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WORM COLORINGS

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Abstract

Given a coloring of the vertices, we say subgraph H is monochromatic if every vertex of H is assigned the same color, and rainbow if no pair of vertices of H are assigned the same color. Given a graph G and a graph F, we define an F-WORM coloring of G as a coloring of the vertices of G without a rainbow or monochromatic subgraph H isomorphic to F. We present some results on this concept especially as regards to the existence, complexity, and optimization within certain graph classes. The focus is on the case that F is the path on three vertices.

Keywords: coloring, rainbow, monochromatic, forbidden, path.

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1. Introduction

Let F be a graph. Consider a coloring of the vertices of G. We say that a copy of F (as a subgraph) is *rainbow* if all its vertices receive different colors. We say that a copy of F (as a subgraph) is *monochromatic* if all its vertices receive the same color. It is easy to avoid monochromatic copies: color every vertex in G a different color. It is also easy to avoid rainbow copies: color every vertex the same color. But things are more challenging if one tries to avoid both simultaneously.

For example, if $G = K_5$, then any coloring of G yields either a monochromatic or a rainbow P_3. On the other hand, if we color $G = K_4$ giving two vertices red and two vertices blue, we avoid both a rainbow and a monochromatic P_3.

So we define an *F-WORM coloring of G* as a coloring of the vertices of G without a rainbow or monochromatic subgraph H isomorphic to F. We assume the graph F has at least 3 vertices, since any subgraph on 1 or 2 vertices is automatically rainbow or monochromatic. For example, if G is bipartite (and F is not empty), then the bipartition is automatically an F-WORM coloring. Indeed, if G is k-colorable with k less than the order of F (and F is nonempty), then a proper k-coloring of G is an F-WORM coloring.

In this paper we explore the concept and establish some basic properties. We also consider, for a graph that has such a coloring, what the range of colors is. To this end, we define $W^+(G, F)$ as the maximum number of colors and $W^-(G, F)$ as the minimum number of colors in an F-WORM coloring of graph G. In this paper we focus on the fundamental results and the case that F is the path P_3. Some further results where a cycle or clique is forbidden are given in [6].

Vertex colorings with various local constraints, especially avoiding monochromatic subgraphs, have been studied extensively; see for example [10]. Edge-colorings that avoid rainbow subgraphs have been studied under the term “anti-Ramsey numbers”; see for example [2]. There are also a few papers on edge-colorings that avoid some monochromatic and some rainbow subgraph; see for example [1].

More recently, vertex colorings that avoid rainbow subgraphs have been considered by Bujtás *et al.* [4], whose 3-*consecutive C-coloring* is equivalent to a coloring without a rainbow P_3, and by Bujtás *et al.* [3] who defined the *star-$[k]$ upper chromatic number* as the maximum number of colors in a coloring of the vertices without a rainbow $K_{1,k}$. It follows that $W^+(G, P_3)$ is at most the star-[2] upper chromatic number. However, our parameter is not equal to theirs (even in graphs where $W^+(G, P_3)$ exists). For example, take the tree S obtained from the star on three edges by subdividing each edge once. Then one can color S with 4 colors while avoiding a rainbow P_3 (color the center and all its neighbors the same color, and color each leaf with a different color). On the other hand, one can easily check that any P_3-WORM coloring of S uses at most three colors (or see Theorem 17 below).

We proceed as follows. In Section 2 we show that if a graph has a P_3-WORM coloring then it has one using two colors, from which it follows that the decision problem is NP-hard. In Sections 3 and 4 we consider the existence and range of P_3-WORM colorings for several graph families including bipartite graphs, Cartesian products, cubic graphs, outerplanar graphs, and trees. Finally, we consider some related complexity results in Section 5 and an extremal question in Section 6.

2. Basics

In this section we consider the maximum and minimum number of colors in a P_3-WORM coloring. In particular, we show that if a graph has a P_3-WORM coloring then it has such a coloring with only two colors. Note that we consider only connected graphs G, since if G is disconnected then the existence and range of colors for G is determined by the existence and range of colors for the components. For example, $W^+(G, P_3)$ is the sum of $W^+(G_i, P_3)$ over all components G_i of G.

We consider first the maximum number of colors that a P_3-WORM coloring may use.

