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GENERALIZED FRACTIONAL AND CIRCULAR TOTAL COLORINGS OF GRAPHS [1]

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Abstract

Let $\mathcal{P}$ and $\mathcal{Q}$ be additive and hereditary graph properties, $r, s \in \mathbb{N}$, $r \geq s$, and $[\mathbb{Z}_r]^s$ be the set of all s-element subsets of $\mathbb{Z}_r$. An (r, s)-*fractional*

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$(\mathcal{P}, \mathcal{Q})$-*total coloring* of G is an assignment $h : V(G) \cup E(G) \to [\mathbb{Z}_r]^s$ such that for each $i \in \mathbb{Z}_r$ the following holds: the vertices of G whose color sets contain color i induce a subgraph of G with property $\mathcal{P}$, edges with color sets containing color i induce a subgraph of G with property $\mathcal{Q}$, and the color sets of incident vertices and edges are disjoint. If each vertex and edge of G is colored with a set of s consecutive elements of $\mathbb{Z}_r$ we obtain an (r, s)-*circular* $(\mathcal{P}, \mathcal{Q})$-*total coloring* of G. In this paper we present basic results on (r, s)-fractional/circular $(\mathcal{P}, \mathcal{Q})$-total colorings. We introduce the fractional and circular $(\mathcal{P}, \mathcal{Q})$-total chromatic number of a graph and we determine this number for complete graphs and some classes of additive and hereditary properties.

Keywords: graph property, $(\mathcal{P}, \mathcal{Q})$-total coloring, circular coloring, fractional coloring, fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number, circular $(\mathcal{P}, \mathcal{Q})$-total chromatic number.

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1. Introduction

In this paper we study graph invariants which combine different types of graph colorings, namely generalized $\mathcal{P}$-colorings, fractional or circular colorings, and total colorings of graphs.

Let $r, s \in \mathbb{N}$, $r \geq s$. Throughout this paper we consider simple, finite and undirected graphs. Let $\mathcal{I}$ denote the set of these graphs. For simplifying the notation we will write $a \in \mathbb{Z}_r$ instead of residue class $a_r \in \mathbb{Z}_r$, if there is no confusion. The set $\{a, a+1, \ldots, b\}$ (elements reduced modulo r) of consecutive elements of $\mathbb{Z}_r = \{0, 1, \ldots, r-1\}$ will be denoted by $[a, b]$, the same notation $[a, b]$ is also used for consecutive integers. We will denote by $[\mathbb{Z}_r]^s$ the set of all s-element subsets of $\mathbb{Z}_r$. Consider an assignment $h : V(G) \cup E(G) \to [\mathbb{Z}_r]^s$ for the graph G. Then the subgraph of G induced by the set $V_{i,h} = \{v \in V(G) : i \in h(v)\}$ or induced by the set $E_{i,h} = \{e \in E(G) : i \in h(e)\}$ is denoted by $G[V_{i,h}]$ or $G[E_{i,h}]$, respectively.

A *graph property* $\mathcal{P}$ is any non-empty isomorphism closed subset of $\mathcal{I}$. The set of graphs without edges is a property denoted by $\mathcal{O}$. A property $\mathcal{P}$ of graphs is called *hereditary* if it is closed under taking subgraphs and *additive* if it is closed under disjoint union of graphs. All graph properties considered in this paper are both hereditary and additive. The *completeness* of a hereditary graph property $\mathcal{P}$ is the number $c(\mathcal{P}) = \max\{k : K_{k+1} \in \mathcal{P}\}$. Obviously, for additive properties $\mathcal{P}$ it holds $c(\mathcal{P}) = 0$ if and only if $\mathcal{P} = \mathcal{O}$. We list several well-known hereditary and additive properties with completeness k and we use the following notations of graph properties introduced in [3].

$\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\}$,
$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\}$,
$\mathcal{D}_k = \{G \in \mathcal{I} : \delta(H) \leq k \text{ for each } H \subseteq G\}$,
$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ contains no } K_{k+2}\}$.

A proper graph coloring requires that for each color i the corresponding color class is an independent set of vertices, i.e., that the subgraph induced by vertices with color i has property $\mathcal{O}$. By using the class of hereditary properties there is a natural generalization of proper colorings. We obtain a *$\mathcal{P}$-coloring* by replacing the property $\mathcal{O}$ in the definition of a proper coloring by any other hereditary graph property $\mathcal{P}$. The concept of $\mathcal{P}$-colorings was introduced by several authors (see e.g. [8]) and investigated in many papers (see [3, 4, 12]).

$(\mathcal{P}, \mathcal{Q})$-total k-colorings were introduced in [2] as colorings of the elements (vertices and edges) of G such that for each color $i \in [0, k-1]$ the set of all vertices of color i induce a subgraph with property $\mathcal{P}$, the set of all edges of color i induce a subgraph with property $\mathcal{Q}$, and incident vertices and edges are colored differently. The *$(\mathcal{P}, \mathcal{Q})$-total chromatic number* of G, denoted by $\chi''_{\mathcal{P},\mathcal{Q}}(G)$, is the minimum number k of colors of a $(\mathcal{P}, \mathcal{Q})$-total k-coloring of G.

Other generalizations of total colorings are fractional or circular total colorings. We describe the definitions of these two colorings simultaneously: Let r, s be positive integers with $r \geq s$. An *(r, s)-fractional/circular total coloring* of G is an assignment of s-element subsets of arbitrary/consecutive elements of $\mathbb{Z}_r$ to the vertices and edges of G such that every two adjacent or incident elements of $V(G) \cup E(G)$ are colored with disjoint color sets. The *fractional total chromatic number* of G, denoted by $\chi''_f(G)$, is defined as

$$\chi''_f(G) = \inf\left\{\frac{r}{s} : G \text{ has an } (r,s)\text{-fractional total coloring}\right\}$$

and the *circular total chromatic number* of G, denoted by $\chi''_c(G)$, is defined as

$$\chi''_c(G) = \inf\left\{\frac{r}{s} : G \text{ has an } (r,s)\text{-circular total coloring}\right\}.$$

Both $\chi''_f(G)$ and $\chi''_c(G)$ are rational. Moreover, in the definition of χ''_f and χ''_c we can replace the infimum by the minimum (see, e.g., [13] p. 4, 30, [7]). For more information and details we refer the reader to [7, 10, 13].

