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GENERALIZED FRACTIONAL TOTAL COLORINGS OF GRAPHS

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Abstract

Let $\mathcal{P}$ and $\mathcal{Q}$ be additive and hereditary graph properties and let r, s be integers such that $r \geq s$. Then an $\frac{r}{s}$*-fractional* $(\mathcal{P}, \mathcal{Q})$*-total coloring* of a finite graph $G = (V, E)$ is a mapping f, which assigns an s-element subset of the set $\{1, 2, \ldots, r\}$ to each vertex and each edge, moreover, for any color i all vertices of color i induce a subgraph with property $\mathcal{P}$, all edges of color i induce a subgraph with property $\mathcal{Q}$ and vertices and incident edges have been assigned disjoint sets of colors. The minimum ratio of an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of G is called *fractional* $(\mathcal{P}, \mathcal{Q})$*-total chromatic number* $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{r}{s}$. We show in this paper that $\chi''_{f,\mathcal{P},\mathcal{Q}}$ of a graph G with $o(V(G))$ vertex orbits and $o(E(G))$ edge orbits can be found as a solution of a linear program with integer coefficients which consists only of $o(V(G)) + o(E(G))$ inequalities.

Keywords: fractional coloring, total coloring, automorphism group.

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1. Introduction

Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E. We denote by n the number of vertices of G and by e the number of edges of G. By *elements* we will mean the vertices and the edges of a graph G.

A *total coloring* of a graph G is a coloring of the vertices and the edges such that each two elements which are adjacent or incident obtain distinct colors. The minimum number of colors of a total coloring of G is called *total chromatic number* $\chi''(G)$ of G.

The following conjecture is known as the Total Coloring Conjecture and it was formulated independently in the 1960s by Behzad [1] and Vizing [12]. It has been verified for several special classes of graphs, including for example complete graphs (see [2, 6] for surveys).

Conjecture 1. If G is a graph with maximum degree $\Delta(G)$, then $\chi''(G) \leq \Delta(G) + 2$.

There are several possibilities how to define the fractional version of total colorings. All definitions can be found in [11]. In this paper we will use only two of them.

An $\frac{r}{s}$-*fractional total coloring* of a graph G is a coloring of the vertices and the edges such that each vertex and each edge has been assigned an s-element subset of the set $\{1, 2, \ldots, r\}$ and each two adjacent or incident elements receive disjoint sets of colors. The *fractional total chromatic number* $\chi''_f(G)$ of G is the infimum ratio $\frac{r}{s}$ of an $\frac{r}{s}$-fractional total coloring of G. Kilakos and Reed [10] proved that $\chi''_f(G) \leq \Delta(G) + 2$ for any graph G.

In this paper we deal with generalized fractional total colorings of graphs. We denote the class of all finite simple graphs by $\mathcal{I}$. A *graph property* $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$ (we mean the disjoint union of graphs). A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$ (see [3, 5]).

We use the following standard notations for specific hereditary properties:

$$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\},$$

$$\mathcal{O}^k = \{G \in \mathcal{I} : \chi(G) \leq k\},$$

$$\mathcal{D}_k = \{G \in \mathcal{I} : \text{each subgraph of } G \text{ contains a vertex of degree at most } k\},$$

$$\mathcal{T} = \{G \in \mathcal{I} : G \text{ is a planar graph}\},$$

$$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\},$$

$$\mathcal{O}_k = \{G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\},$$

$$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\},$$

where $\chi(G)$ is the *chromatic number*. In the following we will only consider additive and hereditary graph properties.

The *completeness* of the property $\mathcal{P}$ is an interesting invariant of a graph property and it is defined as $c(\mathcal{P}) = \sup\{i : K_{i+1} \in \mathcal{P}\}$. Note that $c(\mathcal{I}_k) = c(\mathcal{O}_k) = c(\mathcal{D}_k) = c(\mathcal{S}_k) = k$, $c(\mathcal{T}) = 3$, $c(\mathcal{O}) = 0$.

Borowiecki and Mihók [5] dealt with graph properties and showed that the set of all additive and hereditary properties is a complete distributive lattice $(\mathbb{L}^a, \subseteq)$, where $\mathcal{O}$ is the smallest element of it and $\mathcal{I}$ is the greatest one. The set of properties $\mathcal{P} \in \mathbb{L}^a$ with $c(\mathcal{P}) = k, k \in \mathbb{N}$, is also a complete distributive lattice $(\mathbb{L}^a_k, \subseteq)$ with the smallest element $\mathcal{O}_k$ and the greatest element $\mathcal{I}_k$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two graph properties. We consider a total coloring of a graph G such that adjacent elements can have the same color assigned but we require that subgraphs of G induced by a set of vertices of the same color must have property $\mathcal{P}$ and subgraphs of G induced by a set of edges of the same color must have property $\mathcal{Q}$ and incident elements cannot have the same color assigned. For example, if $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{O}_1$, then this is an ordinary total coloring of a graph G. Such properties have also been studied by Borowiecki *et al.* in [4].