Theorem 1. *If a graph G on n vertices has a P_3-WORM coloring, then*

$$W^+(G, P_3) \leq n/2 + 1.$$

Proof. Note that if we add edges to the graph, then the constraints increase, and so W^+ can only decrease. So it suffices to prove the result for G a tree. The result is by induction. The base case of $n = 1$ is trivial. Further, if G is a star then $W^+(G, P_3) = 2$; so we may assume that G is not a star. Let v be a non-leaf vertex that has at most one non-leaf neighbor w. Let $G' = G - \{v\} - L_v$, where L_v is the set of leaf-neighbors of v. Then G' is connected. Further, $N_G[v]$ receives at most two colors, and if exactly two colors, then one of those colors is the same color as w. It follows that the number of colors in G is at most one more than the number of colors in G', and the bound follows. ∎

One example of equality in Theorem 1 is the case that G is a path.

Observation 2. *For the path on n vertices, $W^+(P_n, P_3) = \lfloor n/2 \rfloor + 1$.*

Proof. Say the vertices of the path are $v_1 v_2 \cdots v_n$. Color v_1 with color 1, color v_2 and v_3 with color 2, color v_4 and v_5 with color 3, and so on. This coloring has neither a rainbow nor a monochromatic P_3 and uses $\lfloor n/2 \rfloor + 1$ colors. ∎

The above theorem can also be deduced from a result of Bujt'as *et al.* [4]. They showed that their 3-consecutive C-coloring number of a connected graph is at most one more than the vertex cover number, which we denote by $\beta(G)$. By definition, a P_3-WORM coloring is a 3-consecutive C-coloring. Thus:

Observation 3. *If a connected graph G has a P_3-WORM coloring, then*

$$W^+(G, P_3) \leq \beta(G) + 1.$$

We next consider the minimum number of colors that a P_3-WORM coloring may use.

Theorem 4. *A graph G has a P_3-WORM coloring if and only if G has a P_3-WORM coloring using only two colors.*

Proof. Consider a P_3-WORM coloring of the graph G. Say an edge is monochromatic if its two ends have the same color. By the lack of monochromatic P_3's, the monochromatic edges form a matching. Let H be the spanning subgraph of G with the monochromatic edges removed. Consider any edge uv in H; say u is color i and v is color j. Then by the lack of rainbow P_3's, every neighbor of u is color j and every neighbor of v is color i. It follows that H is bipartite. If we 2-color G by the bipartition of H, the monochromatic edges still form a matching, and so this is a P_3-WORM coloring. ■

Recall that a 1-defective 2-coloring of a graph G is a 2-coloring such that each vertex has at most one neighbor of its color. It follows that:

A 1*-defective* 2*-coloring is equivalent to a* P_3*-WORM* 2*-coloring.*

For example, Cowen [5] proved that determining whether a graph has a 1-defective 2-coloring is NP-complete. It follows that determining whether a graph has a P_3-WORM coloring is NP-complete.

It is not true, however, that if a graph has a a P_3-WORM coloring using k colors, then it has one using j colors for every $2 < j < k$. Indeed, we now construct a graph H_k that has a P_3-WORM coloring using k colors and one using 2 colors, but for no other number of colors.

For $k \geq 3$ we construct graph H_k as follows. Let $s = \max(3, k-2)$. For every ordered pair of distinct i and j, with $i, j \in \{1, \ldots, k\}$, create disjoint sets B_i^j of s vertices. For each i define $C_i = \bigcup_{j \neq i} B_i^j$. Then add all s^2 possible edges between sets B_i^j and B_j^i for all $i \neq j$. For each triple of distinct integers i, j, j', add exactly one edge between B_i^j and $B_i^{j'}$ such that for each i the subgraph induced by C_i has maximum degree 1. One possibility for the graph H_4 is shown in Figure 1.

Observation 5. *For* $k \geq 3$*, every* P_3*-WORM coloring of* H_k *uses either* 2 *or* k *colors.*

Proof. Consider a P_3-WORM coloring of H_k. Note that the subgraph induced by $B_i^j \cup B_j^i$ is $K_{s,s}$. It is easy to show that for $s \geq 3$ the only P_3-WORM coloring of $K_{s,s}$ is the bipartition. It follows that for each i and j, all s vertices in B_i^j receive the same color; further, the color of B_i^j is different from the color of B_j^i. Because there is a P_3 that goes from B_j^i to B_i^j to $B_i^{j'}$, it must be that $B_i^{j'}$ receives either the color of B_i^j or the color of B_j^i. That is, there are precisely two colors on all of $C_i \cup C_j$.

