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EXTENDING THE MAX ALGORITHM FOR MAXIMUM INDEPENDENT SET

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Abstract

The maximum independent set problem is an NP-hard problem. In this paper, we consider Algorithm MAX, which is a polynomial time algorithm for finding a maximal independent set in a graph G. We present a set of forbidden induced subgraphs such that Algorithm MAX always results in finding a maximum independent set of G. We also describe two modifications of Algorithm MAX and sets of forbidden induced subgraphs for the new algorithms.

Keywords: maximum independent set, stable set, stability number, independence number, reduction, graph transformation, MAX Algorithm, MIN Algorithm, Vertex Order Algorithm.

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1. Introduction

In a simple graph $G = (V, E)$, a set of vertices is *independent* (or *stable*) if no two vertices in this set are adjacent. The cardinality of a maximum size independent set in G is called the *independence number* (or the *stability number*) of G, denoted by $\alpha(G)$. The problem of determining an independent set of maximum cardinality, so-called Maximum Independent Set (or MIS for short) problem, finds important applications in various fields, some examples are computer vision and pattern recognition. It is well-known that the problem is generally NP-hard. Alekseev has shown in [1] that if a graph H has a connected component not of the form $S_{i,j,k}$, where $S_{i,j,k}$ is the graph consisting of three induced paths of lengths i, j and k, with a common initial vertex, then the MIS problem is NP-hard for H-free graphs. It is shown that the MIS problem is solvable in polynomial time for $S_{1,1,1}$-free graphs (claw-free graphs) [28, 34] and for $S_{0,1,2}$-free graphs (P_4-free graphs) [10]. These results were generalized to $S_{1,1,2}$-free graphs (fork-free graphs) [2] and $S_{0,2,2}$ (P_5-free graphs) [21]. Hence, the complexity status of the MIS problem in the graph classes defined by a single forbidden induced subgraph of the form $S_{i,j,k}$ was solved for the case $i + j + k \leq 4$. For larger $i + j + k$ (P_6, $S_{1,2,2}$, $S_{2,2,2}$ and so on), it is still an open question and gained much effort to solve in some subclasses. Some examples are $(P_n, K_{1,n})$-free graphs [22], (P_6, diamond)-free graphs [29], $(P_6, K_{2,3})$-free graphs [30], ($S_{2,2,2}$, banner)-free graphs [12], ($S_{2,2,2}, B_2$, domino, $M_3, K_{1,m}$)-free graphs [20], ($S_{1,2,k}$, banner, $K_{1,n}$)-free graphs [19], ($S_{1,2,5}$, banner)-free graphs [24], $S_{1,2,k}$-free planar graphs, $S_{1,k,k}$-free graphs of low degree [23], and $S_{2,2,2}$-free subcubic graphs [26].

Useful techniques for solving the problem are heuristic algorithms, i.e., the methods giving maximal (inclusion sense) independent sets in polynomial time. Three well-known algorithms are VO (Vertex Ordering) [27], MIN [31], and MAX [15]. Moreover, for some restricted graph classes, these algorithms always result in finding a maximum independent set of G, and hence yield the exact value of the independence number of G in polynomial time.

If $H_1, H_2, \ldots, H_k$ are graphs, then we say that G is $\{H_1, H_2, \ldots, H_k\}$-*free* if G does not contain a copy of any of the graphs $H_1, H_2, \ldots, H_k$ as an induced subgraph. In [27], Mahadev and Reed characterized a class of graphs for which a maximum independent set can be obtained by VO, that is the $\mathcal{F}_1$-free graphs, where $\mathcal{F}_1 = \{F_1, F_2, F_3, F_4, F_5, F_6\}$.

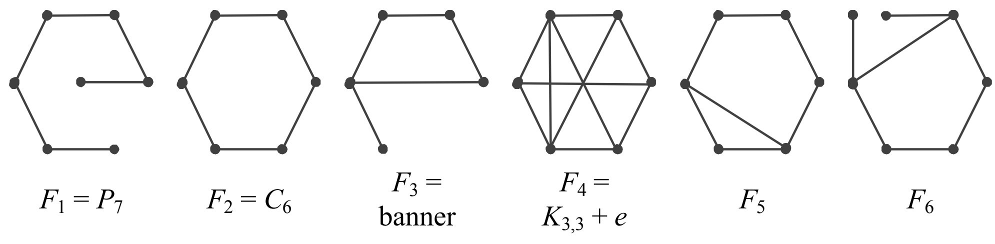

Figure 1. Forbidden subgraphs for Algorithm VO.

A set of forbidden induced subgraphs $\mathcal{F}_2$, under which Algorithm MIN always results in finding a maximum independent is given in [17] by Harant, Schiermeyer, and Ryjáček, where $\mathcal{F}_2 = \{F_1, F_3, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}\}$.

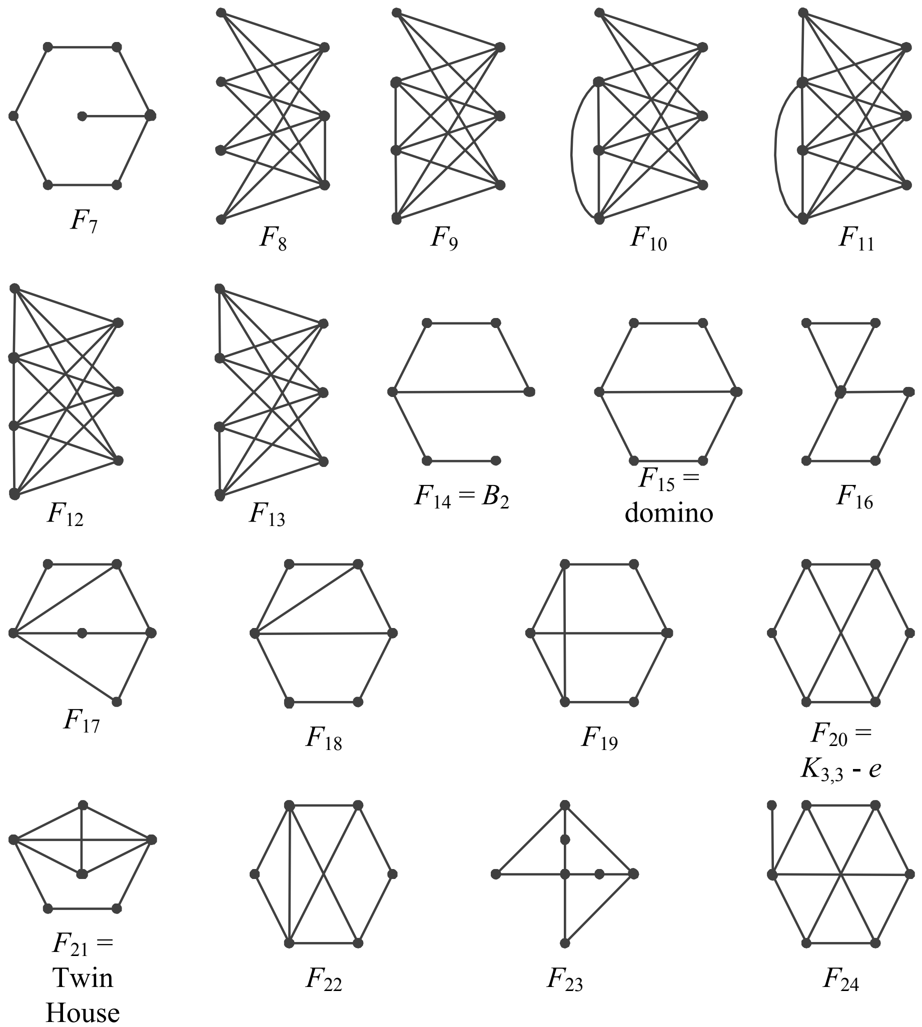

Figure 2. Some forbidden subgraphs for Algorithm MIN.

Zverovich [35] obtained another set, say $\mathcal{F}_3$, of forbidden induced subgraphs for MIN, where $\mathcal{F}_3 = \{F_1, F_4, F_5, F_6, F_7, F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19}, F_{20}, F_{21}, F_{22}, F_{23}, F_{24}\}$.

Following this direction, we will describe a set of forbidden induced subgraphs for Algorithm MAX in Section 2.

