a rainbow nor a monochromatic $P_{m}$.

We next prove the upper bound by induction on $n$ for fixed $m$. The base cases are $n \leq 2 m-2$. For these $n,\lfloor(m-2) n /(m-1)\rfloor+1=n-1$, and the desired conclusion is true. Now let $n \geq 2 m-1$. By the induction hypothesis, the number of colors used by the first $n-m+1$ vertices of $P_{n}$ is at most

$$
\left\lfloor\frac{(m-2)(n-m+1)}{(m-1)}\right\rfloor+1=\left\lfloor\frac{(m-2) n}{(m-1)}\right\rfloor-m+3 .
$$

Also note that the last $m-1$ vertices of $P_{n}$ use at most $m-2$ colors other than those used by the first $n-m+1$ vertices, otherwise we would have a rainbow $P_{m}$. Therefore, the total number of colors used is at most $\lfloor(m-2) n /(m-1)\rfloor+1$. This completes the proof.

### 3.2. Cartesian products

Recall that the Cartesian product of graphs $G$ and $H$, denoted $G \square H$, is the graph whose vertex set is $V(G) \times V(H)$, in which two vertices $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjacent if $u_{1} v_{1} \in E(G)$ and $u_{2}=v_{2}$, or $u_{1}=v_{1}$ and $u_{2} v_{2} \in E(H)$. We next consider a $P_{3}$-WORM coloring of $G \square H$.

Theorem 8. If $G$ and $H$ are nontrivial connected graphs and $G \square H$ has a $P_{3}$ WORM coloring, then it uses only two colors.

Proof. It suffices to prove the result when $G$ and $H$ are trees. We proceed by induction. Clearly when $G=H=K_{2}$, we have $W^{+}\left(C_{4}, P_{3}\right)=2$.

So assume that at least one of the factors, say $G$, has order at least 3. Let $u$ be a leaf of $G$, with neighbor $u^{\prime}$, and let $G^{\prime}=G-\{u\}$. By the inductive hypothesis, every $P_{3}$-WORM coloring of $G^{\prime} \square H$ uses only two colors. Consider any vertex $v$ of $H$. Since $G$ is not $K_{2}$, vertex $u^{\prime}$ has at least one neighbor in $G^{\prime}$, and so vertex $\left(u^{\prime}, v\right)$ is the center of a $P_{3}$ in $G^{\prime} \square H$. This means that the vertex $\left(u^{\prime}, v\right)$ has a neighbor $x$ of a different color in $G^{\prime} \square H$, and thus ( $u, v$ ) must get either the color of $\left(u^{\prime}, v\right)$ or $x$.

### 3.3. Cubic graphs

Let $G$ be a connected cubic graph. We know from [8] that $G$ has a 2-coloring where every vertex has at most one neighbor of the same color. This coloring is
a $P_{3}$-WORM. So the natural question is: What is the minimum and maximum value of $W^{+}\left(G, P_{3}\right)$ as a function of the order $n$ ?

There are many cubic graphs $G$ that have $W^{+}\left(G, P_{3}\right)=2$. One general family is the ladder $C_{m} \square K_{2}$. (See Theorem 8.)

Computer checking of small cases suggests that the maximum value of parameter $W^{+}\left(G, P_{3}\right)$ is $n / 4+1$. This value is achieved by several graphs including the following graph. For $s \geq 2$ create $B_{s}$ by taking $s$ copies of $K_{4}-e$ and adding edges to make the graph cubic and connected. For example, $B_{5}$ is illustrated in Figure 2.


Figure 2. Graph $B_{5}$ conjectured to have maximum $W^{+}\left(G, P_{3}\right)$ for cubic graphs.

Observation 9. For $s \geq 2, W^{+}\left(B_{s}, P_{3}\right)=s+1$.
Proof. Consider a $P_{3}$-WORM coloring of $B_{s}$. It is easy to show that each copy of $K_{4}-e$ has exactly two colors, and one of those colors is also present at the end of each edge leading out of the copy. Thus $s+1$ is an upper bound. An optimal coloring is obtained by coloring each central pair from a $K_{4}-e$ with a new color, and coloring all other vertices the same color.