So suppose some C_i receives two colors. Then every other C_j is colored with a subset of these two colors. Otherwise, assume every C_i is monochromatic. It follows that every C_i is a different color, and thus we use k colors. ■

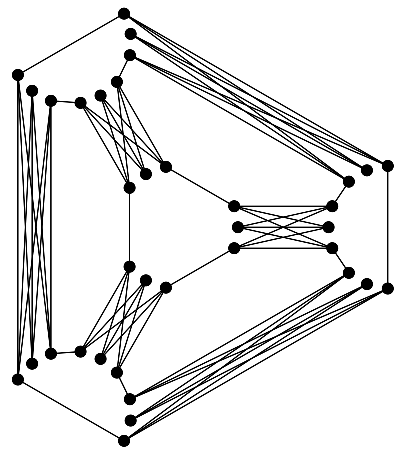

Figure 1. The graph H_4 whose P_3-WORM colorings use either 2 or 4 colors.

3. Some Calculations

Now we consider P_3-WORM colorings for some specific families of graphs.

3.1. Bipartite graphs

As observed earlier, the bipartite coloring of a bipartite graph is automatically a P_3-WORM coloring. So we focus on the maximum number of colors a WORM-coloring may use. We observed above that every P_3-WORM coloring of $K_{m,m}$ uses two colors. Indeed, we now observe the following slightly more general result:

Observation 6. *For $n \geq m \geq 2$, $W^+(K_{n,n}, K_{1,m}) = 2m - 2$.*

Proof. One can achieve $2m - 2$ colors by using disjoint sets of $m - 1$ different colors on each partite set. So we need to prove the upper bound.

Let A and B denote the partite sets of $K_{n,n}$ and consider a $K_{1,m}$-WORM coloring. Suppose that one partite set receives at least m different colors, say A. Then starting with these m vertices, it follows that every vertex v in B must be one of these m colors. Furthermore, v cannot see m distinct colors different from it. That is, the coloring uses exactly m colors. On the other hand, if every partite set has at most $m - 1$ colors, the total number of colors is at most $2(m - 1)$. It follows that $W^+(K_{n,n}, K_{1,m}) \leq \max(m, 2m - 2) = 2m - 2$. ∎

We saw earlier that $W^+(P_n, P_3) = \lfloor n/2 \rfloor + 1$. This result can be generalized slightly to other forbidden paths:

Theorem 7. *For $n \geq m \geq 3$, $W^+(P_n, P_m) = \left\lfloor \frac{(m-2)n}{m-1} \right\rfloor + 1$.*

Proof. Let the path P_n be $v_1v_2\cdots v_n$. Give every vertex a different color except that $v_{a(m-1)}$ and $v_{a(m-1)+1}$ receive the same color for $1 \le a \le \lfloor (n-1)/(m-1) \rfloor$. For example, if $m=4$, then the coloring of P_n starts $1,2,3,3,4,5,5,6,7,7,8,9,\ldots$ Thus, the total number of colors is $n - \lfloor (n-1)/(m-1) \rfloor$, which equals the claimed formula. It is easily checked that the coloring has neither a rainbow nor a monochromatic P_m.

We next prove the upper bound by induction on n for fixed m. The base cases are $n \le 2m-2$. For these n, $\lfloor (m-2)n/(m-1) \rfloor + 1 = n-1$, and the desired conclusion is true. Now let $n \ge 2m-1$. By the induction hypothesis, the number of colors used by the first $n-m+1$ vertices of P_n is at most

$$\left\lfloor \frac{(m-2)(n-m+1)}{(m-1)} \right\rfloor + 1 = \left\lfloor \frac{(m-2)n}{(m-1)} \right\rfloor - m + 3.$$

Also note that the last $m-1$ vertices of P_n use at most $m-2$ colors other than those used by the first $n-m+1$ vertices, otherwise we would have a rainbow P_m. Therefore, the total number of colors used is at most $\lfloor (m-2)n/(m-1) \rfloor + 1$. This completes the proof. ■

3.2. Cartesian products

Recall that the *Cartesian product* of graphs G and H, denoted $G \square H$, is the graph whose vertex set is $V(G) \times V(H)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2v_2 \in E(H)$. We next consider a P_3-WORM coloring of $G \square H$.