Given a graph G, Algorithm MAX chooses a maximum degree vertex, deletes it from the graph together with all incident edges. The process is repeated until there are no edges remaining. Then the remaining vertices form a maximal independent set of G. This idea is based on an assumption that a maximum degree vertex has very small possibility to belong to some maximum independent set and hence can be deleted from the graph without (or with very small possibility of) changing the independence number. Obviously, it will be better if we can find and delete only a vertex not belonging to some maximum independent set. Such vertices are called *α-redundant* vertices [5].

Definition. [5] Given a graph G, a vertex v of G will be called α-redundant if $\alpha(G-v)=\alpha(G)$.

Here, for a graph G, and a vertex v of G, we denote by $G-v$ the graph obtained from G by deleting v and all incident edges. Unfortunately, the problem of finding such vertices, is polynomially equivalent with the MIS problem and hence also NP-hard in general. However, in some cases, α-redundant vertices can be recognized efficiently. In the following, we present some conditions to recognize α-redundant vertices. First, we have the following obvious fact.

Proposition 1. Given a graph G, a vertex v of G is α-redundant if and only if there exists some maximum independent set S such that $v \notin S$.

Lemma 2 [13]**.** *Let G be a graph containing an induced $K_{1,m}$, $u(v_1, v_2, \ldots, v_m)$, where u is the center-vertex. Then either u is α-redundant or there exists some $u_1, u_2, \ldots, u_m$ such that $\{u, u_1, u_2, \ldots, u_m\}$ is independent and there is a perfect matching between $\{u_i\}$ and $\{v_i\}$.*

The α-redundant technique was used successfully in [5, 6, 12, 13, 36, 20] to extend some polynomial results of the MIS problem in some subclasses of P_5-free graphs. Note that, in Lemma 2, for the case $m=1$, we have a so-called *neighborhood reduction* technique [14]: let a and b be adjacent vertices in a graph G. If every neighbor vertex of b is also adjacent to a, then for any independent set S containing a, the set $(S\backslash\{a\})\cup\{b\}$ is also independent, i.e., a is α-redundant. And for the case $m=2$, we have a so-called *vertex deletion* technique [4]: let three vertices a, b, and c induce a P_3 with edges ab and bc. If $(N(a)\cup N(c))\backslash N[b]$ is a clique, then the removal of b does not change the independence number of the graph, i.e., b is α-redundant.

In Section 2, we will describe conditions under which a maximum degree vertex is not α-redundant in order to describe forbidden subgraphs for Algorithm MAX. In Section 3, we will describe some techniques to recognize some α-redundant vertex in the neighborhood of a maximum degree vertex u. In case we succeed, we will delete such a vertex instead of u. It leads us to a new hybrid algorithm for solving the MIS problem. Our motivation is finding an algorithm better than MAX and VO in the sense of forbidden induced subgraph set. In Section 4, we will combine Algorithm MAX with $K_{1,m}$-reduction to use some polynomial time solution of the MIS problem in some subclass of $K_{1,m}$-free graphs. Section 5 is devoted to comparison of the new algorithms with classical heuristic methods. In Section 6, we will give some discussion around the topic and related results.

Now, to some other notions which will be used throughout the paper. In this paper, we consider only finite undirected, simple graphs $G = (V(G), E(G))$. For $u, v \in V(G)$, we write $u \sim v$ if $uv \in E(G)$ and $u \nsim v$ if $uv \notin E(G)$. Denote by $N(x)$ the *neighborhood* of a vertex $x \in V(G)$, by $d(x) = |N_G(x)|$ the *degree* of x in G. For a set $M \subseteq V(G)$, denote by $G[M]$ the *induced subgraph* of G on M. For a vertex x, we also denote $N_M(x) = N(x) \cap M$ and $d_M(x) = |N_M(x)|$. For an induced subgraph H of G, denote $N_H(x) = N_{V(H)}(x)$ and $d_H(x) = |N_H(x)|$. Let $\Delta(G) = \max\{d(u) : u \in V(G)\}$. As usual, we also denote by K_p the complete graph having p vertices and by $K_{m,n}$ the complete bipartite graph with cardinalities m and n of the two parts. For $K_{1,m}$, the vertex of degree m is called the center-vertex. Denote by P_n the induced path having n vertices.

2. Forbidden Subgraphs for Algorithm MAX

First, we recall the well-known greedy Algorithm MAX (see Algorithm 1) for finding a maximal independent set in a graph, which was introduced in [15].

Algorithm 1 Algorithm MAX (Maximum degree).

Input: $G = (V, E)$.
Output: S, a maximal independent set of G.

```
H_n := G; i := n;
while E(H_i) ≠ ∅ do
    Choose a vertex v_i ∈ V(H_i) such that d_{H_i}(v_i) = Δ(H_i);
    H_{i-1} := H_i − v_i; i := i − 1;
end while
S := V(H_i);
return S
```

Obviously, the set S, generated by Algorithm MAX, is a maximal (but not necessarily maximum) independent set in G, and hence $\alpha(G) \geq |S|$. In this section, we describe some set of forbidden induced subgraphs, under which $\alpha(G) = |S|$. Let G be a graph such that the algorithm fails for G, i.e., the algorithm chooses and deletes some vertex of maximum degree u of G but the deletion of u reduces the independence number of G. It implies that u belongs to every maximum independent set of G or, in other words, u is not α-redundant by Proposition 1. Now, we consider some conditions, under which u is not α-redundant.

Lemma 3. *Let G be a graph, u be a vertex, $v \in N(u)$, and S be a maximum independent set such that $u \in S$. Then either u is α-redundant or $d_S(v) \geq 2$.*

Proof. If $d_S(v) = 1$, then $(S \backslash \{u\}) \cup \{v\}$ is a maximum independent set not containing u, i.e., u is α-redundant by Proposition 1. ■

Lemma 4. *Let G be a graph, $E(G) \neq \emptyset$, u be a maximum degree vertex and S be a maximum independent set such that $u \in S$. Then either u is α-redundant or there exist some $v_1, v_2 \in N(u)$ such that $v_1 \nsim v_2$.*

Proof. Let $|N(u)| = p$. Since u is of maximum degree and $E(G) \neq \emptyset$, $p > 0$. Assume that $N(u)$ is a clique. Then for every $v \in N(u)$, $d(v) \geq p - 1 + 1 = p = d(u) \geq d(v)$. Hence, for every $v \in N(u)$, $d(v) = p$, i.e., $N[u]$ is a clique and a connected component of G. Now, u is α-redundant because there exists some maximum independent set S containing a neighbor of u and not containing u. ■

Lemma 5. *Let G be a graph and (v_1, u, v_2) be an induced P_3 (u is the mid-vertex). Then either u is α-redundant or there exist u_1, u_2 such that $\{u_1, v_1, u, v_2, u_2\}$ induces a P_5 (u is the mid-vertex) or a banner (F_3) (v_1 or v_2 is of degree three) or a $K_{2,3}$ (v_1, v_2 are of degree three).*

Proof. Assume that u is not α-redundant. Then by Lemma 2, there exist u_1, u_2 such that $\{u, u_1, u_2\}$ is independent and there exists a perfect matching between $\{u_1, u_2\}$ and $\{v_1, v_2\}$, i.e., $\{u_1, v_1, u, v_2, u_2\}$ induces a P_5, a banner, or a $K_{2,3}$ depending on the adjacency between $\{u_1, u_2\}$ and $\{v_1, v_2\}$. ■

Given a graph G, denote by $k_{MAX}(G)$ the minimum cardinality of a (maximal) independent set given by Algorithm MAX. The following result describes a set of forbidden induced subgraphs, under which Algorithm MAX gives a maximum independent set.

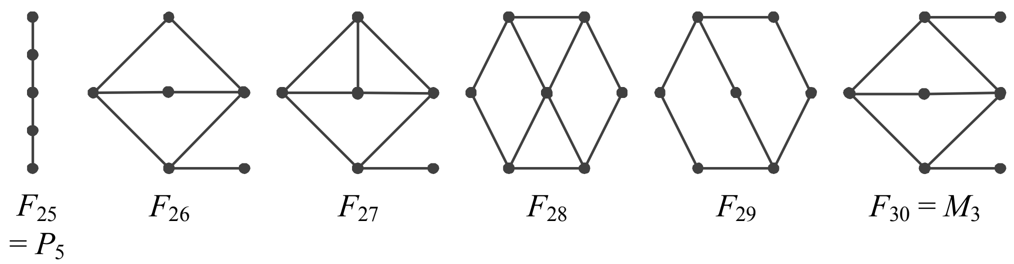

Figure 3. Some forbidden subgraphs for Algorithm MAX and Algorithm MMAX.