Another interesting case is where the forbidden graph is the star on three edges. Here the maximum $W^{+}\left(G, K_{1,3}\right)$ for cubic graphs $G$ of order $n$ is $3 n / 4$, achieved uniquely by the above graph $B_{s}$. The upper bound is given by Proposition 18 of [3]:
Theorem 10 [3]. For an $r$-regular graph $G$ of order $n, W^{+}\left(G, K_{1, r}\right) \leq r n /(r+1)$.

### 3.4. Outerplanar graphs

Recall that a maximal outerplanar graph, or $M O P$, is an outerplanar graph with a maximum number of edges. That is, an outer cycle with chords triangulating the interior. In this section, we determine which maximal outerplanar graphs have a $P_{3}$-WORM coloring. But first we note that if such a graph $G$ has a $P_{3}$-WORM coloring, then $W^{+}\left(G, P_{3}\right)=W^{-}\left(G, P_{3}\right)=2$.

Observation 11. If a MOP has a $P_{3}$-WORM coloring, then that coloring uses two colors.

Proof. Consider some triangle $T_{0}=\{x, y, z\}$. It must have exactly two colors; say red and blue. If this is the whole graph we are done. Otherwise, there is another triangle $T_{1}$ that overlaps $T_{0}$ in two vertices. Say $T_{1}$ has vertices $\{x, y, w\}$. Then if $x$ and $y$ have different colors, $w$ must be one of their colors. Further, if $x$ and $y$ are both red say, since $z x w$ is a $P_{3}$ it must be that $w$ is the same color as $z$. That is, all vertices of $T_{1}$ are red or blue. Repeating the argument we see that all vertices in the graph are red or blue.

We now consider the necessary conditions for a $P_{3}$-WORM coloring. Note that by Theorem 4, this is equivalent to determining which MOPs have a 1 defective 2 -coloring. Let $F_{6}$ denote the fan given by the join $K_{1} \vee P_{6}$. This graph and the Hajós graph (also known as a 3 -sun) are shown in Figure 3.


Figure 3. Two MOPs: the fan $F_{6}$ and the Hajós graph.

Observation 12. Neither the fan $F_{6}$ nor the Hajós graph has a $P_{3}$-WORM coloring.

Proof. Consider a 2-coloring of the fan $F_{6}$. Let $v$ be the central vertex; say $v$ is colored red. Then at most one other vertex can be colored red. It follows that there must be 3 consecutive non-red vertices on the path. Thus, the coloring is not WORM.

Consider a 2-coloring of the Hajós graph. It is immediate that two of the central vertices must be one color, say red, and the other central vertex the other color, say blue. Now let $u$ and $v$ be the two vertices of degree 2 that have a blue neighbor. Then, coloring either of them red creates a red $P_{3}$, but coloring both of them blue creates a blue $P_{3}$.

So it is necessary that the MOP has maximum degree at most 5 and contains no copy of the Hajós graph. For example, one such MOP is drawn in Figure 4. (The vertices of degree 5 are in white.)

The interior graph of a MOP $G$, denoted by $C_{G}$, is the subgraph of $G$ induced by the chords. (This is well-defined as a MOP has a unique Hamiltonian cycle.)

Observation 13. A MOP G contains a copy of Hajós graph if and only if the interior graph $C_{G}$ has a cycle.


Figure 4. A MOP that has no $F_{6}$ or Hajós subgraph.

Proof. If $G$ contains a copy of Hajós graph, then $C_{G}$ contains a triangle. Conversely, assume $C_{G}$ contains a cycle; then it must contain a triangle. Since every chord of $G$ is contained in two adjacent triangles, it follows that $G$ contains a copy of the Hajós graph.

A caterpillar is a tree in which every vertex is within distance one of a central path. Hedetniemi et al. [7] showed that if the interior graph of a MOP is acyclic, then it is a caterpillar. Let $V_{5}$ denote the set of vertices of degree 5 in a MOP $G$. Equivalently, $V_{5}$ is the vertices of degree 3 in $C_{G}$. Define a stem as a path in $C_{G}$ whose ends are in $V_{5}$ and whose interior vertices are not.

Theorem 14. $A M O P G$ has a 1-defective 2 -coloring (equivalently a $P_{3}$-WORM coloring) if and only if
(a) $G$ has maximum degree at most 5 ,
(b) the interior graph $C_{G}$ is a caterpillar, and
(c) every stem of $C_{G}$ has odd length.