Let $r, s \in \mathbb{N}$, $r \geq s$, and $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be two additive and hereditary graph properties. An *(r, s)-fractional/circular $(\mathcal{P}, \mathcal{Q})$-total coloring* of a graph G is a coloring of the vertices and edges of G by s-elements subsets of arbitrary/consecutive elements of $\mathbb{Z}_r$ (reduced modulo r) such that for each color i, $0 \leq i \leq r-1$,the set of vertices colored by subsets containing i induce a subgraph of G with property $\mathcal{P}$, the set of edges colored by subsets containing i induce a subgraph of G with property $\mathcal{Q}$, and color sets of incident vertices and edges are disjoint. The *fractional* $(\mathcal{P},\mathcal{Q})$*-total chromatic number* of G is defined as

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \inf\left\{\frac{r}{s} : G \text{ has an } (r,s)\text{-fractional } (\mathcal{P},\mathcal{Q})\text{-total coloring}\right\}$$

and the *circular* $(\mathcal{P},\mathcal{Q})$*-total chromatic number* of G as

$$\chi''_{c,\mathcal{P},\mathcal{Q}}(G) = \inf\left\{\frac{r}{s} : G \text{ has an } (r,s)\text{-circular } (\mathcal{P},\mathcal{Q})\text{-total coloring}\right\}.$$

If $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{O}_1$, then $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \chi''_f(G)$ and $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) = \chi''_c(G)$ by the definitions. Therefore, (r,s)-fractional/circular $(\mathcal{P},\mathcal{Q})$-total colorings are generalizations of fractional/circular total colorings.

Borowiecki and Mihók [3] showed that the set of all additive and hereditary properties ordered by inclusion is a complete distributive lattice $(\mathbb{L}^a,\subseteq)$ with the least element $\mathcal{O}$ and the greatest element $\mathcal{I}$. Moreover, the set of properties $\mathcal{P} \in \mathbb{L}^a$ with $c(\mathcal{P}) = k$, $k \in \mathbb{N}$, with respect to $\subseteq$ is a complete distributive lattice $(\mathbb{L}^a_k,\subseteq)$. Its least element is $\mathcal{O}_k$ and the greatest $\mathcal{I}_k$. Note that $\mathcal{O}_k \subseteq \mathcal{S}_k \subset \mathcal{D}_k \subset \mathcal{I}_k$. As we will prove in Section 3 it holds that

$$\chi''_{c,\mathcal{P},\mathcal{Q}}(K_{\omega(G)}) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(K_{|V(G)|}) \text{ and}$$
$$\chi''_{f,\mathcal{P},\mathcal{Q}}(K_{\omega(G)}) \leq \chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{f,\mathcal{P},\mathcal{Q}}(K_{|V(G)|})$$

for arbitrary graphs G with clique number $\omega(G)$ and order $|V(G)|$. Therefore, it is an interesting task to determine the fractional and the circular $(\mathcal{P},\mathcal{Q})$-total chromatic number of complete graphs. While the total fractional and the total circular chromatic number of a complete graph are equal to its total chromatic number, it is very difficult to determine these values in general. For example, $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(K_n) = \lfloor n/2\rfloor + 2$, see [2], and we will prove in Section 4 that $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \chi''_{c,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n+1)}{2(n-1)}$ for odd n.

The composition of this paper is as follows: In the second section we present equivalent definitions of the fractional $(\mathcal{P},\mathcal{Q})$-total chromatic number of a graph. Basic properties of (r,s)-fractional/circular $(\mathcal{P},\mathcal{Q})$-total colorings and some fractional/circular $(\mathcal{P},\mathcal{Q})$-total chromatic numbers are determined in the third section. The fractional/circular $(\mathcal{P},\mathcal{Q})$-total chromatic numbers of complete graphs for selected graph properties are studied in Section 4.

2. Equivalent Definitions for $\chi''_{f,\mathcal{P},\mathcal{Q}}$

Fractional $(\mathcal{P},\mathcal{Q})$-total colorings may be viewed in several ways. We present the following equivalent definitions.

Let $G = (V(G),E(G))$ be a graph and N a subset of $V(G) \cup E(G)$. Let us denote $N_V = N \cap V(G)$ and $N_E = N \cap E(G)$. We shall call the set N a

$(\mathcal{P},\mathcal{Q})$-*total independent set* of G if the subgraph $G[N_V]$ induced by N_V has the property $\mathcal{P}$, the subgraph $G[N_E]$ induced by N_E has the property $\mathcal{Q}$, and no vertex of N_V is an end-vertex of an edge in N_E.

Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be two additive hereditary properties and $\Gamma(G)$ be the family of all $(\mathcal{P},\mathcal{Q})$-total independent sets of a graph G. A fractional $(\mathcal{P},\mathcal{Q})$-total coloring is an assignment $\varphi : \Gamma(G) \to \langle 0,1\rangle$ such that

$$\sum_{N\in\Gamma(G):N\ni x} \varphi(N) \geq 1 \quad \text{for all } x \in V(G)\cup E(G), \tag{1}$$

where $\langle 0,1\rangle$ is the closed real interval from 0 to 1. The value of the objective function

$$\sum_{N\in\Gamma(G)} \varphi(N) \to \min \tag{2}$$

in an optimal solution of the linear program (1) is the *fractional* $(\mathcal{P},\mathcal{Q})$-*total chromatic number* of a graph G, denoted by $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$. We show that this definition is equivalent to the one above.

Let us consider a finite graph G and let $\sum_{N_i\in\Gamma(G)} b_i/s_i = r/s$, $s = \mathrm{lcm}_{N_i\in\Gamma(G)}(s_i)$ (note that $\gcd(r,s)$ may be greater than 1) be one of the rational valued optimal solutions of the above presented linear program (1) for G. This means that we have a solution with $\varphi(N_i) = \frac{b_i}{s_i} = \frac{a_i}{s}$ for $i = 1,\ldots,t = |\Gamma(G)|$ and $a_1+\cdots+a_t = r$. Let F be a union of disjoint sets of colors F_i with $|F_i| = a_i$, $i = 1,\ldots,t$. We define an assignment $h : V(G)\cup E(G) \to P(F)$, where $P(F)$ is the power set of F, in such a way that

$$h(x) = \bigcup_{\substack{i\\ x\in N_i\in\Gamma(G)}} F_i$$

for every $x \in V(G)\cup E(G)$. Then

$$|h(x)| = \left| \bigcup_{\substack{i\\ x\in N_i\in\Gamma(G)}} F_i \right| = \sum_{\substack{i\\ x\in N_i\in\Gamma(G)}} |F_i| = \sum_{\substack{i\\ x\in N_i\in\Gamma(G)}} a_i = s \sum_{\substack{i\\ x\in N_i\in\Gamma(G)}} \varphi(N_i) \geq s$$

for all $x \in V(G)\cup E(G)$. Now let $g(x)$ be an arbitrary s-element subset of $h(x)$. Then g is an (r,s)-fractional $(\mathcal{P},\mathcal{Q})$-total coloring of G.