Theorem 6 (Forbidden subgraphs for MAX Algorithm)**.** *Let G be an $\mathcal{F}_4$-free graph of order $n \geq 7$, where $\mathcal{F}_4 = \{F_4, F_{15}, F_{19}, F_{20}, F_{21}, F_{24}, F_{25}, F_{26}, F_{27}\}$. Then $k_{MAX}(G) = \alpha(G)$.*

Proof. Assume that the statement in the theorem is not correct. That means there exists a (without loss of generality) connected graph G satisfying the assumption of the theorem and some vertex $u \in V(G)$ such that

1. u is of maximum degree in G and

2. u belongs to every maximum independent set of G,

i.e., Algorithm MAX will fail.

Let S be a maximum independent set of G (hence $u \in S$) and $T = V(G)\backslash S$. Since G is P_5-free and u is not α-redundant, we obtain the following observation from Lemma 5.

Claim 7. *Let $v_1, v_2 \in N(u)$ be two non-adjacent vertices. Then there exists some $u' \in S$ such that $v_1, v_2 \in N(u) \cap N(u')$.*

Also by Lemma 5, for two non-adjacent neighbors v_1, v_2 of u, there exist some u_1, u_2 such that $\{u, u_1, u_2, v_1, v_2\}$ induces a P_5, or a $K_{2,3}$, or an F_3. The first case cannot happen because G is P_5-free. We consider the two other cases by Claims 8 and 9.

Claim 8. *There are no vertices $v_1, v_2 \in T$ and $u_1, u_2 \in S$ such that $\{v_1, v_2, u, u_1, u_2\}$ induces a $K_{2,3}$.*

Proof. Suppose that $\{v_1, v_2, u, u_1, u_2\}$ induces a $K_{2,3}$, for some $v_1, v_2 \in T$ and $u_1, u_2 \in S$. Let H be a maximal (by inclusion) induced complete bipartite subgraph of G with parts A and B such that $\{u, u_1, u_2\} \subseteq A \subseteq S$ and $\{v_1, v_2\} \subseteq B \subseteq T$.

Case 1. $|B| < |A|$. Since $d(u) \geq d(v_2)$, there exists $v_3 \in T\backslash V(H)$ such that $v_3 \sim u$ and $v_3 \nsim v_2$.

If $N_A(v_3) = A$, then $v_3 \sim v$ for some $v \in B$ (otherwise, H is not maximal). Without loss of generality, let $v = v_1$. Then $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces an F_4, a contradiction.

Hence, there exists some $v_3 \in T\backslash V(H)$ such that $v_3 \sim u$, $v_3 \nsim v_1$, and $v_3 \nsim u'$ for some $u' \in A$, say $v_3 \nsim u_1$.

If $v_3 \nsim u' \, \forall u' \in A\backslash\{u\}$, then, by Claim 7, there exists some $u_3 \in S\backslash A$ such that u_3 is adjacent to v_1 and v_3. Moreover, $v \nsim u_3$ for some $v \in B$ (otherwise we have a contradiction with the maximality of H). Assume that $v_2 \nsim u_3$. Then $v_3 \sim v_2$, otherwise $\{u_2, v_2, u, v_3, u_3\}$ induces a P_5, a contradiction. Now, $\{u, u_1, u_3, v_1, v_2, v_3\}$ induces an F_{28}, a contradiction.

Hence, $u_3 \sim u'$ for some $u' \in A$, say $u' = u_2$. But then $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{20} or an F_{21}, depending on $v_3 \sim v_2$ or not, a contradiction.

Case 2. $|B| \geq |A|$, i.e, there exists $v_3 \in B\backslash\{v_1, v_2\}$. The set $S' = (S\backslash A) \cup B$ cannot be an independent set of G, otherwise, since $|S'| \geq |S|$ and $u \notin S'$, u is α-redundant. Hence, there exists some $u_3 \in S\backslash A$ such that $u_3 \sim v$ for some $v \in B$, say $v = v_1$. Moreover, the maximality of H implies that u_3 cannot be adjacent to every vertex of B. Without loss of generality, assume that $u_3 \nsim v_2$. Then $\{u, u_1, u_2, u_3, v_1, v_2, v_3\}$ induces an F_{24} or an F_{20}, depending on $v_3 \sim u_3$ or not, a contradiction. $\square$

Claim 9. *There are no vertices* $v_1, v_2 \in T$ *and* $u_1, u_2 \in S$ *such that* $\{v_1, v_2, u, u_1, u_2\}$ *induces an* F_3.

Proof. For contradiction, let $v_1, v_2 \in T$ and $u_1, u_2 \in S$ be vertices such that $\{u, u_1, u_2, v_1, v_2\}$ induces an F_3 (without loss of generality, assume that u_2 is the vertex of degree one in the F_3).

Let $H = G[\{u, u_1, u_2, v_1, v_2\}]$. Since $d_H(v_1) > d_H(u)$, there exists some $v_3 \notin V(H)$ such that $u \sim v_3$ and $v_3 \nsim v_1$. By Claim 8, $v_3 \nsim u_1$ or $v_3 \nsim u_2$.

If $v_3 \sim u_1$ and $v_3 \nsim u_2$, then $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{20} or an F_{26}, depending on $v_2 \sim v_3$ or not, a contradiction.

If $v_3 \sim u_2$ and $v_3 \nsim u_1$, then $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{15} or an F_{19}, depending on $v_2 \sim v_3$ or not, a contradiction.

If $v_3 \nsim u_1$ and $v_3 \nsim u_2$, then by Claim 7, there exists some $u_3 \in S$ such that $u_3 \sim v_3, u_3 \sim v_1$. Moreover, by Claim 8, $v_2 \nsim u_3$. Now, $\{u, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{15} or an F_{27}, depending on $v_2 \sim v_3$ or not, a contradiction. □

Lemmas 4, 5 and Claims 7, 8, 9 finish the proof of the theorem. ■

3. A Modification of Algorithm MAX

Algorithms MAX, MIN, VO are heuristic methods, i.e., for a graph G, they give a maximal (and not necessarily maximum) independent set in polynomial time. Some other useful techniques for solving the problem are transformation methods. Transformation methods include graph transformation (changing some part of the graph) with controlled changing of the independence number. Some examples of graph transformation methods are STRUCTION [11], Magnet [16], BAT [18], vertex folding [8], and vertex splitting [2, 32]. Other transformation methods are deletion (or insertion) of an edge [7] or deletion of a vertex (along with deletion of all incident edges) [4], simplicial reduction [33], neighborhood reduction [14], and twin reduction [9]. We can repeat the deletion of a vertex or an edge until we obtain a simple (enough) graph, i.e., a graph for which we already have some efficient algorithm for solving the problem.

In this section, we describe a modification of Algorithm MAX using α-redundant technique. The idea here is to pick a maximum degree vertex u of G, but before deleting u from G, we check if some neighbor of u is α-redundant (in the sense of Lemma 2) and delete it instead of u in the case it is. Algorithm 2 is the pseudo-code of our new algorithm.

Consider an arbitrary simple graph G, let $n = |V(G)|$. Clearly, MMAX gives a maximal independent set. The algorithm repeatedly checks if the remaining graph still contains edges and chooses a maximum degree vertex u (Step 3). In

Algorithm 2 Algorithm MMAX (Modification of MAX).

Input: $G = (V, E)$.

Output: S, a maximal independent set of G.