Proof. We first prove necessity. Let $G$ be a MOP with a $P_{3}$-WORM coloring. By Observations 12 and $13, G$ has maximum degree at most 5 and $C_{G}$ is a caterpillar. Let $P$ denote the central path of the caterpillar $C_{G}$. We are done unless $C_{G}$ has a stem; so assume $P_{u, v}$ is a stem with ends $u$ and $v$.

Let the path through $N(u)$ be $u_{1} u_{2} u_{3} u_{4} u_{5}$. Note that $u_{1}$ and $u_{5}$ are neighbors of $u$ on the outer cycle. Further, by the lack of Hajós subgraph, the edge $u_{2} u_{3}$ is not in $C_{G}$; that is, $u_{2}$ and $u_{3}$ are consecutive on the outer cycle. Similarly, so are $u_{3}$ and $u_{4}$. It follows that $u_{3}$ has degree exactly 3 in $G$, and so $u_{3}$ is a leaf in $C_{G}$.

Now consider the $P_{3}$-WORM coloring of $G$. By Observation 11, this coloring uses two colors, say 1 and 2 . It is easy to see that $u$ must have the same color as $u_{3}$, while $u_{1}, u_{2}, u_{4}, u_{5}$ have the other color. In particular, $u$ has no neighbor on $P$ of the same color. Similarly, $v$ has no neighbor on $P$ of the same color. Further, since the subgraph of $G$ induced by the vertices of $P_{u, v}$ is its square, all other vertices do have neighbors on $P$ of the same color. Indeed, assume $u$ has color 1 ; then the coloring pattern of $P_{u, v}$ must be either $1,2,2,1,1, \ldots, 2,2,1$ or $1,2,2,1,1, \ldots, 1,1,2$. Hence, the stem $P_{u, v}$ must have odd length.

Now we prove sufficiency. Assume $G$ has maximum degree at most 5, and the interior graph $C_{G}$ is a caterpillar with every stem of $C_{G}$ having odd length. We color $P$ with two colors such that every vertex of $V_{5}$ has no neighbor on $P$ of
the same color and all other vertices (except possibly the end-vertices of $P$ ) do have neighbors on $P$ of the same color. Then we give each vertex of $C_{G}-P$ the same color as their neighbor in $C_{G}$.

It remains to color the (at most two) vertices of degree 2 in $G$. Let $x$ be a vertex of degree 2 in $G$, and let $y$ and $z$ be its neighbors. Let $t$ be the other vertex with which $y$ and $z$ forms a triangle. Say $t$ is adjacent to $z$ on the outer cycle. By the construction of the coloring so far, it follows that either $y$ has the same color as $t$, in which case we can give $x$ the same color as $z$, or $y$ has the same color as $z$, in which case we can give $x$ the other color. Thus we can extend the coloring to the whole graph, as required.

## 4. Trees

The following observation will facilitate bounds for $W^{+}\left(T, P_{3}\right)$ when $T$ is a tree.
Observation 15. Consider a tree T. A $P_{3}-W O R M$ coloring of some of the vertices, such that the colored vertices induce a connected subgraph, can be extended to a $P_{3}$-WORM coloring of the whole tree.

Proof. Assume we have a $P_{3}$-WORM coloring of $U \subseteq V(T)$ such that $U$ induces a connected subgraph of $T$. Consider any uncolored vertex $v$ that is adjacent to some colored vertex $w_{v}$ (since $T$ is a tree, $w_{v}$ is unique). If $w_{v}$ sees a color $c$ different from its own color, then assign $v$ the color $c$. If $w_{v}$ has no colored neighbor or its only colored neighbor has the same color as it, then give $v$ any other color. In both cases we do not create a monochromatic or rainbow $P_{3}$. Repeat until all vertices colored.

For example, since a tree $T$ contains a path of the same diameter, it follows from Observation 2 that $W^{+}\left(T, P_{3}\right) \geq \operatorname{diam}(T) / 2+1$.

