# STRONG $f$-STAR FACTORS OF GRAPHS 

ZHENG YAN<br>Yangtze University, Jingzhou<br>Hubei, China<br>e-mail: yanzhenghubei@163.com


#### Abstract

Let $G$ be a graph and $f: V(G) \rightarrow\{2,3, \ldots\}$. A spanning subgraph $F$ is called strong $f$-star of $G$ if each component of $F$ is a star whose center $x$ satisfies $\operatorname{deg}_{F}(x) \leq f(x)$ and $F$ is an induced subgraph of $G$. In this paper, we prove that $G$ has a strong $f$-star factor if and only if oddca $(G-S) \leq$ $\sum_{x \in S} f(x)$ for all $S \subset V(G)$, where oddca $(G)$ denotes the number of odd complete-cacti of $G$.


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## 1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. We write $|G|$ for the order of $G$ (i.e., $|G|=|V(G)|)$. For a vertex $v$ of $G$, we denote by $\operatorname{deg}_{G}(v)$ the degree of $v$ in $G$. For a vertex set $S$ of $G$, let $G-S$ denote the subgraph of $G$ induced by $V(G)-S$. Let $I s o(G)$ and $i s o(G)$ denote the set of isolated vertices and the number of isolated vertices of $G$, respectively. A graph $G$ is called a complete-cactus if $G$ is connected and every block of $G$ is a complete graph. A complete-cactus is called an odd complete-cactus if all its blocks are complete graphs of odd order. Note that $K_{1}$ is an odd complete-cactus.

For a set $\mathcal{S}$ of connected graphs, a spanning subgraph $F$ of a graph $G$ is called an $\mathcal{S}$-factor of $G$ if each component of $F$ is isomorphic to an element of $\mathcal{S}$. A complete bipartite graph $K_{1, n}$ is called a star, and its vertex of degree $n$ is called the center. For $K_{1,1}$, an arbitrarily chosen vertex is its center.

The following theorem was independently obtained by Las Vergnas [6] and by Amahashi and Kano [2].

Theorem $1[2,6]$. Let $n \geq 2$ be an integer. Then a graph $G$ has a $\left\{K_{1,1}, K_{1,2}, \ldots\right.$, $\left.K_{1, n}\right\}$-factor if and only if iso $(G-S) \leq n|S|$ for all $S \subset V(G)$.

Let $G$ be a graph and let $f: V(G) \rightarrow\{2,3,4, \ldots\}$ be a function defined on $V(G)$. Then a spanning subgraph $F$ is called an $f$-star factor of $G$ if each component of $F$ is a star and its center $x$ satisfies $\operatorname{deg}_{F}(x) \leq f(x)$. The following theorem gives a criterion for a graph to have an $f$-star factor.

Theorem 2 [3]. Let $G$ be a graph and let $f: V(G) \rightarrow\{2,3, \ldots\}$ be a function. Then $G$ has an $f$-star factor if and only if $i s o(G-S) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$.

For a set $\mathcal{S}$ of connected graphs, a subgraph $H$ of $G$ is called a strong $\mathcal{S}$ subgraph if every component of $H$ is isomorphic to an element of $\mathcal{S}$ and is an induced subgraph of $G$. A spanning strong $\mathcal{S}$-subgraph is called a strong $\mathcal{S}$-factor. A strong $\left\{K_{1,1}, K_{1,2}, \ldots\right\}$-factor is briefly called a strong star factor. Kelmans [7] and Saito and Watanabe [8] proved independently the following theorem.

Theorem $3[7,8]$. A connected graph $G$ has a strong star factor if and only if $G$ is not an odd complete-cactus.
 odd complete-cacti, and let $\operatorname{oddca}(G)=|O d d C a(G)|$ denote the number of odd complete-cacti of $G$. Egawa, Kano and Kelmans [4] generalized the above theorem as follows by considering a strong $\left\{K_{1,1}, K_{1,2}, \ldots, K_{1, n}\right\}$-factor.

Theorem 4 [4]. Let $n \geq 2$ be an integer. Then a graph $G$ has a strong $\left\{K_{1,1}, K_{1,2}\right.$, $\left.\ldots, K_{1, n}\right\}$-factor if and only if oddca $(G-S) \leq n|S|$ for all $S \subset V(G)$.