On the other hand, suppose that G has an (r,s)-fractional $(\mathcal{P},\mathcal{Q})$-total coloring g. Define an assignment $\psi : \Gamma(G)\times[0,r-1] \to \langle 0,1\rangle$ such that

$$\psi(N,i) = \begin{cases} 1/s & \text{for } i = 0,\ldots,r-1 \text{ and } N = \{x \in V(G)\cup E(G) : g(x) \ni i\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we define $\varphi : \Gamma(G) \to \langle 0, 1\rangle$ as follows:

$$\varphi(N) = \sum_{i=0}^{r-1} \psi(N, i).$$

It is easy to verify that for all $x \in V(G) \cup E(G)$ it holds that

$$\begin{aligned}
\sum_{\substack{N\in\Gamma(G)\\ x\in N}} \varphi(N) &= \sum_{\substack{N\in\Gamma(G)\\ x\in N}} \sum_{i=0}^{r-1} \psi(N, i) \\
&= \sum_{\substack{N\in\Gamma(G)\\ x\in N}} \sum_{i\in g(x)} \psi(N, i) \\
&= \sum_{i\in g(x)} \sum_{\substack{N\in\Gamma(G)\\ x\in N}} \psi(N, i) \\
&= \sum_{i\in g(x)} \frac{1}{s} = s \cdot \frac{1}{s} = 1
\end{aligned}$$

and

$$\sum_{N\in\Gamma(G)} \varphi(N) = \sum_{N\in\Gamma(G)} \sum_{i=0}^{r-1} \psi(N, i) = \sum_{i=0}^{r-1} \sum_{N\in\Gamma(G)} \psi(N, i) \le \sum_{i=0}^{r-1} \frac{1}{s} = \frac{r}{s}.$$

Note that the inequality in the last relation is tight if not all colors are used in the coloring g.

Hence the LP-definition of the fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number is equivalent to the one mentioned above.

Karafová [9] showed that if $c(\mathcal{P}) = k$, then the fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number of a complete graph K_n equals the optimum value of the objective function of the following linear program:

$$\sum_{i=0}^{k+1} z_i \to \min \tag{3}$$

$$\sum_{i=0}^{k+1} i z_i \ge n, \tag{4}$$

$$\sum_{i=0}^{k+1} a_i z_i \ge \binom{n}{2}, \tag{5}$$

$$(6) \qquad z_i \geq 0,$$

for each $i = 0, \ldots, k+1$, where a_i is the maximum number of edges in all $(\mathcal{P}, \mathcal{Q})$-total independent sets of K_n with exactly i vertices.

Theorem 2.1. *Let $n \geq 3$. Then $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n+1)}{2(n-1)}$.*

Proof. Since the maximal graphs in the property $\mathcal{D}_1$ are trees it holds that $a_i = (n-i) - 1$ for $i = 0, 1, 2$ in the linear program (3)–(6). Then the fractional chromatic number of K_n equals the value of the objective function in an optimal solution of the following linear program:

$$z_0 + z_1 + z_2 \to \min$$

$$z_1 + 2z_2 \geq n,$$

$$(n-1)z_0 + (n-2)z_1 + (n-3)z_2 \geq \binom{n}{2}.$$

It is easy to see that $\boldsymbol{z} = (\frac{n}{n-1}, 0, \frac{n}{2})$ is an optimal solution and $\frac{n(n+1)}{2(n-1)}$ is the corresponding value of the objective function. ■

3. Basic Properties

At first we present several introductory results.

Observation 3.1. *For every graph G,*

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi''(G).$$

Observation 3.2. *If the graph G has an (r,s)-fractional/circular $(\mathcal{P}, \mathcal{Q})$-total coloring, then G has a (t,s)-fractional/circular $(\mathcal{P}, \mathcal{Q})$-total coloring for each $t \in \mathbb{N}$, $t \geq r$.*

Lemma 3.3. *Let $G = (V, E)$ be a graph. Then*

(1) $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \chi''_{c,\mathcal{P},\mathcal{Q}}(G) = 1$ *if and only if* $E = \emptyset$,

(2) $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \chi''_{c,\mathcal{P},\mathcal{Q}}(G) = 2$ *if and only if* $G \in \mathcal{P}$ *and* $G \in \mathcal{Q}$, *and* $E \neq \emptyset$,

(3) $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) > 2$ *and* $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) > 2$ *if and only if* $G \notin \mathcal{P}$ *or* $G \notin \mathcal{Q}$.

Proof. (1) It holds that $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) \geq \chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{r}{s} \geq 1$ since $r \geq s$. If $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) = 1$, then all elements of G must have the same color set in an (r,s)-circular $(\mathcal{P},\mathcal{Q})$-total coloring since $r = s$. This implies $E = \emptyset$.

If $E = \emptyset$ then $G \in \mathcal{O} \subseteq \mathcal{P}$ and therefore all vertices can be colored the same which implies $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) = 1$.

(2) If $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \chi''_{c,\mathcal{P},\mathcal{Q}}(G) = 2$, then $E \neq \emptyset$ by (1). Since $r = 2s$, all vertices must obtain the same color set and also all edges must be colored the same in each component of G. Therefore, all components belong to $\mathcal{P} \cap \mathcal{Q}$ which implies $G \in \mathcal{P} \cap \mathcal{Q}$ since $\mathcal{P}$ and $\mathcal{Q}$ are additive properties.