We consider next a tree algorithm. There are general results (see for example [9]) that show that there is a linear-time algorithm to compute the parameter $W^{+}\left(T, P_{3}\right)$ for a tree $T$, and indeed for bounded treewidth. Nevertheless, we give the details of an algorithm below, and then use it to calculate the value of $W^{+}\left(T, P_{3}\right)$ for a spider (sometimes called an octopus). We do the standard postorder traversal algorithm. That is, we root the tree at some vertex $r$ and then calculate a vector at each vertex representing the values of several parameters on the subtree rooted at that vertex.

For vertex $v$, define $T_{v}$ to be the subtree rooted at $v$ and $k(v)$ to be the number of children of $v$. Define $p(v)$ to be the maximum number of colors in a $P_{3}$-WORM coloring of $T_{v}$ with the constraint that $v$ has a child of the same color ("partnered"); and define $s(v)$ to be the maximum number of colors in a
$P_{3}$-WORM coloring of $T_{v}$ with the constraint that $v$ has no child of the same color ("solitary"). By Observation 15, such a coloring exists (that is, $p(v)$ and $s(v)$ are defined) except for the case of $p(v)$ when $k(v)=0$.

Define $\ell_{p}(v)=2$ if $k(v) \geq 2$ and 1 otherwise; define $\ell_{s}(v)=2$ if $k(v) \geq 1$, and 1 otherwise. Note that $\ell_{p}(v)$ and $\ell_{s}(v)$ denote the number of colors in $N[v]$ in a partnered and solitary coloring of $T_{v}$ respectively. Let $P(v)=p(v)-\ell_{p}(v)$ and $S(v)=s(v)-\ell_{s}(v)$.

Theorem 16. If vertex $v$ has children $c_{1}, \ldots, c_{k}, k \geq 1$, then

$$
\begin{aligned}
& p(v)= \begin{cases}\max _{1 \leq i \leq k}\left\{1+s\left(c_{i}\right)+\sum_{j \neq i} \max \left(P\left(c_{j}\right), S\left(c_{j}\right)\right)\right\}, & \text { if } k \geq 2, \\
s\left(c_{1}\right), & \text { otherwise } ;\end{cases} \\
& s(v)=2+\sum_{i=1}^{k} \max \left(P\left(c_{i}\right), S\left(c_{i}\right)\right) .
\end{aligned}
$$

Proof. Consider a $P_{3}$-WORM coloring of $T_{v}$. Say $v$ has children $c_{1}, \ldots, c_{k}$. To maximize the colors, the color-set used in $T_{c_{i}}$ should be as disjoint as possible from the color-set used in $T_{c_{j}}$. But note that there has to be some overlap.

Specifically, if $v$ is solitary, then all its children have the same color. In the tree $T_{c_{i}}$, any child of $c_{i}$ has the same color as either $c_{i}$ or $v$. So the maximum number of colors that appear only in $T_{c_{i}}-\left\{c_{i}\right\}$ is $\max \left(P\left(c_{i}\right), S\left(c_{i}\right)\right)$. Further, if $v$ is partnered, say with $c_{i}$, then there are $s\left(c_{i}\right)$ colors in the subtree $T_{c_{i}}$. There is 1 color for all other children $c_{j}$ of $v$. As above, the maximum number of colors that appear only in $T_{c_{j}}-\left\{c_{j}\right\}$ is $\max \left(P\left(c_{j}\right), S\left(c_{j}\right)\right)$.

Since these maxima can be computed in time proportional to $k(v)$, and $W^{+}\left(T, P_{3}\right)=\max (p(r), s(r))$, we obtain a linear-time algorithm to calculate $W^{+}\left(T, P_{3}\right)$ for a tree $T$.

As an application, we determine the value of $W^{+}\left(T, P_{3}\right)$ for an octopus:
Theorem 17. Let $X$ be a star with $k \geq 2$ leaves, and let $T$ be the subdivision of $X$ where the $i^{\text {th }}$ edge of $X$ is subdivided $a_{i} \geq 0$ times for $1 \leq i \leq k$. Then

$$
W^{+}\left(T, P_{3}\right)=2+\sum_{i=1}^{k}\left\lceil\frac{a_{i}-1}{2}\right\rceil+x,
$$

where $x$ is 1 if at least one $a_{i}$ is odd and 0 otherwise.
Proof. This follows from Theorem 16 by considering the children $c_{1}, \ldots, c_{k}$ of the original center. It is easy to check that $P\left(c_{i}\right)=\left\lceil\left(a_{i}-1\right) / 2\right\rceil$ and $S\left(c_{i}\right)=$ $\left\lceil\left(a_{i}-2\right) / 2\right\rceil$.

