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STRONG f-STAR FACTORS OF GRAPHS

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Abstract

Let G be a graph and $f : V(G) \to \{2, 3, \ldots\}$. A spanning subgraph F is called strong f-star of G if each component of F is a star whose center x satisfies $\deg_F(x) \leq f(x)$ and F is an induced subgraph of G. In this paper, we prove that G has a strong f-star factor if and only if $oddca(G - S) \leq \sum_{x\in S} f(x)$ for all $S \subset V(G)$, where $oddca(G)$ denotes the number of odd complete-cacti of G.

Keywords: f-star factor, strong f-star factor, complete-cactus, factor of graph.

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1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For a vertex set S of G, let $G-S$ denote the subgraph of G induced by $V(G) - S$. Let $Iso(G)$ and $iso(G)$ denote the set of isolated vertices and the number of isolated vertices of G, respectively. A graph G is called a *complete-cactus* if G is connected and every block of G is a complete graph. A complete-cactus is called an *odd complete-cactus* if all its blocks are complete graphs of odd order. Note that K_1 is an odd complete-cactus.

For a set $\mathcal{S}$ of connected graphs, a spanning subgraph F of a graph G is called an *$\mathcal{S}$-factor* of G if each component of F is isomorphic to an element of $\mathcal{S}$. A complete bipartite graph $K_{1,n}$ is called a *star*, and its vertex of degree n is called the *center*. For $K_{1,1}$, an arbitrarily chosen vertex is its center.

The following theorem was independently obtained by Las Vergnas [6] and by Amahashi and Kano [2].

Theorem 1 [2, 6]**.** *Let $n \geq 2$ be an integer. Then a graph G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $iso(G - S) \leq n|S|$ for all $S \subset V(G)$.*

Let G be a graph and let $f : V(G) \to \{2, 3, 4, \ldots\}$ be a function defined on $V(G)$. Then a spanning subgraph F is called an *f-star factor* of G if each component of F is a star and its center x satisfies $\deg_F(x) \leq f(x)$. The following theorem gives a criterion for a graph to have an f-star factor.

Theorem 2 [3]**.** *Let G be a graph and let $f : V(G) \to \{2, 3, \ldots\}$ be a function. Then G has an f-star factor if and only if $iso(G - S) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$.*

For a set $\mathcal{S}$ of connected graphs, a subgraph H of G is called a *strong $\mathcal{S}$-subgraph* if every component of H is isomorphic to an element of $\mathcal{S}$ and is an induced subgraph of G. A spanning strong $\mathcal{S}$-subgraph is called a *strong $\mathcal{S}$-factor*. A strong $\{K_{1,1}, K_{1,2}, \ldots\}$-factor is briefly called a *strong star factor*. Kelmans [7] and Saito and Watanabe [8] proved independently the following theorem.

Theorem 3 [7, 8]**.** *A connected graph G has a strong star factor if and only if G is not an odd complete-cactus.*

For a graph G, let $OddCa(G)$ denote the set of components of G that are odd complete-cacti, and let $oddca(G) = |OddCa(G)|$ denote the number of odd complete-cacti of G. Egawa, Kano and Kelmans [4] generalized the above theorem as follows by considering a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor.

Theorem 4 [4]**.** *Let $n \geq 2$ be an integer. Then a graph G has a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $oddca(G - S) \leq n|S|$ for all $S \subset V(G)$.*

A subgraph H is called a *strong f-star subgraph* of G if each component of H is a star, whose center x satisfies $\deg_H(x) \leq f(x)$, and H is an induced subgraph of G. A spanning f-star subgraph of G is called a *strong f-star factor* of G. Obviously, if $f(x) = n$ for all $x \in V(G)$, then a strong f-star factor of G is a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor. In this paper, we obtain the following result which is a generalization of Theorem 4.

Theorem 5. *Let G be a graph and let $f : V(G) \to \{2, 3, \ldots\}$ be a function. Then G has a strong f-star factor if and only if*

$$oddca(G - S) \leq \sum_{x \in S} f(x), \text{ for all } S \subset V(G). \tag{1}$$

A strong f-star subgraph H of a graph G is said to be *maximum* if G has no strong f-star subgraph H' such that $|H'| > |H|$. A formula for the order of a maximum strong f-star subgraph of a graph is easily obtained as a maximum matching, which is given in the following theorem.

Theorem 6. *Let G be a graph and let $f : V(G) \to \{2, 3, 4, \ldots\}$ be a function. Then the order of a maximum strong f-star subgraph H of G is given by*

$$|H| = |G| - \max_{X \subset V(G)} \left\{ oddca(G - X) - \sum_{x \in X} f(x) \right\}. \quad (2)$$

Finally, we consider a problem of covering a given vertex subset with a strong f-star subgraph. The condition for the existence of such a subgraph, which is given in the following theorem, is a natural extension of the criterion for the existence of a strong f-star factor.