If $E \neq \emptyset$ then $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) \geq \chi''_{f,\mathcal{P},\mathcal{Q}}(G) \geq 2$ by (1). If $G \in \mathcal{P} \cap \mathcal{Q}$ then there exists a $(2,1)$-circular $(\mathcal{P},\mathcal{Q})$-total coloring of G which implies $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq 2$.

(3) Obviously, if $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) > 2$ and $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) > 2$, then $G \notin \mathcal{P} \cap \mathcal{Q}$ by (1) and (2), and vice versa. ■

Lemma 3.4. *If $H \subseteq G$, then $\chi''_{f,\mathcal{P},\mathcal{Q}}(H) \leq \chi''_{f,\mathcal{P},\mathcal{Q}}(G)$, and $\chi''_{c,\mathcal{P},\mathcal{Q}}(H) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G)$.*

Proof. Let h be an (r,s)-fractional/circular $(\mathcal{P},\mathcal{Q})$-total coloring of G. By restricting this coloring to the set $V(H) \cup E(H)$ we obtain an (r,s)-fractional/circular $(\mathcal{P},\mathcal{Q})$-total coloring of the graph H. ■

Lemma 3.5. *If $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$, then $\chi''_{f,\mathcal{P}_1,\mathcal{Q}_1}(G) \geq \chi''_{f,\mathcal{P}_2,\mathcal{Q}_2}(G)$, and $\chi''_{c,\mathcal{P}_1,\mathcal{Q}_1}(G) \geq \chi''_{c,\mathcal{P}_2,\mathcal{Q}_2}(G)$.*

Proof. Let h be an (r,s)-fractional/circular $(\mathcal{P}_1,\mathcal{Q}_1)$-total coloring of G. Then $G[V_{i,h}] \in \mathcal{P}_1$ and $G[E_{i,h}] \in \mathcal{Q}_1$ for all $i \in \mathbb{Z}_r$. Since $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$, it holds that $G[V_{i,h}] \in \mathcal{P}_2$ and $G[E_{i,h}] \in \mathcal{Q}_2$ for all $i \in \mathbb{Z}_r$; thus, the coloring h is also an (r,s)-fractional/circular $(\mathcal{P}_2,\mathcal{Q}_2)$-total coloring of G, and so $\chi''_{f,\mathcal{P}_1,\mathcal{Q}_1}(G) \geq \chi''_{f,\mathcal{P}_2,\mathcal{Q}_2}(G)$ and $\chi''_{c,\mathcal{P}_1,\mathcal{Q}_1}(G) \geq \chi''_{c,\mathcal{P}_2,\mathcal{Q}_2}(G)$. ■

Because the considered invariants are monotone and $K_{\omega(G)} \subseteq G \subseteq K_{|V(G)|}$ for every graph G, we can formulate the following statements.

Corollary 3.6. $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_{\omega(G)}) \leq \chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{f,\mathcal{P},\mathcal{Q}}(K_{|V(G)|})$,

$$\chi''_{c,\mathcal{P},\mathcal{Q}}(K_{\omega(G)}) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(K_{|V(G)|}).$$

Corollary 3.7. $\chi''_{f,\mathcal{I}_{c(\mathcal{P})},\mathcal{I}_{c(\mathcal{Q})}}(K_{\omega(G)}) \leq \chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{f,\mathcal{O}_{c(\mathcal{P})},\mathcal{O}_{c(\mathcal{Q})}}(K_{|V(G)|})$,
$\chi''_{c,\mathcal{I}_{c(\mathcal{P})},\mathcal{I}_{c(\mathcal{Q})}}(K_{\omega(G)}) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{c,\mathcal{O}_{c(\mathcal{P})},\mathcal{O}_{c(\mathcal{Q})}}(K_{|V(G)|})$.

Lemma 3.8. *Let $n \in \mathbb{N}$ be an arbitrary integer. Then the graph G has an (r,s)-circular $(\mathcal{P},\mathcal{Q})$-total coloring if and only if G has an (nr,ns)-circular $(\mathcal{P},\mathcal{Q})$-total coloring.*

Proof. Let $n \in \mathbb{N}$ and $h : V(G) \cup E(G) \to [\mathbb{Z}_r]^s$ be an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring of G. Define a new (nr, ns)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring $g : V(G) \cup E(G) \to [\mathbb{Z}_{nr}]^{ns}$ of G in the following way: for each $x \in V(G) \cup E(G)$ let $g(x) = [na, na+ns-1]$ if $h(x) = [a, a+s-1]$. For each element $x \in V(G) \cup E(G)$ and color $i \in \mathbb{Z}_r$, we have that $i \in h(x)$ if and only if $ni, \dots, n(i+1)-1 \in g(x)$ (reduced modulo nr). Consequently, for each color $i \in \mathbb{Z}_r$, the graph $G[V_{i,h}]$ (or $G[E_{i,h}]$) is isomorphic to the graph $G[V_{j,g}]$ (or $G[E_{j,g}]$) for each $j \in [ni, n(i+1)-1]$. If $G[V_{i,h}] \in \mathcal{P}$ (or $G[E_{i,h}] \in \mathcal{Q}$), then $G[V_{j,g}] \in \mathcal{P}$ (or $G[E_{j,g}] \in \mathcal{Q}$) for each $j \in [ni, n(i+1)-1]$. Moreover, color sets of incident vertices and edges given by the coloring g are disjoint since $h(x) \cap h(y) = \emptyset$ if and only if $g(x) \cap g(y) = \emptyset$ for each $x, y \in V(G) \cup E(G)$.

Conversely, suppose that G has an (nr, ns)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring g. Define an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring h of G in the following way: for each $x \in V(G) \cup E(G)$ let $h(x) = [\lceil a/n \rceil, \lceil a/n \rceil + s - 1]$ if $g(x) = [a, a + ns - 1]$. For each $i \in \mathbb{Z}_r$, $G[V_{i,h}] \cong G[V_{ni,g}]$ and $G[E_{i,h}] \cong G[E_{ni,g}]$, thus $G[V_{i,h}] \in \mathcal{P}$ and $G[E_{i,h}] \in \mathcal{Q}$. Finally, color sets of incident vertices and edges given by h are disjoint by the same argument as above. ■

The following result is an immediate consequence of Lemma 3.8.

Corollary 3.9. *If the graph G has an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring, then it has an (a, b)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring with $a/b = r/s$ and* $\gcd(a, b) = 1$.