A subgraph $H$ is called a strong $f$-star subgraph of $G$ if each component of $H$ is a star, whose center $x$ satisfies $\operatorname{deg}_{H}(x) \leq f(x)$, and $H$ is an induced subgraph of $G$. A spanning $f$-star subgraph of $G$ is called a strong $f$-star factor of $G$. Obviously, if $f(x)=n$ for all $x \in V(G)$, then a strong $f$-star factor of $G$ is a strong $\left\{K_{1,1}, K_{1,2}, \ldots, K_{1, n}\right\}$-factor. In this paper, we obtain the following result which is a generalization of Theorem 4.

Theorem 5. Let $G$ be a graph and let $f: V(G) \rightarrow\{2,3, \ldots\}$ be a function. Then $G$ has a strong $f$-star factor if and only if

$$
\begin{equation*}
\text { oddca }(G-S) \leq \sum_{x \in S} f(x), \text { for all } S \subset V(G) \tag{1}
\end{equation*}
$$

A strong $f$-star subgraph $H$ of a graph $G$ is said to be maximum if $G$ has no strong $f$-star subgraph $H^{\prime}$ such that $\left|H^{\prime}\right|>|H|$. A formula for the order of a maximum strong $f$-star subgraph of a graph is easily obtained as a maximum matching, which is given in the following theorem.

Theorem 6. Let $G$ be a graph and let $f: V(G) \rightarrow\{2,3,4, \ldots\}$ be a function. Then the order of a maximum strong $f$-star subgraph $H$ of $G$ is given by

$$
\begin{equation*}
|H|=|G|-\max _{X \subset V(G)}\left\{\operatorname{oddca}(G-X)-\sum_{x \in X} f(x)\right\} \tag{2}
\end{equation*}
$$

Finally, we consider a problem of covering a given vertex subset with a strong $f$-star subgraph. The condition for the existence of such a subgraph, which is given in the following theorem, is a natural extension of the criterion for the existence of a strong $f$-star factor.

Theorem 7. Let $G$ be a graph and let $f: V(G) \rightarrow\{2,3,4, \ldots\}$ be a function. Let $W$ be a subset of $V(G)$. Then $G$ has a strong $f$-star subgraph covering $W$ if and only if

$$
\begin{equation*}
\text { oddca }(G-S \mid W) \leq \sum_{x \in S} f(x), \text { for all } S \subset V(G) \tag{3}
\end{equation*}
$$

where oddca $(G-S \mid W)$ denotes the number of odd complete-cacti of $G-S$ contained in $W$.

## 2. Proof of the Results

We need some other notations. For two sets $X$ and $Y, X \subset Y$ means that $X$ is a proper subset of $Y$. Let $G$ be a graph. For two vertices $x$ and $y$ of $G$, we write $x y$ or $y x$ for an edge joining $x$ to $y$. For a vertex $v$ of $G$, we denote by $N_{G}(v)$ the neighborhood of $v$. For a subset $S$ of $V(G)$, we define $N_{G}(S):=\bigcup_{x \in S} N_{G}(x)$. For convenience, we briefly call a complete-cactus a cactus in the following proofs. Analogously, an odd complete-cactus is called an odd cactus. Every block of a cactus is a complete graph, and we call it an odd block or even block according to its order.

In order to prove Theorem 5, we need the following lemmas.
Lemma 8 [4]. (i) Let $G$ be an odd complete-cactus. Then for every vertex $v$ of $G, G-v$ has a 1-factor.
(ii) An odd complete-cactus does not have a strong star factor.

Lemma 9 [5]. Let $G$ be a bipartite graph with bipartition $(A, B)$, and let $g$, $f$ : $V(G) \rightarrow \mathbb{Z}$ be functions such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then $G$ has a $(g, f)$-factor if and only if

$$
\gamma^{*}(X, Y)=\sum_{x \in X} f(x)+\sum_{x \in Y}\left(\operatorname{deg}_{G}(x)-g(x)\right)-e_{G}(X, Y) \geq 0
$$

and

$$
\gamma^{*}(Y, X)=\sum_{x \in Y} f(x)+\sum_{x \in X}\left(\operatorname{deg}_{G}(x)-g(x)\right)-e_{G}(Y, X) \geq 0
$$

for all subsets $X \subseteq A$ and $Y \subseteq B$.