1: $H_n := G; i := n;\ S := \emptyset$
2: **while** $E(H_i) \neq \emptyset$ **do**
3: Choose a vertex $u \in V(H_i)$ such that $d_{H_i}(u) = \Delta(H_i)$;
4: **for** $v \in N_{H_i}(u)$ **do**
5: **if** there exists some $u_1 \in N_{H_i}(v) \backslash N_{H_i}[u]$ such that there exists no $v_1 \in N_{H_i}(u_1) \backslash N_{H_i}[v]$ **then**
6: $H_{i-1} := H_i - v;\ i := i - 1;$ **break**;
7: **end if**
8: **if** there exist some $u_1, u_2 \in N_{H_i}(v) \backslash N[u]$ and $u_1 \nsim u_2$ such that there exist no $v_1, v_2, v_3 \in N_{H_i}(u) \backslash N_{H_i}[v]$ and $v_1 \sim u$, $v_2 \sim u_1$, $v_3 \sim u_2$ **then**
9: $H_{i-1} := H_i - v;\ i := i - 1;$ **break**;
10: **end if**
11: **end for**
12: $H_{i-1} := H_i - u;\ i := i - 1;$
13: **end while**
14: $S := V(H_i)$;
15: **return** S

Steps 5 and 8, the algorithm checks and removes a vertex $v \in N(u)$ if v is α-redundant by applying Lemma 2 for the case $m = 1$ and $m = 3$, respectively. If no vertex in $N(u)$ is α-redundant in this sense, then Step 12 removes u with the assumption that u is α-redundant. In the case that there is no remaining edge, the remaining vertices form a maximal independent set (Step 14).

We can find a maximum degree vertex u of G in time $\mathrm{O}(n^2)$. Then $|N(u)|$ is at most $n - 1$. For $v \in N(u)$, we can check if v is α-redundant in time $\mathrm{O}(n^2)$ in Step 5, and in time $\mathrm{O}(n^5)$ in Step 8. The removal of vertices will be performed at most n times. Therefore, we obtain the following result.

Theorem 10. *For an arbitrary graph G, Algorithm MMAX finds a maximal independent set in time $\mathrm{O}(n^7)$.*

Given a graph G, denote by $k_{MMAX}(G)$ the minimum cardinality of a maximal independent set found by Algorithm MMAX. The following theorem provides a set of forbidden induced subgraphs for Algorithm MMAX.

Theorem 11. *Let G be an $\mathcal{F}_5$-free graph of order $n \geq 7$, where $\mathcal{F}_5 = \{F_1, F_5, F_7, F_8, F_{14}, F_{15}, F_{18}, F_{20}, F_{21}, F_{24}, F_{28}, F_{29}, F_{30}\}$. Then ${}_{MMAX}(G) = \alpha(G)$.*

Proof. We basically follow the idea of the proof of Theorem 6. Assume that the statement in the theorem is not correct, that means there exists some graph G satisfying the assumption of Theorem 11 and a vertex $u_0 \in V(G)$ such that:

1. u_0 is of maximum degree in G,

2. for every vertex $v \in N(u_0)$, v is not α-redundant (as described in Steps 4.1 and 4.2), and

3. u_0 belongs to every maximum independent set of G,

i.e., the MMAX algorithm will fail.

Let S be a maximum independent set of G, i.e., $u_0 \in S$. Denote $T = V(G)\backslash S$, i.e., $N(u_0) \subset T$. We have the following observations.

Claim 12. *There exists $v_0 \in N(u_0)$ and $u_1, u_2 \in S$ such that $\{v_0, u_0, u_1, u_2\}$ induces $K_{1,3}$, where v_0 is the center-vertex.*

Proof. By Lemma 4, there exist $v_1, v_2 \in N(u_0)$ such that $v_1 \nsim v_2$. Since u_0 is not α-redundant, by Lemma 5, the only case we have to consider is that there exists $u_1, u_2 \in S$ such that $\{u_1, v_1, u_0, v_2, u_2\}$ induces a P_5, because for the remaining cases (say $K_{2,3}$ and banner), we have a desired $K_{1,3}$. Since v_1, v_2 are not α-redundant, by applying Lemma 2 for the case $m = 1$, $N(u_1)\backslash N[v_1]$ and $N(u_2)\backslash N[v_2]$ are not empty. We process by considering the following cases.

Case 1. There exists some $v_3 \in (N(u_1)\backslash N[v_1]) \cap (N(u_2)\backslash N[v_2])$. If $v_3 \sim u_0$, then v_3 is such a vertex v_0 of the conclusion of the claim. Hence, we assume that $v_3 \nsim u_0$. Since $(S\backslash\{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_3\}$ is not independent, there exists some $u_3 \in S\backslash\{u_0, u_1, u_2\}$ such that u_3 is adjacent to at least one vertex among v_1, v_2, v_3. Now, u_3 is adjacent to at least two vertices among $\{v_1, v_2, v_3\}$, otherwise $\{u_0, u_1, u_2, u_3, v_1, v_2\}$ induces an F_7, a contradiction. Hence, v_1 or v_2 is such a vertex v_0 of the conclusion of the claim.

Case 2. There exists some $v_3 \in N(u_1)\backslash(N[v_1] \cup N(u_2))$ and $v_4 \in N(u_2)\backslash(N[v_2] \cup N(u_1))$.

Subcase 2.1. $v_3 \sim v_4$.

Subcase 2.1.1. $v_3 \sim u_0$ (similar for the case $v_4 \sim u_0$). Then $v_3 \sim v_2$, otherwise $\{u_2, v_2, u_0, v_1, u_1, v_3\}$ induces an F_{14}, a contradiction. Moreover, $v_4 \nsim v_1$, otherwise $\{u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_{15}, a contradiction. Hence, $v_4 \sim u_0$, otherwise $\{u_2, v_4, v_3, u_1, v_1, u_0\}$ induces an F_{14}, a contradiction. But now, $\{u_1, v_1, u_0, v_2, u_2, v_4\}$ induces an F_{14}, a contradiction.

Subcase 2.1.2. u_0 is adjacent to neither v_3 nor v_4. If $v_3 \sim v_2$ (similar for the case $v_4 \sim v_1$), then $v_1 \sim v_4$, otherwise $\{v_1, u_0, v_2, u_2, v_4, v_3\}$ induces an F_{14}, a contradiction. Now, $\{u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_{15}, a contradiction.

Hence, we assume that $v_1 \nsim v_4$ and $v_2 \nsim v_3$. Since $(S \backslash \{u_1, u_0, u_2\}) \cup \{v_1, v_2, v_3\}$ (and $(S \backslash \{u_1, u_0, u_2\}) \cup \{v_1, v_2, v_4\}$ either) is not indepedent, there exists some $u_3 \in S \backslash \{u_0, u_1, u_2\}$ such that u_3 is adjacent to at least one vertex among $\{v_1, v_2, v_3, v_4\}$. Hence, u_3 is adjacent to v_1 or v_2, otherwise $\{u_0, u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ induces an F_1, a contradiction. Now, v_1 or v_2 is such a vertex v_0 of the conclusion of the claim.

Subcase 2.2. $v_3 \nsim v_4$.

Subcase 2.2.1. $v_3 \sim u_0$ (similar for the case $v_4 \sim u_0$). Then $v_3 \sim v_2$, otherwise $\{u_2, v_2, u_0, v_1, u_1, v_3\}$ induces an F_{14}, a contradiction. Now, $v_1 \sim v_4$ if and only if $v_4 \sim u_0$, otherwise $\{u_2, v_4, u_0, v_1, u_1, v_3\}$ induces an F_{14}, a contradiction. Hence, we have the two following subcases.

(i) v_4 is adjacent to v_1, u_0. Then $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_{28}, a contradiction.

(ii) v_4 is not adjacent to u_0 and v_1. Since $(S \backslash \{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_4\}$ is not independent, there exists some $u_3 \in S \backslash \{u_0, u_1, u_2\}$ such that u_3 is adjacent to at least one vertex among v_1, v_2, v_4. Hence u_3 is adjacent to v_1 or v_2, otherwise $u_3 \sim v_4$ and $\{u_3, v_4, u_2, v_2, u_0, v_1, u_1\}$ induces an F_1, a contradiction. Now, v_1 or v_2 is such a vertex v_0 of the conclusion of the lemma.

Subcase 2.2.2. u_0 is not adjacent to v_3, v_4.

(i) $v_3 \sim v_2$ (similar for the case $v_4 \sim v_1$). Then $v_4 \nsim v_1$, otherwise $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$ induces an F_{29}, a contradiction. Since $(S \backslash \{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_4\}$ is not independent, there exists some $u_3 \in S \backslash \{u_0, u_1, u_2\}$ such that u_3 is adjacent to at least one vertex among v_1, v_2, v_4. Hence u_3 is adjacent to v_1 or v_2, otherwise $u_3 \sim v_4$ and $\{u_3, v_4, u_2, v_2, u_0, v_1, u_1\}$ induces an F_1, a contradiction. Now, v_1 or v_2 is such a vertex v_0 of the conclusion of the claim.