Theorem 8. *If G and H are nontrivial connected graphs and $G \square H$ has a P_3-WORM coloring, then it uses only two colors.*

Proof. It suffices to prove the result when G and H are trees. We proceed by induction. Clearly when $G = H = K_2$, we have $W^+(C_4, P_3) = 2$.

So assume that at least one of the factors, say G, has order at least 3. Let u be a leaf of G, with neighbor u', and let $G' = G - \{u\}$. By the inductive hypothesis, every P_3-WORM coloring of $G' \square H$ uses only two colors. Consider any vertex v of H. Since G is not K_2, vertex u' has at least one neighbor in G', and so vertex (u', v) is the center of a P_3 in $G' \square H$. This means that the vertex (u', v) has a neighbor x of a different color in $G' \square H$, and thus (u, v) must get either the color of (u', v) or x. ■

3.3. Cubic graphs

Let G be a connected cubic graph. We know from [8] that G has a 2-coloring where every vertex has at most one neighbor of the same color. This coloring is a P_3-WORM. So the natural question is: What is the minimum and maximum value of $W^+(G, P_3)$ as a function of the order n?

There are many cubic graphs G that have $W^+(G, P_3) = 2$. One general family is the ladder $C_m \square K_2$. (See Theorem 8.)

Computer checking of small cases suggests that the maximum value of parameter $W^+(G, P_3)$ is $n/4 + 1$. This value is achieved by several graphs including the following graph. For $s \geq 2$ create B_s by taking s copies of $K_4 - e$ and adding edges to make the graph cubic and connected. For example, B_5 is illustrated in Figure 2.

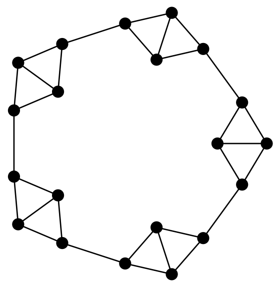

Figure 2. Graph B_5 conjectured to have maximum $W^+(G, P_3)$ for cubic graphs.

Observation 9. *For $s \geq 2$, $W^+(B_s, P_3) = s + 1$.*

Proof. Consider a P_3-WORM coloring of B_s. It is easy to show that each copy of $K_4 - e$ has exactly two colors, and one of those colors is also present at the end of each edge leading out of the copy. Thus $s + 1$ is an upper bound. An optimal coloring is obtained by coloring each central pair from a $K_4 - e$ with a new color, and coloring all other vertices the same color. ■

Another interesting case is where the forbidden graph is the star on three edges. Here the maximum $W^+(G, K_{1,3})$ for cubic graphs G of order n is $3n/4$, achieved uniquely by the above graph B_s. The upper bound is given by Proposition 18 of [3]:

Theorem 10 [3]**.** *For an r-regular graph G of order n, $W^+(G, K_{1,r}) \leq rn/(r+1)$.*

3.4. Outerplanar graphs

Recall that a *maximal outerplanar graph*, or *MOP*, is an outerplanar graph with a maximum number of edges. That is, an outer cycle with chords triangulating the interior. In this section, we determine which maximal outerplanar graphs have a P_3-WORM coloring. But first we note that if such a graph G has a P_3-WORM coloring, then $W^+(G, P_3) = W^-(G, P_3) = 2$.

Observation 11. *If a MOP has a P_3-WORM coloring, then that coloring uses two colors.*

Proof. Consider some triangle $T_0 = \{x, y, z\}$. It must have exactly two colors; say red and blue. If this is the whole graph we are done. Otherwise, there is another triangle T_1 that overlaps T_0 in two vertices. Say T_1 has vertices $\{x, y, w\}$. Then if x and y have different colors, w must be one of their colors. Further, if x and y are both red say, since zxw is a P_3 it must be that w is the same color as z. That is, all vertices of T_1 are red or blue. Repeating the argument we see that all vertices in the graph are red or blue. ■

We now consider the necessary conditions for a P_3-WORM coloring. Note that by Theorem 4, this is equivalent to determining which MOPs have a 1-defective 2-coloring. Let F_6 denote the fan given by the join $K_1 \vee P_6$. This graph and the Hajós graph (also known as a 3-sun) are shown in Figure 3.

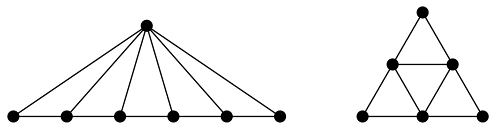

Figure 3. Two MOPs: the fan F_6 and the Hajós graph.