Theorem 7. *Let G be a graph and let $f : V(G) \to \{2, 3, 4, \ldots\}$ be a function. Let W be a subset of $V(G)$. Then G has a strong f-star subgraph covering W if and only if*

$$oddca(G - S|W) \le \sum_{x \in S} f(x), \ \text{for all } S \subset V(G), \quad (3)$$

where $oddca(G - S|W)$ denotes the number of odd complete-cacti of $G - S$ contained in W.

2. Proof of the Results

We need some other notations. For two sets X and Y, $X \subset Y$ means that X is a proper subset of Y. Let G be a graph. For two vertices x and y of G, we write xy or yx for an edge joining x to y. For a vertex v of G, we denote by $N_G(v)$ the neighborhood of v. For a subset S of $V(G)$, we define $N_G(S) := \bigcup_{x \in S} N_G(x)$. For convenience, we briefly call a complete-cactus a cactus in the following proofs. Analogously, an odd complete-cactus is called an odd cactus. Every block of a cactus is a complete graph, and we call it an odd block or even block according to its order.

In order to prove Theorem 5, we need the following lemmas.

Lemma 8 [4]**.** (i) *Let G be an odd complete-cactus. Then for every vertex v of G, $G - v$ has a 1-factor.*

(ii) *An odd complete-cactus does not have a strong star factor.*

Lemma 9 [5]**.** *Let G be a bipartite graph with bipartition (A, B), and let $g, \ f : V(G) \to \mathbb{Z}$ be functions such that $g(x) \le f(x)$ for all $x \in V(G)$. Then G has a $(g, \ f)$-factor if and only if*

$$\gamma^*(X, Y) = \sum_{x \in X} f(x) + \sum_{x \in Y} (\deg_G(x) - g(x)) - e_G(X, \ Y) \ge 0,$$

and

$$\gamma^*(Y, X) = \sum_{x \in Y} f(x) + \sum_{x \in X} (\deg_G(x) - g(x)) - e_G(Y, \ X) \ge 0,$$

for all subsets $X \subseteq A$ and $Y \subseteq B$.

Lemma 10. *Let G be a bipartite graph with bipartition (A, B). Let $f : V(G) \to \{1, 2, 3, \ldots\}$ be a function such that $f(x) \geq 2$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$. Then G has a $(1, f)$-factor if and only if*

$$(4) \qquad \begin{aligned} |N_G(X)| \geq |X| \text{ for all } X \subseteq A, \text{ and} \\ \sum\nolimits_{x \in N_G(Y)} f(x) \geq |Y| \text{ for all } Y \subseteq B. \end{aligned}$$

Proof. If G has a $(1, f)$-factor F, then (4) follows from

$$|N_G(X)| \geq |N_F(X)| \geq |X| \text{ and } \sum\nolimits_{x \in N_G(Y)} f(x) \geq \sum\nolimits_{x \in N_F(Y)} f(x) \geq |Y|.$$

Conversely, assume that (4) holds. We may assume that G is connected, since otherwise each component satisfies (4) and has a $(1, f)$-factor by induction, and hence G itself has a $(1, f)$-factor. For any subsets $X \subseteq A$ and $Y \subseteq B$, it follows from (4) that

$$\begin{aligned} \gamma^*(X, Y) &= \sum\nolimits_{x \in X} f(x) + \sum\nolimits_{y \in Y} (\deg_{G-X}(y) - 1) \\ &\geq \sum\nolimits_{x \in X} f(x) - |S| \geq \sum\nolimits_{x \in N_G(S)} f(x) - |S| \geq 0, \end{aligned}$$

where $S = Iso(G - X) \cap Y \subseteq B$, $N_G(S) \subseteq X$.

$$\begin{aligned} \gamma^*(Y, X) &= \sum\nolimits_{y \in Y} f(y) + \sum\nolimits_{x \in X} (\deg_{G-Y}(x) - 1) \\ &\geq |Y| - |T| \geq |N_G(T)| - |T| \geq 0, \end{aligned}$$

where $T = Iso(G - Y) \cap X \subseteq A$, $N_G(T) \subseteq Y$.

Therefore by Lemma 9, G has the desired $(1, f)$-factor. ∎

***Proof of Theorem* 5.** Suppose that G has a strong f-star factor F. Let $\emptyset \neq S \subset V(G)$. Since every odd cactus D of $G - S$ does not have a strong f-star by Lemma 8, F has an edge joining D to S. It is obvious that for every vertex $s \in S$, F has at most $f(x)$ edges joining x to odd cacti in $G - S$. Hence $oddca(G - S) \leq \sum_{x \in S} f(x)$.