(ii) $v_3 \nsim v_2$, i.e., v_3 is not adjacent to u_0, v_2, and v_4 and similarly v_4 is not adjacent to v_1 and u_0. Now, $\{v_3, u_1, v_1, u_0, v_2, u_2, v_4\}$ induces an F_1, a contradiction. □

Claim 13. *Let* $v_0 \in N(u_0)$ *and* $u_1, u_2 \in N_S(v_0) \backslash \{u_0\}$ *be two non-adjacent vertices. Then there exist some* v_1, v_2 *such that* $\{u_0, u_1, u_2, v_0, v_1, v_2\}$ *induces a* $K_{3,3}$.

Proof. Since $\{v_0, u_0, u_1, u_2\}$ induces a $K_{1,3}$ and v_0 is not α-redundant, by Lemma 2 (for the case $m = 3$), there exist some $v_1, v_2, v_3 \in V(G)$ such that $\{v_0, v_1, v_2, v_3\}$ is independent and $u_i \sim v_{i+1}$ for $i = 0, 1, 2$. Let $X = \{u_0, u_1, u_2\}$. By the symmetry, we consider the following cases.

Case 1. $|N_X(v_i)| = 3$ for at least two positive integers i. Then $\{u_0, u_1, u_2, v_0, v_1, v_2, v_3\}$ induces a $K_{3,3}$.

Case 2. $|N_X(v_2)| = 2$ and $N_X(v_2) = \{u_1, u_2\}$. Then $\{u_0, u_1, u_2, v_0, v_1, v_2\}$ induces an F_{14} in the case $|N_X(v_1)| = 1$, an F_{15} in the case $|N_X(v_1)| = 2$ or an F_{20} in the case $|N_X(v_1)| = 3$, a contradiction.

Case 3. $|N_X(v_1)| = |N_X(v_2)| = 1$ and $|N_X(v_3)| = 3$. Then $\{u_0, u_1, u_2, v_0, v_1, v_2, v_3\}$ induces an F_{30}, a contradiction.

Case 4. $|N_X(v_i)| = 1$ for $i = 1, 2, 3$. Let H be a maximal (inclusion sense) graph consisting of k induced paths of length two of the form $v_0u_iv_{i+1}$ where v_0 is the common initial vertex. Since $(S\backslash\{u_0, u_i\}) \cup \{v_0, v_{i+1}\}$ is not independent for every i, for each i $(2 \leq i \leq k)$, there exists some $w_i \in S\backslash\{u_0, u_{i-1}\}$ such that w_i is adjacent to v_0 or v_i. The rest of the proof is processed by considering the following subcases.

Subcase 4.1. There exists some i such that $w_i \sim v_i$, without loss of generality, assume that $w_2 \sim v_2$. By Lemma 3, there exists some $u \in N_S(v_1)\backslash\{u_0\}$. If $u = w_2$, then $\{u_0, u_1, u_2, w_2, v_0, v_1, v_2\}$ induces an F_{15} or an F_7 depending on $w_2 \sim v_0$ or not, a contradiction. If $u \neq w_2$, then $\{u, v_1, u_0, v_0, u_1, v_2, w_2\}$ induces an F_1, an F_7, an F_{14}, or an F_{15} depending on the adjacency between $\{u, w_2\}$ and $\{v_0, v_1, v_2\}$, a contradiction.

Subcase 4.2. There exists some $w \in S$ such that w is not adjacent to any v_i for $i \geq 2$ and $w \sim v_0$. Then $w \nsim v_1$, otherwise $\{v_2, u_1, v_0, u_0, v_1, w\}$ induces an F_{15}, a contradiction. Since v_0 is not α-redundant, by Lemma 2 (for the case $m = 1$), there exists some $t \in N(w)\backslash N[v_0]$ and by the maximality of H, t is adjacent to some vertex u_i or v_i. Moreover, by Lemma 3, there exists some $u \in S\backslash V(H)$ such that $v_1 \sim u$.

Subcase 4.2.1. t is not adjacent to any v_i. Then $t \sim u_i$ for some u_i and t is adjacent to all others u_j, otherwise $\{v_{j+1}, u_j, v_0, u_i, t, w\}$ induces an F_{14}, a contradiction. Thus, $\{u, u_0, w, v_0, v_1, t\}$ induces an F_{15} or an F_{14} depending on $t \sim u$ or not, a contradiction.

Subcase 4.2.2. $t \sim v_i$ for some $i \geq 1$.

(i) $t \nsim v_1$. Then, without loss of generality, assume that $t \sim v_2$. Hence, t is not adjacent to u_0, u_1, and u, otherwise $\{u_0, u_1, u, w, v_0, v_1, t\}$ induces an F_7, an F_{14}, or an F_{15}, a contradiction. Now, $\{t, v_2, u_1, v_0, u_0, v_1, u\}$ induces an F_1, a contradiction.

(ii) $t \sim v_1$. We consider the three following subcases.

(a) t is adjacent to u_i for some $i \geq 1$. Then $t \sim u$, otherwise $\{u, v_1, t, w, v_0, u_i\}$ induces an F_{14}, a contradiction. Now, $\{t, u, v_0, v_1, u_0, u_i\}$ induces an F_{18} or an F_5 depending on $t \sim u_0$ or not, a contradiction.

(b) $t \nsim u_i$ for $i \geq 1$ and $t \sim u_0$. Then $t \sim v_{i+1}$ for every $i \geq 1$, otherwise $\{v_{i+1}, u_i, v_0, u_0, t, w\}$ induces an F_{14}, a contradiction. Now, $\{v_1, t, v_0, v_2, v_3, u_1, u_2\}$ induces an F_7, a contradiction.

(c) $t \nsim u_i$ for every i. Then $t \nsim u$, otherwise $\{t, w, v_0, u_0, v_1, u\}$ induces an F_5, a contradiction. Now, $\{u, v_1, t, w, v_0, u_1, v_2\}$ induces an F_1, a contradiction.

□

Claim 14. *There exist no* $u_1, u_2 \in S$ *and* v_1, v_2, v_3 *such that* $\{u_0, u_1, u_2, v_1, v_2, v_3\}$ *induces a* $K_{3,3}$.

Proof. Indeed, suppose there exist $u_1, u_2 \in S$ and v_1, v_2, v_3 such that $\{u_0, u_1, u_2, v_1, v_2, v_3\}$ induces a $K_{3,3}$. Let H be a maximal (inclusion sense) induced complete bipartite subgraph of G with parts A and B such that $A = \{u_0, u_1, \ldots, u_p\} \subseteq S$ and $B = \{v_1, v_2, \ldots, v_q\} \subseteq T$ ($p \geq 2$ and $q \geq 3$).

Case 1. $p < q$. The set $S' = (S \backslash A) \cup B$ is not an independent set of G, otherwise we have a maximum independent set not containing u, a contradiction. Hence, there exists some $u \in S \backslash A$ such that $u \sim v_i$ for some $v_i \in B$, say $u \sim v_1$. Moreover, the maximality of H implies that u is not adjacent to some vertex of B. Without loss of generality, assume that $u \nsim v_2$. Then u is not adjacent to any vertex $v_i \in B \backslash \{v_1\}$, otherwise $\{u, u_0, u_1, v_1, v_2, v_i\}$ induces an F_{20}, a contradiction. Now, $\{u, u_0, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{24}, a contradiction.

Case 2. $3 \leq q \leq p$. Since $d(u_0) \geq d(v_2)$, there exists $v \in T \backslash V(H)$ such that $v \sim u_0$ and $v \nsim v_2$.

Subcase 2.1. $N_A(v) = A$. By the maximality of H, $v \sim v_i$ for some $v_i \in B$. Without loss of generality, assume that $v \sim v_1$. Now, $\{u_0, u_1, u_2, u_3, v_1, v_2, v\}$ induces an F_8, a contradiction.

Subcase 2.2. $v \nsim u_i$ for some u_i, without loss of generality, assume that $v \nsim u_1$. Then v is not adjacent to any vertex u_i, $i \geq 2$, otherwise $\{u_0, u_1, u_i, v, v_1, v_2\}$ induces an F_{21} or an F_{20} depending on $v \sim v_1$ or not, a contradiction. Now, by Lemma 3, there exists some $u \in N_S(v) \backslash A$.