Lemma 10. Let $G$ be a bipartite graph with bipartition $(A, B)$. Let $f: V(G) \rightarrow$ $\{1,2,3, \ldots\}$ be a function such that $f(x) \geq 2$ for all $x \in A$, and $f(x)=1$ for all $x \in B$. Then $G$ has a $(1, f)$-factor if and only if

$$
\begin{align*}
\left|N_{G}(X)\right| & \geq|X| \text { for all } X \subseteq A, \text { and } \\
\sum_{x \in N_{G}(Y)} f(x) & \geq|Y| \text { for all } Y \subseteq B . \tag{4}
\end{align*}
$$

Proof. If $G$ has a $(1, f)$-factor $F$, then (4) follows from

$$
\left|N_{G}(X)\right| \geq\left|N_{F}(X)\right| \geq|X| \text { and } \sum_{x \in N_{G}(Y)} f(x) \geq \sum_{x \in N_{F}(Y)} f(x) \geq|Y|
$$

Conversely, assume that (4) holds. We may assume that $G$ is connected, since otherwise each component satisfies (4) and has a $(1, f)$-factor by induction, and hence $G$ itself has a $(1, f)$-factor. For any subsets $X \subseteq A$ and $Y \subseteq B$, it follows from (4) that

$$
\begin{aligned}
\gamma^{*}(X, Y) & =\sum_{x \in X} f(x)+\sum_{y \in Y}\left(\operatorname{deg}_{G-X}(y)-1\right) \\
& \geq \sum_{x \in X} f(x)-|S| \geq \sum_{x \in N_{G}(S)} f(x)-|S| \geq 0
\end{aligned}
$$

where $S=I$ so $(G-X) \cap Y \subseteq B, N_{G}(S) \subseteq X$.

$$
\begin{aligned}
\gamma^{*}(Y, X) & =\sum_{y \in Y} f(y)+\sum_{x \in X}\left(\operatorname{deg}_{G-Y}(x)-1\right) \\
& \geq|Y|-|T| \geq\left|N_{G}(T)\right|-|T| \geq 0
\end{aligned}
$$

where $T=I s o(G-Y) \cap X \subseteq A, N_{G}(T) \subseteq Y$.
Therefore by Lemma $9, G$ has the desired $(1, f)$-factor.
Proof of Theorem 5. Suppose that $G$ has a strong $f$-star factor $F$. Let $\emptyset \neq$ $S \subset V(G)$. Since every odd cactus $D$ of $G-S$ does not have a strong $f$ star by Lemma $8, F$ has an edge joining $D$ to $S$. It is obvious that for every vertex $s \in S, F$ has at most $f(x)$ edges joining $x$ to odd cacti in $G-S$. Hence oddca $(G-S) \leq \sum_{x \in S} f(x)$.

We shall prove the sufficiency of Theorem 5 by induction on $\sum_{x \in V(G)} f(x)$. We may assume that $|G| \geq 3$ and $G$ is connected, since otherwise by applying the induction hypothesis to each component, we can obtain the desired strong star-factor of $G$. By taking $S=\emptyset$, it follows that $G$ is not an odd cactus.

Obviously, $\sum_{x \in V(G)} f(x) \geq 2|G|$, since $f(x) \geq 2$ for all $x \in V(G)$. If $\sum_{x \in V(G)} f(x)=2|G|$, then $f(x)=2$ for all $x \in V(G)$. Thus the condition (1) becomes

$$
\text { oddca }(G-S) \leq 2|S|, \text { for all } S \subset V(G)
$$

By Theorem 4, $G$ has a strong $\left\{K_{1,1}, K_{1,2}\right\}$-factor, which is the desired strong $f$-factor of $G$. So we may assume that $\sum_{x \in V(G)} f(x) \geq 2|G|+1$. Then there exists a vertex $w \in V(G)$ such that $f(w) \geq 3$.

Let us define the number $\beta$ by

$$
\beta=\min _{\emptyset \neq X \subset V(G)}\left\{\sum_{x \in X} f(x)-o d d c a(G-X)\right\}
$$

Then $\beta \geq 0$ by (1), and it follows from the definition of $\beta$ that

$$
\begin{equation*}
o d d c a(G-Y) \leq \sum_{x \in Y} f(x)-\beta, \text { for all } \emptyset \neq Y \subset V(G) \tag{5}
\end{equation*}
$$

Take a maximal subset $S$ of $V(G)$ such that

$$
\begin{equation*}
\sum_{x \in S} f(x)-o d d c a(G-S)=\beta \tag{6}
\end{equation*}
$$

Claim 1. $\beta=0$.
Proof. Suppose that $\beta \geq 1$. Define $f^{*}: V(G) \rightarrow\{2,3,4, \ldots\}$ by

$$
f^{*}(x)= \begin{cases}f(x)-1 & \text { if } x=w \\ f(x) & \text { otherwise }\end{cases}
$$

Let $\emptyset \neq X \subset V(G)$. Then we have

$$
o d d c a(G-X) \leq \sum_{x \in X} f(x)-\beta \leq \sum_{x \in X} f(x)-1 \leq \sum_{x \in X} f^{*}(x)
$$

Hence, $G$ has a strong star-factor $F^{*}$ with respect to $f^{*}$ by induction, which is also the strong $f$-star factor of $G$.