We shall prove the sufficiency of Theorem 5 by induction on $\sum_{x \in V(G)} f(x)$. We may assume that $|G| \geq 3$ and G is connected, since otherwise by applying the induction hypothesis to each component, we can obtain the desired strong star-factor of G. By taking $S = \emptyset$, it follows that G is not an odd cactus.

Obviously, $\sum_{x \in V(G)} f(x) \geq 2|G|$, since $f(x) \geq 2$ for all $x \in V(G)$. If $\sum_{x \in V(G)} f(x) = 2|G|$, then $f(x) = 2$ for all $x \in V(G)$. Thus the condition (1) becomes

$$oddca(G - S) \leq 2|S|, \text{ for all } S \subset V(G).$$

By Theorem 4, G has a strong $\{K_{1,1}, K_{1,2}\}$-factor, which is the desired strong f-factor of G. So we may assume that $\sum_{x\in V(G)} f(x) \geq 2|G| + 1$. Then there exists a vertex $w \in V(G)$ such that $f(w) \geq 3$.

Let us define the number β by

$$\beta = \min_{\emptyset\neq X\subset V(G)} \left\{\sum_{x\in X} f(x) - oddca(G - X)\right\}.$$

Then $\beta \geq 0$ by (1), and it follows from the definition of β that

$$oddca(G - Y) \leq \sum_{x\in Y} f(x) - \beta, \text{ for all } \emptyset \neq Y \subset V(G). \tag{5}$$

Take a maximal subset S of $V(G)$ such that

$$\sum_{x\in S} f(x) - oddca(G - S) = \beta. \tag{6}$$

Claim 1. $\beta = 0$.

Proof. Suppose that $\beta \geq 1$. Define $f^* : V(G) \to \{2, 3, 4, \ldots\}$ by

$$f^*(x) = \begin{cases} f(x) - 1 & \text{if } x = w, \\ f(x) & \text{otherwise.} \end{cases}$$

Let $\emptyset \neq X \subset V(G)$. Then we have

$$oddca(G - X) \leq \sum_{x\in X} f(x) - \beta \leq \sum_{x\in X} f(x) - 1 \leq \sum_{x\in X} f^*(x).$$

Hence, G has a strong star-factor F^* with respect to f^* by induction, which is also the strong f-star factor of G. □

Hereafter we assume $\beta = 0$.

Claim 2. *Every component of $G - S$ which is not an odd cactus has a strong f-star factor.*

Proof. Let D be a component of $G - S$ which is not an odd cactus, and let $\emptyset \neq X \subset V(D)$. Then by (5), we have

$$\begin{aligned} oddca(G - S) + oddca(D - X) &= oddca(G - S \cup X) \\ &\leq \sum_{x\in S\cup X} f(x) = \sum_{x\in S} f(x) + \sum_{x\in X} f(x). \end{aligned}$$

Thus $oddca(D - X) \leq \sum_{x\in X} f(x)$ by (6), which implies that D has a strong f-star factor by induction. □

We construct a bipartite graph B with bipartition $(S, OddCa(G-S))$ in which two vertices $x \in S$ and a component $C \in OddCa(G-S)$ are joined by an edge of B if and only if x is adjacent to C in G.

Claim 3. *For every $\emptyset \neq X \subseteq S$ and $\emptyset \neq Y \subseteq OddCa(G-S)$, it follows that $|N_B(X)| \geq |X|$ and $\sum_{x \in N_B(Y)} f(x) \geq |Y|$.*

Proof. Let $\emptyset \neq X \subseteq S$. By (5) and $\beta = 0$, we obtain

$$\begin{aligned}\sum_{x \in S-X} f(x) \geq oddca(G-(S-X)) &\geq oddca(G-S) - |N_B(X)| \\ &\geq \sum_{x \in S} f(x) - |N_B(X)|,\end{aligned}$$

which means $|N_B(X)| \geq \sum_{x \in X} f(x) \geq |X|$. Let $\emptyset \neq Y \subseteq OddCa(G-S)$. Then $N_B(Y) \subseteq S$, and by (1) we have

$$|Y| \leq oddca(G - N_B(Y)) \leq \sum_{x \in N_B(Y)} f(x).$$

Therefore Claim 3 holds. □

By Claim 3, B has a strong f-star factor H given in Lemma 10, which is a $(1, f)$-factor with minimal edge set, and every component of $OddCa(G-S)$ has degree one in H. Consequently, by Lemma 8(i) and Claim 2, we can obtain a strong f-star factor of G from H. ■