Subcase 2.2.1. u is adjacent to some vertex v_i of B, without loss of generality, assume that $u \sim v_1$. By the maximality of H, u is not adjacent to some vertex v_i of B different from v_1, without loss of generality, assume that $u \nsim v_2$. Thus, $\{u, u_0, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{20} or an F_{24} depending on $u \sim v_3$ or not, a contradiction.

Subcase 2.2.2. u is not adjacent to any vertex v_i of B. So, v is adjacent to every vertex v_i of $B \backslash \{v_2\}$, otherwise $\{u, v, u_0, v_2, u_1, v_i\}$ induces an F_{14}. Now, $\{v, u_1, u_2, v_1, v_2, v_3\}$ induces an F_{20}, a contradiction. □

Together, all above claims give us the theorem. ■

4. ALGORITHM MAX AND $K_{1,m}$ REDUCTION

In the literature, there are some results about polynomial time solution for the MIS problem in subclasses of $K_{1,m}$-free graphs, for example $(P_k, K_{1,m})$-free graphs [22], $(S_{1,2,j}$, banner, $K_{1,m})$-free graphs, $(S_{1,2,3}, B_k, K_{1,m})$-free graphs [19] and

$(S_{2,2,2}, B_2(F_{14}), \text{domino}(F_{15}), M_3(F_{30}))$-free graphs [20]. So one possible heuristic approach for the MIS problem is to remove all maximum degree vertices which are the centers of some $K_{1,m}$ and then apply one polynomial solution for some subclass of $K_{1,m}$-free graphs. In this section, we describe the method of combining Algorithm MAX and $K_{1,m}$ reduction. Recall that in [20] we have shown that the MIS problem is solvable in time $\mathrm{O}(n^{m+2})$ for $(S_{2,2,2}, F_{14}, F_{15}, F_{30}, K_{1,m})$-free graphs. In this section, we fix $m \geq 3$. Now, we combine Algorithm MAX and this result to obtain the Algorithm MMAX-l (see Algorithm 3).

Algorithm 3 Algorithm MMAX-l (Combination of MAX and $K_{1,l}$ reduction).

Input: $G = (V, E)$, an $(S_{2,2,2}, F_{14}, F_{15}, F_{30})$-free graph.
Output: S, an independent set of G.

1: $H_n := G; i := n;\ S := \emptyset$;
2: **while** H_i contains a $K_{1,l}$ **do**
3: Choose a vertex $u \in V(H_i)$ such that u is the center-vertex of some $K_{1,l}$ and u is of maximum degree among center vertices of all induced copies of $K_{1,l}$ in H_i;
4: $i := i - 1;\ H_i := H_{i+1} - u$;
5: **end while**
6: Let S be the maximum independent set of H_i obtained by the technique for $(S_{2,2,2}, F_{14}, F_{15}, F_{30}, K_{1,l})$-free graphs described in [20];
7: **return** S

To find and remove center vertices of all $K_{1,l}$'s can be done in time $\mathrm{O}(n^{l+2})$ and Step 6 can be performed in time $\mathrm{O}(n^{l+2})$ [20]. Hence, Algorithm MMAX-l can be performed in time $\mathrm{O}(n^{l+2})$ and gives us an independent set. Given a graph G, we denote by $k_{MMAX-l}(G)$ the minimum cardinality of an independent set given by Algorith MMAX-l. The following theorem describes a set of forbidden induced subgraphs for this algorithm.

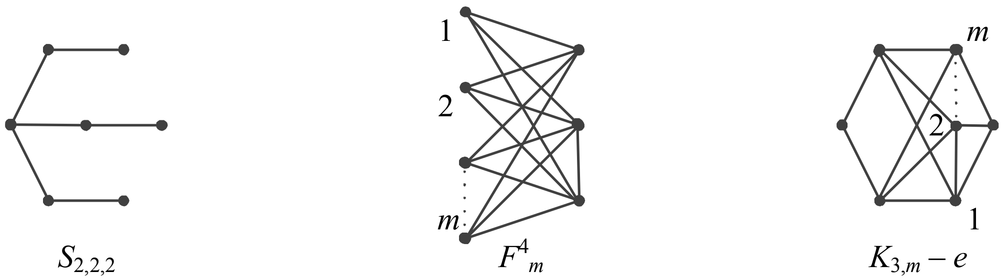

Figure 4. Some forbidden subgraphs for Algorithm MMAX-l ($l \geq 2m - 2$).

Theorem 15 (Forbidden subgraphs for Algorithm MMAX-l)**.** *Let G be an $\mathcal{F}_6$-free graph of order $n \geq 7$, where $\mathcal{F}_6 = \{S_{2,2,2}, F_{14}, F_{15}, F_{30}, K_{3,m} - e, F_4^m\}$. Then*

$k_{MMAX-l}(G) = \alpha(G)$ *for* $l \geq 2m - 2$ *and* $m \geq 4$.

Proof. We will follow the proof of Theorem 6. Assume that the statement in the theorem is not correct. That means there exists a (without loss of generality) connected graph G satisfying the assumption of the theorem and some vertex $u \in V(G)$ such that

1. u is the center-vertex of some $K_{1,l}$ $\{u, v_1, v_2, \ldots, v_l\}$,
2. u is of maximum degree among such center vertices of $K_{1,l}$'s in G, and
3. u belongs to every maximum independent set of G,

i.e., the MMAX-l algorithm will fail.

Let S be a maximum independent set of G (hence, $u \in S$) and $T = V(G)\backslash S$. Let $W = \{v_1, \ldots, v_l, \ldots, v_p\}$ be the maximal (inclusion sense) independent set such that $W \subset N(u) \subset T$. By Lemma 2, there exist $u_1, u_2, \ldots, u_p \in S$ such that there exists a perfect matching between $\{u_i\}$ and $\{v_i\}$. Without loss of generality, assume that $u_i \sim v_i$ for $1 \leq i \leq p$. Let $B = \{u, u_1, \ldots, u_l\}$. Then $H = (W, B, E)$ is a bipartite graph.

Claim 16. *There exists some* $u_i \in B\backslash\{u\}$ *such that* u_i *is adjacent to every vertex of* W.

Proof. Since H does not induce an $S_{2,2,2}$, there exist some u_i, v_j such that $i \neq j$ and $u_i \sim v_j$. Let u_i be a vertex such that $|N_W(u_i)|$ is maximum. Assume that $u_i \nsim v_k$ for some k. If u_k is not adjacent to any vertex of $N_W(u_i)$, then $\{u_i, v_i, v_j, u, v_k, u_k\}$ induces an F_{14} for some $v_j \in N_W(u_i)\backslash\{v_i\}$, a contradiction. Now, assume that $u_k \sim v_j$ for some $v_j \in N_W(u_i)$. By the choice of u_i, $u_k \nsim v_l$ for some $v_l \in N_W(u_i)$. Now, $\{u, v_l, u_i, v_k, u_k, v_j\}$ induces an F_{15}, a contradiction. Hence, u_i is adjacent to every vertex of W. □

Without loss of generality, assume that u_p is adjacent to every vertex of W.

Claim 17. H *is* $2K_2$*-free.*

Proof. Assume that H contains an induced $2K_2$ $u_iv_j + u_kv_r$, for some $u_i, u_k \in B\backslash\{u\}$ and $v_j, v_r \in W$. Note that $i, k \neq p$. Now, $W \cup \{u, u_p, u_i, u_k, v_j, v_r, v_p\}$ induces an F_{30} in the case u_i, u_k are not adjacent to v_p. If $u_i \sim v_p$ (similar for the case $u_k \sim v_p$), then $\{u, u_i, u_k, v_j, v_l, v_p\}$ induces an F_{15} or an F_{14} depending on $u_k \sim v_p$ or not, a contradiction. □

Also by an observation in [13], since H is a $2K_2$-free connected bipartite graph, we have the following observation.

Claim 18 [13]**.** *The vertices of each part of H can be linearly ordered under inclusion of their neighborhood.*

This leads us to the following observation.

Claim 19. *H induces a $K_{p,p}$.*

Proof. By Claim 18, without loss of generality, assume that $N_W(u_i) \subset N_W(u_j)$ for $i < j$, i.e., $u_j \sim v_i$ for every i, j such that $j \geq i$. Moreover, by the existence of a perfect matching between $B \backslash \{u\}$ and W, we have that $|N_W(u_i)| \geq i$ for $i = 1, 2, \ldots$.