Hereafter we assume $\beta=0$.
Claim 2. Every component of $G-S$ which is not an odd cactus has a strong $f$-star factor.

Proof. Let $D$ be a component of $G-S$ which is not an odd cactus, and let $\emptyset \neq X \subset V(D)$. Then by (5), we have

$$
\begin{aligned}
& o d d c a(G-S)+o d d c a(D-X)=\operatorname{oddca}(G-S \cup X) \\
& \quad \leq \sum_{x \in S \cup X} f(x)=\sum_{x \in S} f(x)+\sum_{x \in X} f(x)
\end{aligned}
$$

Thus oddca $(D-X) \leq \sum_{x \in X} f(x)$ by (6), which implies that $D$ has a strong $f$-star factor by induction.

We construct a bipartite graph $B$ with bipartition ( $S, \operatorname{OddCa}(G-S)$ ) in which two vertices $x \in S$ and a component $C \in \operatorname{OddCa}(G-S)$ are joined by an edge of $B$ if and only if $x$ is adjacent to $C$ in $G$.

Claim 3. For every $\emptyset \neq X \subseteq S$ and $\emptyset \neq Y \subseteq O d d C a(G-S)$, it follows that $\left|N_{B}(X)\right| \geq|X|$ and $\sum_{x \in N_{B}(Y)} f(x) \geq|Y|$.
Proof. Let $\emptyset \neq X \subseteq S$. By (5) and $\beta=0$, we obtain

$$
\begin{aligned}
\sum_{x \in S-X} f(x) & \geq \operatorname{oddca}(G-(S-X)) \geq \operatorname{oddca}(G-S)-\left|N_{B}(X)\right| \\
& \geq \sum_{x \in S} f(x)-\left|N_{B}(X)\right|,
\end{aligned}
$$

which means $\left|N_{B}(X)\right| \geq \sum_{x \in X} f(x) \geq|X|$. Let $\emptyset \neq Y \subseteq O d d C a(G-S)$. Then $N_{B}(Y) \subseteq S$, and by (1) we have

$$
|Y| \leq \operatorname{oddca}\left(G-N_{B}(Y)\right) \leq \sum_{x \in N_{B}(Y)} f(x) .
$$

Therefore Claim 3 holds.
By Claim 3, $B$ has a strong $f$-star factor $H$ given in Lemma 10 , which is a $(1, f)$-factor with minimal edge set, and every component of $\operatorname{OddCa(G-S)\text {has}}$ degree one in $H$. Consequently, by Lemma 8(i) and Claim 2, we can obtain a strong $f$-star factor of $G$ from $H$.

Proof of Theorem 6. Let $d=\max _{X \subset V(G)}\left\{\right.$ oddca $\left.(G-X)-\sum_{x \in X} f(x)\right\}$.
Then $d \geq 0$ by considering the case $X=\emptyset$. Moreover, if $d=0$, then (2) follows from Theorem 5. Hence we may assume $d \geq 1$. Let $S$ be a subset of $V(G)$ such that

$$
\operatorname{oddca}(G-S)-\sum_{x \in S} f(x)=d .
$$

Then by considering $\langle S \cup O d d C a(G-S)\rangle_{G}$, which is the subgraph of $G$ induced by $S \cup O d d C a(G-S)$, we have that every strong $f$-star subgraph of $G$ cannot cover at least oddca $(G-S)-\sum_{x \in S} f(x)$ odd cacti of $\operatorname{OddCa}(G-S)$. Hence $|H| \leq|G|-d$, when $H$ is a maximum strong $f$-star subgraph of $G$.