Observation 12. *Neither the fan F_6 nor the Hajós graph has a P_3-WORM coloring.*

Proof. Consider a 2-coloring of the fan F_6. Let v be the central vertex; say v is colored red. Then at most one other vertex can be colored red. It follows that there must be 3 consecutive non-red vertices on the path. Thus, the coloring is not WORM.

Consider a 2-coloring of the Hajós graph. It is immediate that two of the central vertices must be one color, say red, and the other central vertex the other color, say blue. Now let u and v be the two vertices of degree 2 that have a blue neighbor. Then, coloring either of them red creates a red P_3, but coloring both of them blue creates a blue P_3. ■

So it is necessary that the MOP has maximum degree at most 5 and contains no copy of the Hajós graph. For example, one such MOP is drawn in Figure 4. (The vertices of degree 5 are in white.)

The *interior graph* of a MOP G, denoted by C_G, is the subgraph of G induced by the chords. (This is well-defined as a MOP has a unique Hamiltonian cycle.)

Observation 13. *A MOP G contains a copy of Hajós graph if and only if the interior graph C_G has a cycle.*

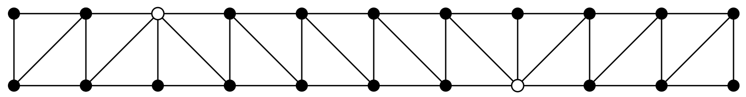

Figure 4. A MOP that has no F_6 or Hajós subgraph.

Proof. If G contains a copy of Hajós graph, then C_G contains a triangle. Conversely, assume C_G contains a cycle; then it must contain a triangle. Since every chord of G is contained in two adjacent triangles, it follows that G contains a copy of the Hajós graph. ∎

A *caterpillar* is a tree in which every vertex is within distance one of a central path. Hedetniemi *et al.* [7] showed that if the interior graph of a MOP is acyclic, then it is a caterpillar. Let V_5 denote the set of vertices of degree 5 in a MOP G. Equivalently, V_5 is the vertices of degree 3 in C_G. Define a *stem* as a path in C_G whose ends are in V_5 and whose interior vertices are not.

Theorem 14. *A MOP G has a* 1*-defective* 2*-coloring* (*equivalently a P_3-WORM coloring*) *if and only if*

(a) *G has maximum degree at most* 5*,*

(b) *the interior graph C_G is a caterpillar, and*

(c) *every stem of C_G has odd length.*

Proof. We first prove necessity. Let G be a MOP with a P_3-WORM coloring. By Observations 12 and 13, G has maximum degree at most 5 and C_G is a caterpillar. Let P denote the central path of the caterpillar C_G. We are done unless C_G has a stem; so assume $P_{u,v}$ is a stem with ends u and v.

Let the path through $N(u)$ be $u_1u_2u_3u_4u_5$. Note that u_1 and u_5 are neighbors of u on the outer cycle. Further, by the lack of Hajós subgraph, the edge u_2u_3 is not in C_G; that is, u_2 and u_3 are consecutive on the outer cycle. Similarly, so are u_3 and u_4. It follows that u_3 has degree exactly 3 in G, and so u_3 is a leaf in C_G.

Now consider the P_3-WORM coloring of G. By Observation 11, this coloring uses two colors, say 1 and 2. It is easy to see that u must have the same color as u_3, while u_1, u_2, u_4, u_5 have the other color. In particular, u has no neighbor on P of the same color. Similarly, v has no neighbor on P of the same color. Further, since the subgraph of G induced by the vertices of $P_{u,v}$ is its square, all other vertices do have neighbors on P of the same color. Indeed, assume u has color 1; then the coloring pattern of $P_{u,v}$ must be either $1, 2, 2, 1, 1, \ldots, 2, 2, 1$ or $1, 2, 2, 1, 1, \ldots, 1, 1, 2$. Hence, the stem $P_{u,v}$ must have odd length.

Now we prove sufficiency. Assume G has maximum degree at most 5, and the interior graph C_G is a caterpillar with every stem of C_G having odd length. We color P with two colors such that every vertex of V_5 has no neighbor on P of the same color and all other vertices (except possibly the end-vertices of P) do have neighbors on P of the same color. Then we give each vertex of $C_G - P$ the same color as their neighbor in C_G.