Lemma 3.10. *Let $n \in \mathbb{N}$. If the graph G has an (r, s)-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring, then it has an (nr, ns)-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring.*

Proof. Let $n \in \mathbb{N}$ and assume that G has an (r, s)-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring h. Define an (nr, ns)-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring $g : V(G) \cup E(G) \to [\mathbb{Z}_{nr}]^{ns}$ of G in the following way: for each $x \in V(G) \cup E(G)$ let $g(x) = [nh_1, n(h_1+1)-1] \cup [nh_2, n(h_2+1)-1] \cup \cdots \cup [nh_s, n(h_s+1)-1]$ if $h(x) = \{h_1, h_2, \dots, h_s\}$. By the definition of g, for each element $x \in V(G) \cup E(G)$ it holds that $i \in h(x)$ if and only if $[ni, n(i+1)-1] \subseteq g(x)$. Hence, $G[V_{i,h}] \in \mathcal{P}$ (or $G[E_{i,h}] \in \mathcal{Q}$) if and only if $G[V_{j,g}] \in \mathcal{P}$ (or $G[E_{j,g}] \in \mathcal{Q}$, respectively) for each $j \in [ni, n(i+1)-1]$. Moreover, color sets of incident vertices and edges given by the coloring g are disjoint. ■

The converse of this lemma is not true. The *Kneser graph* $KG_{n,k}$ is the graph with vertex set $[\mathbb{Z}_n]^k$ and with edges joining disjoint k-element subsets. Then $KG_{6,2}$ has a $(6, 2)$-fractional $(\mathcal{O}, \mathcal{I})$-total coloring by definition, but it has no $(3, 1)$-fractional $(\mathcal{O}, \mathcal{I})$-total coloring. Lovász proved Kneser's conjecture in [11] somewhat later Bárány [1] gave a simple proof $\chi(KG_{n,k}) = n - 2k + 2$, i.e., $\chi(KG_{6,2}) = 4$.

Lemma 3.11. *Let $a, b, r, s \in \mathbb{N}$, $a \geq b$ and $r \geq s$. If G has an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring, then it has also an (a, b)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring for each $a/b \geq r/s$.*

Proof. Suppose that a graph G has an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring, $a/b \geq r/s$ and $ms = nb = \mathrm{lcm}(s, b)$. By Lemma 3.8, G has an (mr, ms)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring g. Since $a/b \geq r/s$, we have $na \geq mr$, thus g is also an (na, nb)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring of G (see Observation 3.2). Then, by Lemma 3.8, the graph G has an (a, b)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring. ■

Theorem 3.12. $\chi''_{\mathcal{P},\mathcal{Q}}(G) - 1 < \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G)$.

Proof. Since each $(\mathcal{P}, \mathcal{Q})$-total r-coloring of a graph G is also an $(r, 1)$-circular $(\mathcal{P}, \mathcal{Q})$-total coloring, we have $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G)$.

If $\chi''_{\mathcal{P},\mathcal{Q}}(G) - 1 \geq \chi''_{c,\mathcal{P},\mathcal{Q}}(G)$, then there exists an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring of G such that $r/s \leq \chi''_{\mathcal{P},\mathcal{Q}}(G) - 1$. Then, by Lemma 3.11, there exists a $(\chi''_{\mathcal{P},\mathcal{Q}}(G) - 1, 1)$-circular $(\mathcal{P}, \mathcal{Q})$-total coloring of G which is also a $(\mathcal{P}, \mathcal{Q})$-total $(\chi''_{\mathcal{P},\mathcal{Q}}(G) - 1)$-coloring contradicting the definition of such colorings. ■

Lemma 3.13. *Let G have an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring h with* $\gcd(r, s) = 1$*, and suppose that there exist an $i \in \mathbb{Z}_r$ such that $[i, i + s - 1] \neq h(x)$ for each $x \in V(G) \cup E(G)$. Then G has an (a, b)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring g with $a/b < r/s$ and $a < r$.*

Proof. Suppose that a graph G has an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring $h_1 = h : V(G) \cup E(G) \to [\mathbb{Z}_r]^s$ and define $\hat{h} : V(G) \cup E(G) \to \mathbb{Z}_r$ by $h(x) = [\hat{h}(x), \hat{h}(x) + s - 1]$. In this coloring at least one color set is not used; without loss of generality let it be the set $[s, 2s - 1]$ (otherwise the graph can be cyclically recolored).

If $s \geq 2$ then assign to each element $x \in V(G) \cup E(G)$ with color set $[2s, 3s - 1]$ the new color set $[2s - 1, 3s - 2]$. By this recoloring, we obtain a coloring h_2 which satisfies $G[V_{i,h_2}] \in \mathcal{P}$ for all $i \in \mathbb{Z}_r$: Obviously, $V_{i,h_2} = V_{i,h}$ for $i \neq 2s - 1, 3s - 1$ and $V_{3s-1,h_2} \subseteq V_{3s-1,h}$ which implies $G[V_{i,h_2}] \in \mathcal{P}$ for $i \neq 2s - 1$. Since $s \geq 2$ and since the set $[s, 2s - 1]$ does not occur in the coloring h and therefore not in h_2, each element $x \in V(G) \cup E(G)$ with $2s - 1 \in h_2(x)$ must also have $2s \in h_2(x)$. It follows that $V_{2s-1,h_2} \subseteq V_{2s,h_2} = V_{2s,h}$ and therefore $G[V_{2s-1,h_2}] \in \mathcal{P}$. Analogously, $G[E_{i,h_2}] \in \mathcal{Q}$ for all $i \in \mathbb{Z}_r$. It also holds that $h_2(x) \cap h_2(y) = \emptyset$ for incident elements $x \in V(G)$, $y \in E(G)$, since $h(x) \cap h(y) = \emptyset$ and since the set $[s, 2s - 1]$ does not occur in h_2. Therefore, h_2 is an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring of G. Note that the coloring h_2 uses at most $r - 2$ color sets.