## 5. WORM is Easy Sometimes

We observed earlier that determining whether a graph has a $P_{3}$-WORM coloring is NP-hard. There are at least a few cases of forbidden graphs where the problem has a polynomial-time algorithm. The first case is trivial:

Observation 18. For $F=m K_{1}$, graph $G$ has an $F$-WORM-coloring if and only if $G$ has at most $(m-1)^{2}$ vertices.

Here is another forbidden graph with a characterization:
Observation 19. Let $F$ be the one-edge graph on three vertices. A graph $G$ has an $F$-WORM coloring if and only if $G$ is
(a) bipartite,
(b) a subgraph of the join $K_{2} \vee m K_{1}$ for some $m \geq 1$, or
(c) $K_{4}$.

Proof. Clearly a graph with order at most 4 has an F-WORM coloring using only 2 colors. We already know that a proper 2-coloring of a bipartite graph is an $F$-WORM coloring. If $G$ is a subgraph of $K_{2} \vee m K_{1}$, color the vertices of the $K_{2}$ with one color and the other vertices a second color. This gives an $F$-WORM coloring.

Conversely, let $G$ be a graph with an $F$-WORM coloring and suppose $G$ is not bipartite. Then consider any vertex $u$ with at least two neighbors. Since there is no monochromatic $F$, it must be that $u$ has a different color to at least one of its neighbors, say $v$. Since there is no rainbow $F$, it follows that every other vertex has the same color as either $u$ or $v$.

But since $G$ is not bipartite, this means there exists an edge $x y$ where $x$ and $y$ have the same color, say color 1 . By the lack of monochromatic $F$ it follows that every other vertex has the other color, say 2 . If the vertices of color 2 form an independent set, then we have a subgraph of the join $K_{2} \vee m K_{1}$ for some $m>0$. Otherwise, by the same reasoning there are exactly two vertices of color 2 and we have a subgraph of $K_{4}$.

## 6. Extremal Questions

The classical Turán problem ask for the maximum number of edges of a graph with $n$ vertices that does not contain some given subgraph $H$. This maximum is called the Turán number of $H$. Here we consider an analogue of the classical Turán problem: what is the maximum number of edges of a graph with $n$ vertices if the graph admits a $F$-WORM coloring? We will let wex $(n, F)$ denote this maximum. We have the following result when $F=P_{3}$.

Theorem 20. For $n \geq 1$,

$$
\text { wex }\left(n, P_{3}\right)= \begin{cases}\frac{n(n+2)}{4}, & \text { if } n \text { is a multiple of } 4, \\ \frac{n^{2}+2 n-4}{4}, & \text { if } n \equiv 2(\bmod 4) \\ \frac{(n-1)(n+3)}{4}, & \text { otherwise }\end{cases}
$$

Proof. Consider a graph $G$ that has a $P_{3}$-WORM coloring. By Theorem 4, there is such a coloring using only two colors, say red and blue. Such a coloring is a $P_{3}$ WORM coloring if and only if there is no monochromatic $P_{3}$. It follows that the maximum number of edges in $G$ is obtained by taking some complete bipartite graph and adding a maximum matching within each partite set.

When $n$ is a multiple of 4 , then the number of edges is obviously maximized when the two colors are used equally. When $n$ is odd, the maximum is when the two colors are used as equally as possible. When $n$ is even but not a multiple of 4 , there are actually two extremal graphs: adding $n / 2$ edges to $K_{n / 2-1, n / 2+1}$ or adding $2(n / 2-1) / 2$ edges to $K_{n / 2, n / 2}$. We omit the calculations.

## 7. Conclusion

We have considered WORM colorings where the forbidden graph is $P_{3}$ and provided existence results and bounds on the maximum number of colors for several graph families. The next step would be to consider other forbidden graphs, such as $K_{3}$. Indeed a natural generalization is to consider sets of forbidden graphs, such as all cycles. Some results in this direction are given in [6].

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