If $u_i \nsim v_j$, for some i, j such that $j > i \geq m-1$, then $\{u, u_{2m-2}, u_i, v_j, v_1, v_2, \ldots, v_{m-1}\}$ induces a $K_{3,m} - e$, a contradiction. Hence, $N_W(u_i) = W$ for $i \geq m-1$.

If $u_i \nsim v_j$ for some i, j such that $j > i$ and $m-2 \geq i \geq 2$, then $\{v_1, v_i, v_j, u_i, u_{m-1}, u_m, \ldots, u_{2m-3}\}$ induces a $K_{3,m} - e$, a contradiction. Thus, $N_W(u_i) = W$ for $i \geq 2$ and hence, $\{u, u_2, \ldots, u_p\} \cup W$ induces a $K_{p,p}$. □

Now, without loss of generality, let H be a maximal (inclusion sense) induced complete bipartite subgraph of G with parts A and W such that $\{u, u_2, \ldots, u_p\} \subseteq A \subseteq S$. Let $A = \{u, u_2, \ldots, u_p, \ldots, u_q\}$.

Claim 20. *There exists some vertex $v \in T \backslash W$ such that $v \sim u$ and $v \nsim v_i$ for some $v_i \in W$.*

Proof. If $p = |W| < |A|$, then since $d(u) \geq d(v_1)$ and v_1 is also a center-vertex of some $K_{1,l}$ (say $v_1(u, u_1, \ldots, u_{l-1})$), there exists $v \in T \backslash V(H)$ such that $v \sim u$ and $v \nsim v_1$.

Consider the case $p = |W| = |A|$. Then the set $S' = (S \backslash A) \cup W$ cannot be an independent set of G, otherwise, S' is a maximum independent set not containing u. Hence, there exists some $u' \in S \backslash A$ such that $u' \sim v$ for some $v \in W$. Assume that $u' \sim v_1$. Now, again, since $d(u) \geq d(v_1)$, there exists $v \in T \backslash V(H)$ such that $v \sim u$ and $v \nsim v_1$. □

Without loss of generality, assume that $v \nsim v_2$. Moreover, by the maximality of W, $v \sim v_i$ for some $v_i \in W$. Wihout loss of generality, assume that $v \sim v_1$. Note that v has at most $m-1$ neighbors in A, otherwise m neighbors of v in A, together with v, v_1, v_2 induce an F_m^4, a contradiction. Moreover, v has at most $m-1$ neighbors in W, otherwise $m-1$ neighbors of v in W, together with v, u_i, u_j, and v_2 induces a $K_{3,m} - e$ for u_i, u_j are two non-neighbors of v in A. We consider the two following cases.

Case 1. v is not adjacent to any vertex of $A \backslash \{u\}$, then there exists some $u' \in S \backslash A$ such that $u' \sim v$, otherwise $(S \backslash u) \cup \{v\}$ is a maximum independent set not containing u, a contradiction. By the maximality of H, there exists some vertex $v_i \in W$ such that $u' \nsim v_i$. If u' is adjacent to two vertices $v_j, v_k \in W$, then

$\{u, u', u_2, \ldots, u_{m-1}, v_i, v_j, v_k\}$ induces a $K_{3,m} - e$, a contradiction. Hence, u' is adjacent to at most one vertex v' of W. Now, $\{v_j, u_2, v_k, u, v, u'\}$ induces an F_{14}, for v_j, v_k are two non-neighbors of both v and u', a contradiction.

Case 2. v is adjacent to some vertex of $A\backslash\{u\}$, without loss of generality, assume that $v \sim u_2$. Then $m-1$ non-neighbors of v in W, together with v, u, u_2, and u_i induces a $K_{3,m} - e$, for some non-neighbor u_i of v in A, a contradiction. ■

5. Comparison

First, we have the following simple observations.

Claim 21. (1) *F_7, F_{29}, and F_{30} induce F_2 and F_{25}.*

(2) *F_{18} induces F_3 and F_{25}.*

(3) *$F_{14}, F_{15}, F_{19}, F_{20}, F_{21}, F_{24}$, and F_{30} induce F_3.*

(4) *F_8 induces F_4.*

(5) *$F_{39}, S_{2,2,2}, F_{14}$, and F_{15} induce F_{25}.*

(6) *F_4^m induces F_8.*

(7) *$K_{3,m} - e$ induces F_{20}.*

Hence, we have the following results.

Proposition 22. (1) *The class of $\mathcal{F}_1$-free graphs and the class of $\mathcal{F}_4$-free graphs are subclasses of the $\mathcal{F}_5$-free graph class.*

(2) *The class of $\mathcal{F}_4$-free graphs is a subclass of the $\mathcal{F}_6$-free graph class.*

Proposition 22 shows that Algorithm MMAX and Algorithm MMAX-l are at least as good as VO and MAX in the sense of forbidden subgraph classes. Now, we compare the heuristic algorithms MAX, MMAX, MIN, VO in the aspect of performance for particular graphs. For the MIN algorithm and the VO algorithm, the full instances can be found in [31] and [27], respectively. For the sake of proofs, we will only recall the main ideas of the algorithms.

For each step, the MIN algorithm chooses a vertex of minimum degree in the remaining graph, puts it in the current (maximal) independent set and deletes the vertex and all its neighbours. The process will be repeated until there is no more remaining vertex.

The VO algorithm first makes a list of vertices of the graph, in which the vertices are ordered in an increasing order of degree. The algorithm sequentially puts vertices into the independent set in the list only if they are not adjacent with any (already) chosen vertex in the current independent set. The process will be repeated until we process through all the list.

We denote by $k_{VO}(G), k_{MIN}(G)$ the minimum cardinalities of independent sets obtained when applying VO, MIN on G, respectively.

Theorem 23. *For every integer $p > 2$ there is a graph G such that*

$$k_{MAX}(G) - k_{MIN}(G) = k_{MAX}(G) - k_{VO}(G) = k_{MMAX}(G) - k_{MIN} =$$

$$= k_{MMAX} - k_{VO} = p - 2.$$

Proof. Let G_1 be an induced coppy of K_p, a complete graph of order p, G_2 be the graph consisting of p independent vertices, and G_3 be the single vertex graph. Let G be the graph obtained by joining an edge for all pairs of vertices x, y for $x \in V(G_i), y \in V(G_{i+1})$, for $i = 1, 2$. Then, clearly, $k_{MIN}(G) = k_{VO}(G) = 2$, while $k_{MMAX}(G) = k_{MAX}(G) = p$. ■

The following result shows that the difference between k_{MMAX} on one hand, and k_{MAX}, k_{VO} and k_{MIN} on the other hand, can be arbitrarily large.

Theorem 24. *For every integer $p > 2$, there is a graph G such that*

$$k_{MMAX}(G) - k_{MAX}(G) = k_{MMAX}(G) - k_{VO(G)} = k_{MMAX}(G) - k_{MIN}(G) = p - 2.$$

Proof. Let G_1, G_3 be induced copies of K_p and G_2 be the graph consisting of p independent vertices. Let G be the graph obtained by joining an edge for all pairs of vertices x, y for $x \in V(G_i), y \in V(G_{i+1})$, for $i = 1, 2$. Then, clearly, $k_{MAX}(G) = k_{MIN}(G) = k_{VO}(G) = 2$, while $k_{MMAX}(G) = p$. ■

Remark. Note that the graph G constructed in the proof of Theorem 24 is K_{p+1} free. Hence, Algorithm MMAX-l always finds a maximum independent set (of cardinality p) for $l \geq p + 1$.

6. Conclusion

Recently, the MIS problem in P_5-free graphs has been the subject of many researchs. In [5], it is proved that the problem can be solved for (P_5, $K_{3,3} - e$ (F_{20}), twin-house (F_{21}))-free graphs in time O(n^6). In [25], the authors described a method solving the problem for $(P_5, K_{3,3} - e)$-free graph. In [21], the first approach for solving the problem for P_5-free graphs in time O(n^{12}) is presented. The three results are generalizations of the forbidden subgraphs for Algorithm MAX. On the other hand, it is obvious that Algorithm MAX has the complexity at most O(n^2).

Moreover, there are not many results about polynomial time solution for the MIS problem in some subclasses of P_7-free graphs, except for (P_7, banner(F_3))-free graphs [3] and $(P_7, K_{1,m})$-free graphs [22]. We have a similar situation for

$S_{2,2,2}$-free graphs, except ($S_{2,2,2}$, banner)-free graphs [12], ($S_{2,2,2}, B_2$, domino,M_3, $K_{m,m}$)-free graphs [20], and $S_{2,2,2}$-free subcubic graphs [26].