Now perform the described recoloring for color sets $[2s, 3s - 1], [3s, 4s - 1], \ldots, [\sigma s, (\sigma+1)s - 1]$, where $\sigma s \equiv 1 \pmod{r}$ (such a σ exists because $\gcd(r, s) = 1$). Note that the values in the color sets are considered modulo r. If $s = 1$ then $\sigma = 1$ and no recoloring is needed. The coloring h_σ uses at most $r - \sigma$

color sets and the corresponding $\hat{h}_\sigma$ uses no color from $H = \{s, 2s, \ldots, \sigma s\}$. Let $t = \frac{\sigma s-1}{r}$. Rename the colors used by $\hat{h}_\sigma$ by the bijection $\varphi : \mathbb{Z}_r \backslash H \to \mathbb{Z}_{r-\sigma}$, $\varphi(i) = i - |H \cap [0, i]| = i - |\{c \in H : c < i\}|$. Then we obtain a new assignment $\hat{g} : V(G) \to \mathbb{Z}_{r-\sigma}$ with $\hat{g}(x) = \varphi(\hat{h}_\sigma(x))$ and a corresponding coloring $g : V(G) \to [\mathbb{Z}_{r-\sigma}]^{s-t}$ with $g(x) = [\hat{g}(x), \hat{g}(x)+s-t-1]$. For each $i \in \mathbb{Z}_r$ consider $M_i = [i, i+s-1]$. Note that each set M_i, $i \neq 1$, contains exactly t values of H and that M_1 contains $t+1$ values of H, but M_1 is not used in the coloring h_σ. It follows that $i \in h_\sigma(x)$ if and only if $\varphi(i) \in g(x)$ for each $i \in \mathbb{Z}_r \backslash H$.

Hence, g is an (a, b)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring of G with $a = r - \sigma$ and $b = s - t$. This implies that $a < r$ since $\sigma \geq 1$ and that $\frac{a}{b} = \frac{r-\sigma}{s-t} = \frac{r-\sigma}{s-\frac{\sigma s-1}{r}} = \frac{r(r-\sigma)}{s(r-\sigma)+1} < \frac{r}{s}$. ■

The following theorem is an immediate consequence of Lemma 3.13.

Theorem 3.14. *Let* $|V(G)| = n$ *and* $|E(G)| = m$. *Then* $\chi''_{c,\mathcal{P},\mathcal{Q}}(G) = \min\left\{\frac{r}{s} : G \text{ has an } (r,s)\text{-circular } (\mathcal{P},\mathcal{Q})\text{-total coloring and } r \leq n+m\right\}$.

Proof. Let h be an (r, s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring of G with $\gcd(r, s) = 1$ which is no restriction according to Corollary 3.9.

If $r > n + m = |V(G)| + |E(G)|$, then there exists an unused color set $[i, i+s-1]$, $i \in \mathbb{Z}_r$. This implies by Lemma 3.13 that G has an (a, b)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring with $a/b < r/s$ and $a < r$. Therefore, one can restrict oneself in the computation of $\chi''_{c,\mathcal{P},\mathcal{Q}}(G)$ to the case $r \leq n + m$. Since $s \leq r$ by definition there are just finitely many fractions r/s which implies that the infinum in the definition of $\chi''_{c,\mathcal{P},\mathcal{Q}}(G)$ can be replaced by the minimum. ■

4. RESULTS FOR K_n

In this section we consider total colorings of complete graphs.

It is easy to see that, for every n,

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(K_n) \leq \chi''_{\mathcal{P},\mathcal{Q}}(K_n) \leq \chi''(K_n).$$

The generalized edge-chromatic number of a graph G, denoted by $\chi'_{\mathcal{P}}(G)$, is defined as the least integer k for which there exists a decomposition of $E(G) = \{E_1, \ldots, E_k\}$ such that $G[E_i] \in \mathcal{P}$ for each $i = 1, \ldots, k$ (see, e.g., [6]). Because $\chi'_{\mathcal{O}_1}(G) = \chi'(G)$ we can say that generalized $\mathcal{P}$-edge colorings are generalizations of proper edge colorings. We use the following lemma for the proof of the next theorem.

Lemma 4.1. $\chi'_{\mathcal{I}_1}(K_n) \leq \lceil \log_2 n \rceil$.

Proof. The proof is by induction on $k = \lceil \log_2 n \rceil$. For $k \le 1$ the statement holds trivially. In the induction step let $n \in [2^k + 1, 2^{k+1}]$. We partition the vertices of K_n into two parts: X with 2^k vertices and Y with $n - 2^k \le 2^k$ vertices. Color the edges of $G[X] \cong K_{2^k}$ and of $G[Y] \subseteq K_{2^k}$ inductively by colors from the set $\{1, \ldots, k\}$. A new color $k+1$ is used for all edges between X and Y. This coloring contains no monochromatic triangle which completes the proof. ■

Note that there are better estimations of $\chi'_{\mathcal{I}_1}(K_n)$ because this parameter is connected to Ramsey numbers (see, e.g., [5]). In the following theorem we determine $\chi''_{f,\mathcal{P},\mathcal{Q}}$ and $\chi''_{c,\mathcal{P},\mathcal{Q}}$ for complete graphs with an arbitrary property $\mathcal{P}$ for the vertices and property $\mathcal{Q} = \mathcal{I}_l$ with (an arbitrary l) for the edges. This result is indepedent of l and the proof is based on the fact that $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots$.

Theorem 4.2. *For each $k \in \mathbb{N}$ there exists a $T(k)$ such that for each $n \ge T(k)$, for each $l \in \mathbb{N}$ and for each $\mathcal{P}$ with $c(\mathcal{P}) = k$ it holds that*

$$\chi''_{c,\mathcal{P},\mathcal{I}_l}(K_n) = \chi''_{f,\mathcal{P},\mathcal{I}_l}(K_n) = \frac{n}{k+1}.$$

Proof. Let $c(\mathcal{P}) = k$. For any (r,s)-fractional $(\mathcal{P},\mathcal{I}_l)$-total coloring of K_n it holds that at most $(k+1)$ vertices are colored with color sets containing i, for each $i \in \mathbb{Z}_r$, and every vertex is colored with an s-element color set. This implies that $r(k+1) \ge ns$ and therefore $\chi''_{c,\mathcal{P},\mathcal{I}_l}(K_n) \ge \chi''_{f,\mathcal{P},\mathcal{I}_l}(K_n) \ge \frac{n}{k+1}$.