It remains to color the (at most two) vertices of degree 2 in G. Let x be a vertex of degree 2 in G, and let y and z be its neighbors. Let t be the other vertex with which y and z forms a triangle. Say t is adjacent to z on the outer cycle. By the construction of the coloring so far, it follows that either y has the same color as t, in which case we can give x the same color as z, or y has the same color as z, in which case we can give x the other color. Thus we can extend the coloring to the whole graph, as required. ■

4. Trees

The following observation will facilitate bounds for $W^+(T, P_3)$ when T is a tree.

Observation 15. *Consider a tree T. A P_3-WORM coloring of some of the vertices, such that the colored vertices induce a connected subgraph, can be extended to a P_3-WORM coloring of the whole tree.*

Proof. Assume we have a P_3-WORM coloring of $U \subseteq V(T)$ such that U induces a connected subgraph of T. Consider any uncolored vertex v that is adjacent to some colored vertex w_v (since T is a tree, w_v is unique). If w_v sees a color c different from its own color, then assign v the color c. If w_v has no colored neighbor or its only colored neighbor has the same color as it, then give v any other color. In both cases we do not create a monochromatic or rainbow P_3. Repeat until all vertices colored. ■

For example, since a tree T contains a path of the same diameter, it follows from Observation 2 that $W^+(T, P_3) \geq \mathrm{diam}(T)/2 + 1$.

We consider next a tree algorithm. There are general results (see for example [9]) that show that there is a linear-time algorithm to compute the parameter $W^+(T, P_3)$ for a tree T, and indeed for bounded treewidth. Nevertheless, we give the details of an algorithm below, and then use it to calculate the value of $W^+(T, P_3)$ for a spider (sometimes called an octopus). We do the standard postorder traversal algorithm. That is, we root the tree at some vertex r and then calculate a vector at each vertex representing the values of several parameters on the subtree rooted at that vertex.

For vertex v, define T_v to be the subtree rooted at v and $k(v)$ to be the number of children of v. Define $p(v)$ to be the maximum number of colors in a P_3-WORM coloring of T_v with the constraint that v has a child of the same color ("partnered"); and define $s(v)$ to be the maximum number of colors in a

P_3-WORM coloring of T_v with the constraint that v has no child of the same color ("solitary"). By Observation 15, such a coloring exists (that is, $p(v)$ and $s(v)$ are defined) except for the case of $p(v)$ when $k(v) = 0$.

Define $\ell_p(v) = 2$ if $k(v) \geq 2$ and 1 otherwise; define $\ell_s(v) = 2$ if $k(v) \geq 1$, and 1 otherwise. Note that $\ell_p(v)$ and $\ell_s(v)$ denote the number of colors in $N[v]$ in a partnered and solitary coloring of T_v respectively. Let $P(v) = p(v) - \ell_p(v)$ and $S(v) = s(v) - \ell_s(v)$.

Theorem 16. *If vertex v has children $c_1, \ldots, c_k$, $k \geq 1$, then*

$$p(v) = \begin{cases} \max\limits_{1 \leq i \leq k} \left\{ 1 + s(c_i) + \sum_{j \neq i} \max\left(P(c_j), S(c_j)\right) \right\}, & \text{if } k \geq 2, \\ s(c_1), & \text{otherwise}; \end{cases}$$

$$s(v) = 2 + \sum_{i=1}^{k} \max\left(P(c_i), S(c_i)\right).$$

Proof. Consider a P_3-WORM coloring of T_v. Say v has children $c_1, \ldots, c_k$. To maximize the colors, the color-set used in T_{c_i} should be as disjoint as possible from the color-set used in T_{c_j}. But note that there has to be some overlap.

Specifically, if v is solitary, then all its children have the same color. In the tree T_{c_i}, any child of c_i has the same color as either c_i or v. So the maximum number of colors that appear only in $T_{c_i} - \{c_i\}$ is $\max(P(c_i), S(c_i))$. Further, if v is partnered, say with c_i, then there are $s(c_i)$ colors in the subtree T_{c_i}. There is 1 color for all other children c_j of v. As above, the maximum number of colors that appear only in $T_{c_j} - \{c_j\}$ is $\max(P(c_j), S(c_j))$. ■

Since these maxima can be computed in time proportional to $k(v)$, and $W^+(T, P_3) = \max(p(r), s(r))$, we obtain a linear-time algorithm to calculate $W^+(T, P_3)$ for a tree T.