Next we prove the inverse inequality $|H| \geq|G|-d$ for a maximum strong $f$-star subgraph $H$ of $G$. Add $2 d$ new vertices $\left\{v_{i}, u_{i}: 1 \leq i \leq d\right\}$ together with $d$ new edges $\left\{v_{i} u_{i}: 1 \leq i \leq d\right\}$ to $G$. Then join every $v_{i}$ to every vertex of $G$ by new edges. Denote the resulting graph by $G^{*}$, and define a function $f^{*}: V\left(G^{*}\right) \rightarrow\{2,3,4, \ldots\}$ by $f^{*}\left(u_{i}\right)=f^{*}\left(v_{i}\right)=2$ for all $1 \leq i \leq d$, and $f^{*}(x)=f(x)$ for all $x \in V(G)$.

Let $Y$ be a non-empty subset of $V\left(G^{*}\right)$. We may assume that $Y$ contains no vertices of $\left\{u_{1}, \ldots, u_{d}\right\}$, when we estimate $o d d c a\left(G^{*}-Y\right)$. If $\left|\left\{v_{1}, \ldots, v_{d}\right\} \cap Y\right|<d$, then

$$
\text { oddca }\left(G^{*}-Y\right) \leq\left|Y \cap\left\{v_{1}, \ldots, v_{d}\right\}\right|+1 \leq \sum_{x \in Y} f(x) .
$$

If $\left\{v_{1}, \ldots, v_{d}\right\} \subset Y$, then all the vertices of $\left\{u_{1}, \ldots, u_{d}\right\}$ become isolated vertices of $G^{*}-Y$, and so by the definition of $d$, we obtain

$$
\begin{aligned}
\operatorname{oddca}\left(G^{*}-Y\right) & \leq \operatorname{oddca}(G-(Y \cap V(G)))+d \\
& \leq \sum_{x \in Y \cap V(G)} f(x)+d+d=\sum_{x \in Y} f(x) .
\end{aligned}
$$

Hence by Theorem 5, $G^{*}$ has a strong $f$-star factor $F^{*}$. Then $H=F^{*}-\left\{u_{i}, v_{i}\right.$ : $1 \leq i \leq d\}$ is a strong $f$-star subgraph of $G$, which covers at least $|G|-d$ vertices. Hence $|H| \geq|G|-d$. Consequently, the theorem is proved.

Proof of Theorem 7. First suppose that $G$ has a strong $f$-star subgraph $F$ covering $W$. Then for every odd cactus $C$ of $G-S$ contained in $W, F$ has at least one edge joining $C$ to $S$. Hence

$$
\operatorname{oddca}(G-S \mid W) \leq \sum_{x \in S} \operatorname{deg}_{F}(x)
$$

Next we assume that (3) holds. We may assume that $G$ is connected, since otherwise, by applying the induction hypothesis to each component of $G$, we can obtain the desired factor of $G$. By Theorem 5, we may assume that $W$ is a proper subset of $V(G)$, and so $V(G)-W \neq \emptyset$. Let $n=|V(G)-W|$. We construct a new graph $H$ from $G$ by adding two new vertices $w_{1}, w_{2}$ and by joining $w_{i}(i=1,2)$ to every vertex in $V(G)-W$. Define $f^{*}: V(H) \rightarrow\{2,3, \ldots\}$ by

$$
f^{*}(x)= \begin{cases}f(x) & \text { if } x \in V(G), \\ \max \{2, n\} & \text { if } x \in\left\{w_{1}, w_{2}\right\} .\end{cases}
$$

It is easy to see that $G$ has a strong $f$-star subgraph covering $W$ if and only if $H$ has a strong $f^{*}$-star factor.

Let $X \subset V(H)$. If $w_{1}, w_{2} \in X$, let $S=X-\left\{w_{1}, w_{2}\right\}$, then

$$
\operatorname{oddca}(H-X) \leq \operatorname{oddca}(G-S \mid W)+n \leq \sum_{x \in S} f(x)+n<\sum_{x \in X} f^{*}(x)
$$

If $w_{1} \in X$ and $w_{2} \notin X$, let $S=X-\left\{w_{1}\right\}$, then

$$
\operatorname{oddca}(H-X) \leq \operatorname{oddca}(G-S \mid W)+1 \leq \sum_{x \in S} f(x)+1<\sum_{x \in X} f^{*}(x) .
$$

If $w_{1}, w_{2} \notin X$, then

$$
\operatorname{oddca}(H-X)=\operatorname{oddca}(G-X \mid W) \leq \sum_{x \in X} f(x)
$$

Therefore, by Theorem 5, $H$ has a strong $f^{*}$-star factor, and thus $G$ has the desired strong $f$-star subgraph which covers $W$.

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