To prove the inequality $\chi''_{c,\mathcal{P},\mathcal{I}_l}(K_n) \le \frac{n}{k+1}$ we construct an $(n, k+1)$-circular $(\mathcal{P},\mathcal{I}_1)$-total coloring h of K_n since $\chi''_{c,\mathcal{P},\mathcal{I}_l}(K_n) \le \chi''_{c,\mathcal{P},\mathcal{I}_1}(K_n)$. Let $V(K_n) = \mathbb{Z}_n$. Define the assignment $h : V(K_n) \cup E(K_n) \to [\mathbb{Z}_n]^{k+1}$ firstly for vertices by $h(i) = [i, k+i]$ for each $i \in \mathbb{Z}_n$. For each color $i \in \mathbb{Z}_n$ the graph $G[V_{i,h}]$ has exactly $k+1$ vertices and therefore has property $\mathcal{O}_k \subseteq \mathcal{P}$.

Next we denote $V_0 = [n-k, k]$, $V_1 = [k+1, \lfloor n/2 \rfloor]$, and $V_2 = V(K_n) \setminus (V_0 \cup V_1) = [\lfloor n/2 \rfloor + 1, n-k-1]$. Then $|V_0| = 2k+1$ and $|V_2| \le |V_1| \le \frac{n-2k}{2}$. Colors from $[0,k]$ were used only on vertices from V_0 and thus the color set $[0,k]$ can be assigned to all edges between V_1 and V_2.

For an arbitrary p there is a sufficiently large n_p such that for $n \ge n_p$ this coloring uses colors from $[2k+1, 2k+p(k+1)]$ only on vertices from V_1. We can also state that colors from $[n-k-p(k+1), n-(k+1)]$ are used only on vertices from V_2. Note that it is sufficient to take $n_p = 2p(k+1) + 4k + 1$. Our goal is to use disjoint color sets $[j(k+1)+k, j(k+1)+2k]$, $j = 1, \ldots, p-1$, for edges from $E(V_2)$ and from $E(V_0)$, color set $[p(k+1)+k, p(k+1)+2k]$ for all edges between V_0 and V_2, disjoint color sets $[n-k-j(k+1), n-j(k+1)]$, $j = 2, \ldots, p$, for edges from $E(V_1)$ and finally $[n-2k-1, n-k-1]$ for all edges between V_0 and V_1. Therefore, if we take $p - 1 \ge \max\{\chi'_{\mathcal{I}_1}(K_{|V_i|}); i = 0,1,2\}$, then the coloring can be created in a such way that for each color $i \in \mathbb{Z}_n$ the graph $G[E_{i,h}]$ has no triangle and therefore is from $\mathcal{I}_1$. Moreover, this coloring uses disjoint color sets for incident vertices and edges. Therefore, this is an $(n, k+1)$-circular $(\mathcal{P}, \mathcal{I}_1)$-total coloring of K_n.

To finish the proof it is sufficient to show the existence of a number p with $\left\lfloor \frac{n-4k-1}{2(k+1)} \right\rfloor \geq p \geq \left\lceil \log_2 \frac{n-2k}{2} \right\rceil + 1$. Such a p does exist because of the properties of linear and logarithmic functions for big values of n. ■

In the previous theorem we can replace property $\mathcal{I}_l$ by an arbitrary property $\mathcal{Q} \supseteq \mathcal{I}_1$.

Remark 4.3. If $n \geq T(c(\mathcal{P}))$, then $\chi''_{c,\mathcal{P},\mathcal{I}_b}(K_n) = \chi''_{f,\mathcal{P},\mathcal{I}_b}(K_n) = \frac{n}{c(\mathcal{P})+1}$. If $n < T(c(\mathcal{P}))$, then $\chi''_{c,\mathcal{P},\mathcal{I}_b}(K_n) \geq \chi''_{f,\mathcal{P},\mathcal{I}_b}(K_n) \geq \frac{n}{c(\mathcal{P})+1}$. Thus, we have $\chi''_{f,\mathcal{P},\mathcal{I}_b}(K_n) \geq \frac{n}{c(\mathcal{P})+1}$ for each $n \in \mathbb{N}$ which implies for an arbitrary graph G

$$\chi''_{f,\mathcal{P},\mathcal{I}_b}(G) \geq \chi''_{f,\mathcal{P},\mathcal{I}_b}(K_{\omega(G)}) \geq \frac{\omega(G)}{c(\mathcal{P})+1}.$$

Borowiecki *et al.* [2] showed that $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(K_n) = \lfloor n/2 \rfloor + 2$. We determine this number for fractional and circular $(\mathcal{D}_1, \mathcal{D}_1)$-total colorings and odd order n. Note that $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n)$ is already calculated in Theorem 2.1. Here we present a different proof.

Theorem 4.4. *Let n be odd, $n \geq 3$. Then*

$$\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \chi''_{c,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n+1)}{2(n-1)}.$$

Proof. It is sufficient to prove $\frac{n(n+1)}{2(n-1)} \leq \chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n)$ and $\chi''_{c,\mathcal{D}_1,\mathcal{D}_1}(K_n) \leq \frac{n(n+1)}{2(n-1)}$. For each (r, s)-fractional $(\mathcal{D}_1, \mathcal{D}_1)$-total coloring of K_n and for each $i \in \mathbb{Z}_r$, the following holds: at most two vertices are colored with sets containing the color i, and at most $n-1$ vertices or edges may be colored with sets containing i. On the other hand, each vertex and each edge is assigned with an s-element color set. This implies that

$$(n-1)r \geq \left(n + \binom{n}{2}\right) s,$$

hence,

$$\frac{r}{s} \geq \frac{n(n+1)}{2(n-1)}$$

and therefore $\frac{n(n+1)}{2(n-1)} \leq \chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n)$ for all n.

To prove the inequality $\chi''_{c,\mathcal{D}_1,\mathcal{D}_1}(K_n) \leq \frac{n(n+1)}{2(n-1)}$ for odd n, say $n = 2t+1$, we construct an $(n(n+1)/2, n-1)$-circular $(\mathcal{D}_1, \mathcal{D}_1)$-total coloring of K_n.