The Algorithm MMAX, as well as MIN and VO, show that the problem can be solved efficiently in some subclasses of class of P_7-free graphs. Our method is a combination between a heuristic method (Algorithm MAX) and a conditional exact method (α-redundant technique). The class of forbidden induced subgraphs of the new algorithm, Algorithm MMAX, is a super class of both MAX and VO. The class of forbidden induced subgraphs of Algorithm MMAX-l also gives us a polynomial solution for the MIS problem in a subclass of $S_{2,2,2}$-free graphs. Note that the complexity of the problem for the class of P_7-free graphs or the class of $S_{2,2,2}$-free graphs is still an open question.

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REFERENCES

[1] V.E. Alekseev, *On the ocal restrictions effect on the complexity of finding the graph independence number*, in: Combinatorial-Algebraic Methods in Applied Mathematics, A. Markov (Ed(s)), (Gorkiy University, 1983) 3–13, in Russian.

[2] V.E. Alekseev, *A polynomial algorithm for finding maximum independent sets in fork-free graphs*, Discrete Anal. Operation Research Serie 1 **6(4)** (1999) 3–19, in Russian.

[3] V.E. Alekseev, *Augmenting graphs for independent sets*, Discrete Appl. Math. **145** (2004) 3–10.
doi:10.1016/j.dam.2003.09.003

[4] A. Billionnet, *Reductions and optimality conditions in the problem of a maximum weighted stable set*, RAIRO Recherche Opérationnelle **15(3)** (1999) 213–231, in French.

[5] A. Brandstädt and V.V. Lozin, *A note on α-redundant vertices in graphs*, Discrete Appl. Math. **108** (2001) 301–308.
doi:10.1016/S0166-218X(00)00239-0

[6] A. Brandstädt, H.O. Le, V.B. Le, *On α-redundant vertices in P_5-free graphs*, Inform. Process. Lett. **82** (2002) 119–122.
doi:10.1016/S0020-0190(01)00265-4

[7] L. Butz, P.L. Hammer and D. Haussmann, *Reduction methods for the vertex packing problem*, in: Proceedings of the 17th Conference on Probability Theory, Brasov, Romania, 1982, M. Iosifescu, T. Postelnicu, S. Grigorescu (Ed(s)), (Utrecht, VNU Science Press, 1985) 73–79.

[8] J. Chen, I.A. Kanji and W. Jia, *Vertex cover: further observations and further improvement*, J. Algorithms **41** (2001) 280–301.
doi:10.1006/jagm.2001.1186

[9] D.G. Corneil, H. Lerchs and L.S. Burlingham, *Complement reducible graphs*, Discrete Appl. Math. **3** (1981) 163–174.
doi:10.1016/0166-218X(81)90013-5

[10] D.G. Corneil, Y. Perl and L.K. Stewart, *A linear recognition algorithm for cographs*, SIAM J. Comput. **14** (1985) 926–934.
doi:10.1137/0214065

[11] C. Ebenegger, P.L. Hammer and D. Werra, *Pseudo-Boolean functions and stability of graphs*, Ann. Discrete Math. **19** (1984) 83–98.

[12] M.U. Gerber and V.V. Lozin, *Robust algorithms for the stable set problem*, Graphs Combin. **19** (2003) 347–356.
doi:10.1007/s00373-002-0517-5

[13] M.U. Gerber and V.V. Lozin, *On the stable set problem in special P_5-free graphs*, Discrete Appl. Math. **125** (2003) 215–224.
doi:10.1016/S0166-218X(01)00321-3

[14] M.C. Golumbic and P.L. Hammer, *Stability in circular arc graphs*, J. Algorithms **9** (1988) 314–320.
doi:10.1016/0196-6774(88)90023-5

[15] J.R. Griggs, *Lower bounds on the independence number in terms of the degree*, J. Combin. Theory (B) **34** (1983) 22–39.
doi:10.1016/0095-8956(83)90003-5

[16] P.L. Hammer and A. Hertz, On a transformation which preserves the stability number(RUTCOR Research report, Rutgers University1991) 69–91.

[17] J. Harant, Z. Ryjáček and I. Schiermeyer, *Forbidden subgraphs implying the MIN-algorithm gives a maximum independent set*, Discrete Math. **256** (2002) 193–201.
doi:10.1016/S0012-365X(02)00571-X

[18] A. Hertz, *On the use Boolean methods for the computation of the stability number*, Discrete Appl. Math. **76** (1997) 183–203.
doi:10.1016/S0166-218X(96)00124-2

[19] A. Hertz and V.V. Lozin, *The maximum independent set problem and augmenting graphs*, in: Graph Theory Combin. Optim., D. Avis, A. Hertz, O. Marcotte (Ed(s)), (Springer, 2005) 69–99.

[20] N.C. Lê, C. Brause and I. Schiermeyer, *New sufficient conditions for α-redundant vertices*, Discrete Math. (2014) in press.
doi:10.1016/j.disc.2014.07.002

[21] D. Lokshtanov, M. Vatshelle and Y. Villanger, Independent set in P_5-free graphs in polynomial time, Technical Report (2013).
www.ii.uib.no/∼martinv/Papers/ISinP5free.pdf

[22] V.V. Lozin and D. Rautenbach, *Some results on graphs without long induced paths*, Inform. Process. Lett. **88** (2004) 167–171.
doi:10.1016/j.ipl.2003.07.004

[23] V.V. Lozin and M. Milanič, *Maximum independent sets in graphs of low degree*, in: Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms SoDA' 07, H. Gabow (Ed(s)), (Society for Industrial and Applied Mathematics, 2007) 874–880.

[24] V.V. Lozin and M. Milanič, *On finding augmenting graphs*, Discrete Appl. Math. **156** (2008) 2517–2529.
doi:10.1016/j.dam.2008.03.008

[25] V.V. Lozin and R. Mosca, *Maximum independent sets in subclasses of P_5-free graphs*, Inform. Process. Lett. **109** (2009) 319–324.
doi:10.1016/j.ipl.2008.11.005

[26] V.V. Lozin, J. Monnot and B. Ries, *On the maximum independent set problem in subclasses of subcubic graphs*, in: International Workshop on Combinatorial Algorithms IWOCA 2013, T. Lecroq, L. Mouchard (Ed(s)), (Springer, 2013) 314–326.

[27] N.V.R. Mahadev and B.A. Reed, *A note on vertex orders for stability number*, J. Graph Theory **30** (1999) 113–120.
doi:10.1002/(SICI)1097-0118(199902)30:2⟨113::AID-JGT5⟩3.0.CO;2-#

[28] G.J. Minty, *On maximal independent sets of vertices in claw-free graphs*, J. Combin. Theory (B) **28** (1980) 284–304.
doi:10.1016/0095-8956(80)90074-X

[29] R. Mosca, *Independent sets in (P_6,diamond)-free graphs*, Discrete Math. Theor. Comput. Sci. **11** (2009) 125–140.

[30] R. Mosca, *Stable sets for $(P_6, K_{2,3})$-free graphs*, Discuss. Math. Graph Theory **32** (2012) 387–401.
doi:10.7151/dmgt.1598

[31] O.J. Murphy, *Lower bounds on the stability number of graphs computed in terms of degrees*, Discrete Math. **90** (1991) 207–211.
doi:10.1016/0012-365X(91)90357-8

[32] O.J. Murphy, *Computing independent sets in graphs with large girth*, Discrete Appl. Math. **35** (1992) 167–170.
doi:0.1016/0166-218X(92)90041-8

[33] D.J. Rose, R.E. Tarjan and G.S. Lueker, *Algorithmic aspects of vertex elimination of graphs*, SIAM J. Comput. **5** (1976) 266–283.
doi:10.1137/0205021

[34] N. Sbihi, *Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans Étoile*, Discrete Math. **29** (1980) 53–76.

[35] I.E. Zverovich, *Minimum degree algorithms for stability number*, Discrete Appl. Math. **132** (2004) 211–216.
doi:10.1016/0012-365X(90)90287-R

[36] I.E. Zverovich and O.I. Zverovich, *Stability number in subclasses of P_5-free graphs*, Appl. Math. J. Chinese Univ. Ser. B **19** (2004) 125–132.
doi:10.1007/s11766-004-0045-6

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