As an application, we determine the value of $W^+(T, P_3)$ for an octopus:

Theorem 17. *Let X be a star with $k \geq 2$ leaves, and let T be the subdivision of X where the i^{th} edge of X is subdivided $a_i \geq 0$ times for $1 \leq i \leq k$. Then*

$$W^+(T, P_3) = 2 + \sum_{i=1}^{k} \left\lceil \frac{a_i - 1}{2} \right\rceil + x,$$

where x is 1 if at least one a_i is odd and 0 otherwise.

Proof. This follows from Theorem 16 by considering the children $c_1, \ldots, c_k$ of the original center. It is easy to check that $P(c_i) = \lceil (a_i - 1)/2 \rceil$ and $S(c_i) = \lceil (a_i - 2)/2 \rceil$. ■

5. WORM is Easy Sometimes

We observed earlier that determining whether a graph has a P_3-WORM coloring is NP-hard. There are at least a few cases of forbidden graphs where the problem has a polynomial-time algorithm. The first case is trivial:

Observation 18. *For $F = mK_1$, graph G has an F-WORM-coloring if and only if G has at most $(m-1)^2$ vertices.*

Here is another forbidden graph with a characterization:

Observation 19. *Let F be the one-edge graph on three vertices. A graph G has an F-WORM coloring if and only if G is*
(a) *bipartite,*
(b) *a subgraph of the join $K_2 \vee mK_1$ for some $m \geq 1$, or*
(c) *K_4.*

Proof. Clearly a graph with order at most 4 has an F-WORM coloring using only 2 colors. We already know that a proper 2-coloring of a bipartite graph is an F-WORM coloring. If G is a subgraph of $K_2 \vee mK_1$, color the vertices of the K_2 with one color and the other vertices a second color. This gives an F-WORM coloring.

Conversely, let G be a graph with an F-WORM coloring and suppose G is not bipartite. Then consider any vertex u with at least two neighbors. Since there is no monochromatic F, it must be that u has a different color to at least one of its neighbors, say v. Since there is no rainbow F, it follows that every other vertex has the same color as either u or v.

But since G is not bipartite, this means there exists an edge xy where x and y have the same color, say color 1. By the lack of monochromatic F it follows that every other vertex has the other color, say 2. If the vertices of color 2 form an independent set, then we have a subgraph of the join $K_2 \vee mK_1$ for some $m > 0$. Otherwise, by the same reasoning there are exactly two vertices of color 2 and we have a subgraph of K_4. ■

6. Extremal Questions

The classical Turán problem ask for the maximum number of edges of a graph with n vertices that does not contain some given subgraph H. This maximum is called the Turán number of H. Here we consider an analogue of the classical Turán problem: what is the maximum number of edges of a graph with n vertices if the graph admits a F-WORM coloring? We will let $wex(n, F)$ denote this maximum. We have the following result when $F = P_3$.

Theorem 20. *For* $n \geq 1$,

$$wex(n, P_3) = \begin{cases} \frac{n(n+2)}{4}, & \textit{if } n \textit{ is a multiple of } 4, \\ \frac{n^2+2n-4}{4}, & \textit{if } n \equiv 2 \pmod 4), \\ \frac{(n-1)(n+3)}{4}, & \textit{otherwise.} \end{cases}$$

Proof. Consider a graph G that has a P_3-WORM coloring. By Theorem 4, there is such a coloring using only two colors, say red and blue. Such a coloring is a P_3-WORM coloring if and only if there is no monochromatic P_3. It follows that the maximum number of edges in G is obtained by taking some complete bipartite graph and adding a maximum matching within each partite set.

When n is a multiple of 4, then the number of edges is obviously maximized when the two colors are used equally. When n is odd, the maximum is when the two colors are used as equally as possible. When n is even but not a multiple of 4, there are actually two extremal graphs: adding $n/2$ edges to $K_{n/2-1,n/2+1}$ or adding $2(n/2-1)/2$ edges to $K_{n/2,n/2}$. We omit the calculations. ∎

7. Conclusion

We have considered WORM colorings where the forbidden graph is P_3 and provided existence results and bounds on the maximum number of colors for several graph families. The next step would be to consider other forbidden graphs, such as K_3. Indeed a natural generalization is to consider sets of forbidden graphs, such as all cycles. Some results in this direction are given in [6].

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