Let $V(K_{2t+1}) = \mathbb{Z}_{2t+1} = [0, 2t]$. We define $h : V(K_{2t+1}) \cup E(K_{2t+1}) \to [\mathbb{Z}_r]^s$ with $r = (2t+1)(t+1)$ and $s = 2t$ as follows:

$$\begin{aligned} h(i) &= [\hat{h}(i), \hat{h}(i) + s - 1] && \text{for } \; i \in \mathbb{Z}_{2t+1}, \\ h(i,j) &= [\hat{h}(i,j), \hat{h}(i,j) + s - 1] && \text{for } \; i, j \in \mathbb{Z}_{2t+1}, i \neq j, \end{aligned}$$

where the corresponding function $\hat{h} : V(K_{2t+1}) \cup E(K_{2t+1}) \to \mathbb{Z}_r$ is defined below. Note that arguments of h and $\hat{h}$ are taken modulo $2t+1$ while the values of $\hat{h}$ are taken modulo $r = 2t^2 + 3t + 1$.

$$\begin{aligned} \hat{h}(i) &= i(t+1) && \text{for } \; i \in \mathbb{Z}_{2t+1}, \\ \hat{h}(0,j) &= \begin{cases} 2t(t+1) - j(t-1) & \text{for } \; j \in [1,t], \\ 2t(j-t) & \text{for } \; j \in [t+1, 2t], \end{cases} \\ \hat{h}(i+1, j+1) &= \hat{h}(i,j) + (t+1) && \text{for } \; 0 \leq i < j < 2t, \\ \hat{h}(i,j) &= \hat{h}(j,i) && \text{for } \; 0 \leq j < i \leq 2t. \end{aligned}$$

Now we want to prove that h is an (r,s)-circular $(\mathcal{P}, \mathcal{Q})$-total coloring of K_n, or equivalently, we need to prove for each color $c \in \mathbb{Z}_r$ that

(A) $K_n[V_{c,h}] \in \mathcal{D}_1$,

(B) $K_n[E_{c,h}] \in \mathcal{D}_1$, and

(C) if $c \in h(i,j)$ for some edge (i,j), then $c \notin h(i)$ and $c \notin h(j)$.

We first show that $\hat{h}$ has the *shifting property*, i.e., $\hat{h}(i+1) = \hat{h}(i) + (t+1)$ for each $i \in \mathbb{Z}_n$ and $\hat{h}(i+1, j+1) = \hat{h}(i,j) + (t+1)$ for each $i, j \in \mathbb{Z}_n, i \neq j$. Obviously, we need to prove only the second equation just for $j = 2t$ since in the other cases it follows directly from the definition of $\hat{h}$. It holds that $\hat{h}(i+1, 0) = \hat{h}(0, i+1)$ and $\hat{h}(i, 2t) + (t+1) = \hat{h}(0, 2t-i) + i(t+1) + (t+1)$ if $i < j = 2t$. By distinguishing the cases for $i \in [0, t-1]$ and $i \in [t, 2t-1]$ we obtain in the first case the common value $2t^2 + t + 1 - i(t-1)$, in the second case the common value $-2t^2 + 2t + i2t = 5t + 1 + i2t$ (considered modulo r, $r = 2t^2 + 3t + 1$). Therefore, it is enough to show properties (A)–(C) for colors $c \in [0, t]$ only.

(A) Colors from $[0, t]$ are only used on vertices 0 and $2t$, i.e., at most two vertices contain color c and therefore $K_n[V_{c,h}] \in \mathcal{D}_1$.

(C) From the shifting property of $\hat{h}$ we have $h(i+1, j+1) \cap h(i+1) \neq \emptyset$ if and only if $h(i,j) \cap h(i) \neq \emptyset$. Therefore, we prove (C) only for edges $\{0, j\}$ and vertex 0. In this case $h(0) = [0, s-1] = [0, 2t-1]$ and we want to show that $\hat{h}(0,j) \in [s, r-s] = [2t, 2t^2 + t + 1]$. For $j \in [1, t]$ ($j \in [t+1, 2t]$, respectively) $\hat{h}(0,j) = 2t^2 + 2t - j(t-1) \in [t^2 + 3t, 2t^2 + t + 1] \subseteq [2t, 2t^2 + t + 1]$ ($\hat{h}(0,j) = -2t^2 + j2t \in [2t, 2t^2] \subseteq [2t, 2t^2 + t + 1]$, respectively) and the required property holds.

(B) Finally, we want to show that $K_n[E_{c,h}]$, $c \in [0, t]$, contains no cycle. To prove this we give the following orientation of the edges of $K_n[E_{c,h}]$: for an edge $\{i, j\}$ with $i < j$ (and $c \in h(i,j)$) we choose the orientation which starts at i and ends in j (denoted by $\overrightarrow{(i,j)}$). Now we will prove that in this oriented graph each vertex is an end of at most one arc and thus in this oriented graph there is no oriented cycle.

We fix vertex $j \in [0, 2t]$ and count the number of arcs $\overrightarrow{(i,j)}$ (with $c \in h(\overrightarrow{(i,j)})$). If $j - i \in [t+1, 2t]$ (and $i < j$), then $i \in [0, t-1], j \in [t+1, 2t]$ and $\hat{h}(i,j) = \hat{h}(0, j-i) + i(t+1) = 2t(j-i-t) + i(t+1) \in [2t(t+1-t)+0, 2t(2t-0-t)+0(t+1)] = [2t, 2t^2] \subseteq [2t, 2t^2+t+1] = [s, r-s]$ and $c \notin h(i,j)$. In the other case, $j - i \in [1, t]$ (and $i < j$). Then $\hat{h}(i,j) = \hat{h}(0, j-i) + i(t+1) = 2t(t+1) - (j-i)(t-1) + i(t+1) = 2t(t+1) - j(t-1) + i2t$. Therefore, the color sets $h(i,j)$ for fixed j are pairwise disjoint since $s = 2t$ and at most one of them contains the color c. ∎

As $K_n \subseteq K_{n+1}$ we obtain by Lemma 3.4 the following result.

Corollary 4.5. *Let n be even, $n \geq 4$. Then*

$$\frac{n(n+1)}{2(n-1)} \leq \chi''_{c,\mathcal{D}_1,\mathcal{D}_1}(K_n) \leq \frac{(n+1)(n+2)}{2n}.$$

We conjecture that Theorem 4.4 holds for all $n \geq 3$, i.e., also for even n.

Conjecture 4.6. *Let n be even, $n \geq 4$. Then*

$$\chi''_{c,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n+1)}{2(n-1)}.$$

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