Proof of Theorem 6. Let $d = \max_{X \subset V(G)} \left\{ oddca(G-X) - \sum_{x \in X} f(x) \right\}$. Then $d \geq 0$ by considering the case $X = \emptyset$. Moreover, if $d = 0$, then (2) follows from Theorem 5. Hence we may assume $d \geq 1$. Let S be a subset of $V(G)$ such that

$$oddca(G-S) - \sum_{x \in S} f(x) = d.$$

Then by considering $\langle S \cup OddCa(G-S) \rangle_G$, which is the subgraph of G induced by $S \cup OddCa(G-S)$, we have that every strong f-star subgraph of G cannot cover at least $oddca(G-S) - \sum_{x \in S} f(x)$ odd cacti of $OddCa(G-S)$. Hence $|H| \leq |G| - d$, when H is a maximum strong f-star subgraph of G.

Next we prove the inverse inequality $|H| \geq |G| - d$ for a maximum strong f-star subgraph H of G. Add $2d$ new vertices $\{v_i, u_i : 1 \leq i \leq d\}$ together with d new edges $\{v_i u_i : 1 \leq i \leq d\}$ to G. Then join every v_i to every vertex of G by new edges. Denote the resulting graph by G^*, and define a function $f^* : V(G^*) \to \{2, 3, 4, \ldots\}$ by $f^*(u_i) = f^*(v_i) = 2$ for all $1 \leq i \leq d$, and $f^*(x) = f(x)$ for all $x \in V(G)$.

Let Y be a non-empty subset of $V(G^*)$. We may assume that Y contains no vertices of $\{u_1, \ldots, u_d\}$, when we estimate $oddca(G^* - Y)$. If $|\{v_1, \ldots, v_d\} \cap Y| < d$, then

$$oddca(G^* - Y) \leq |Y \cap \{v_1, \ldots, v_d\}| + 1 \leq \sum_{x \in Y} f(x).$$

If $\{v_1, \ldots, v_d\} \subset Y$, then all the vertices of $\{u_1, \ldots, u_d\}$ become isolated vertices of $G^* - Y$, and so by the definition of d, we obtain

$$\begin{aligned} oddca(G^* - Y) &\le oddca(G - (Y \cap V(G))) + d \\ &\le \sum\nolimits_{x \in Y \cap V(G)} f(x) + d + d = \sum\nolimits_{x \in Y} f(x). \end{aligned}$$

Hence by Theorem 5, G^* has a strong f-star factor F^*. Then $H = F^* - \{u_i, v_i : 1 \le i \le d\}$ is a strong f-star subgraph of G, which covers at least $|G| - d$ vertices. Hence $|H| \ge |G| - d$. Consequently, the theorem is proved. ■

Proof of Theorem 7. First suppose that G has a strong f-star subgraph F covering W. Then for every odd cactus C of $G - S$ contained in W, F has at least one edge joining C to S. Hence

$$oddca(G - S|W) \le \sum\nolimits_{x \in S} \deg_F(x).$$

Next we assume that (3) holds. We may assume that G is connected, since otherwise, by applying the induction hypothesis to each component of G, we can obtain the desired factor of G. By Theorem 5, we may assume that W is a proper subset of $V(G)$, and so $V(G) - W \neq \emptyset$. Let $n = |V(G) - W|$. We construct a new graph H from G by adding two new vertices w_1, w_2 and by joining $w_i (i = 1, 2)$ to every vertex in $V(G) - W$. Define $f^* : V(H) \to \{2, 3, \ldots\}$ by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G), \\ \max\{2, n\} & \text{if } x \in \{w_1, w_2\}. \end{cases}$$

It is easy to see that G has a strong f-star subgraph covering W if and only if H has a strong f^*-star factor.

Let $X \subset V(H)$. If $w_1, w_2 \in X$, let $S = X - \{w_1, w_2\}$, then

$$oddca(H - X) \le oddca(G - S|W) + n \le \sum\nolimits_{x \in S} f(x) + n < \sum\nolimits_{x \in X} f^*(x).$$

If $w_1 \in X$ and $w_2 \notin X$, let $S = X - \{w_1\}$, then

$$oddca(H - X) \le oddca(G - S|W) + 1 \le \sum\nolimits_{x \in S} f(x) + 1 < \sum\nolimits_{x \in X} f^*(x).$$

If $w_1, w_2 \notin X$, then

$$oddca(H - X) = oddca(G - X|W) \le \sum\nolimits_{x \in X} f(x).$$

Therefore, by Theorem 5, H has a strong f^*-star factor, and thus G has the desired strong f-star subgraph which covers W. ■

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