An $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of a finite graph G is a mapping f, which assigns an s-element subset of the set $\{1, 2, \ldots, r\}$ to each vertex and each edge $\left(f : V \cup E \to \binom{\{1,2,\ldots,r\}}{s}\right)$, moreover, for any color i all vertices of color i induce a subgraph with property $\mathcal{P}$, all edges of color i induce a subgraph with property $\mathcal{Q}$ and vertices and incident edges have been assigned disjoint sets of colors. The infimum ratio $\frac{r}{s}$ of an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of G is called the fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \frac{r}{s}$ of G.

An *automorphism* π of a graph G is an isomorphism from G to itself. Thus, the vertex bijection π_V is a permutation on $V(G)$, and the edge bijection π_E is a permutation on $E(G)$. The action of the automorphism group $\mathcal{A}(G)$ on a graph G partitions $V(G)$ into *vertex-orbits*. That is, the vertices u and v are in the same orbit if there exists an automorphism π such that $\pi(u) = v$. Similary, $\mathcal{A}(G)$ partitions $E(G)$ into *edge-orbits*. More about automorphism and orbits can be found in [7].

The vertices and edges of each graph can be separated into several disjoint orbits. We denote the number of vertex orbits by $o(V(G))$ and the number of edge orbits by $o(E(G))$. Let $o(G) = o(V(G)) + o(E(G))$.

There are two equivalent definitions of fractional colorings and also of generalized $(\mathcal{P}, \mathcal{Q})$-total colorings of a graph. The equivalence has been shown in [9], and so we state both of these definitions here.

Definition 1. Let G be a simple graph. Let $r, s \in \mathbb{N}$ and $r \geq s$. An $\frac{r}{s}$-*fractional* $(\mathcal{P},\mathcal{Q})$-*total coloring* of G is a mapping $f : V \cup E \to \binom{\{1,2,\ldots,r\}}{s}$ such that for each color i all vertices of color i induce a subgraph with property $\mathcal{P}$, all edges of color i induce a subgraph with property $\mathcal{Q}$, moreover, each incident vertex and edge have been assigned disjoint sets of colors. The *fractional* $(\mathcal{P},\mathcal{Q})$-*total chromatic number* is $\chi''_{1,f,\mathcal{P},\mathcal{Q}}(G) = \inf\{\frac{r}{s} :$ G has an $\frac{r}{s}$ -fractional $(\mathcal{P}, \mathcal{Q})$-total coloring$\}$.

For the second definition we need to define generalized independent sets of graphs. A $(\mathcal{P},\mathcal{Q})$-*independent set* is a subset of $V \cup E$ such that the vertices in this set induce a graph with property $\mathcal{P}$, the edges induce a graph with property $\mathcal{Q}$ and, moreover, vertices and edges are not incident.

Definition 2. Let $I_1, I_2, \ldots, I_t,\ \ t \in \mathbb{N}$, be all (maximal) ($\mathcal{P}$,$\mathcal{Q}$)-independent sets in G. A *fractional* ($\mathcal{P}$,$\mathcal{Q}$)*-total coloring* of G is a mapping g, which assigns a non-negative weight $g(I_j)$ to each set $I_j,\ \ j = 1, \ldots, t$, such that $\sum_{u \in I_j} g(I_j) \geq 1$ for each element $u \in V \cup E$. The *fractional* ($\mathcal{P}$,$\mathcal{Q}$)*-total chromatic number* $\chi''_{2,f,\mathcal{P},\mathcal{Q}}(G)$ of G is the least total weight of a fractional ($\mathcal{P}$,$\mathcal{Q}$)-total coloring of G.

As we have mentioned above, Definitions 1 and 2 of the fractional total chromatic numbers $\chi''_{1,f,\mathcal{P},\mathcal{Q}}(G)$ and $\chi''_{2,f,\mathcal{P},\mathcal{Q}}(G)$ are equivalent. In the following we will use the common notation of the fractional chromatic number $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$.

In Definition 2 we consider only the maximal ($\mathcal{P}$, $\mathcal{Q}$)-independent sets $I_j,\ j = 1, \ldots, t$, because we can take a maximal independent set instead of a non-maximal one and so more elements (vertices and edges) can get the same weight.

For determining $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ according to Definition 2, we have to solve the following linear program:

$$
\begin{aligned}
&h_1 : \sum\nolimits_{j=1}^{t} f(I_j) \to \min \\
&\sum\nolimits_{I_j \ni u} f(I_j) \geq 1, \quad u \in V \cup E, \qquad (1)\\
&f(I_j) \geq 0, \quad j = 1, \ldots, t.
\end{aligned}
$$

Our aim in the next two sections is to prove that we can reduce the number of conditions in this linear program to $o(G)$ and also reduce the number of the variables to μ (with $\mu \leq t$) defined below in Section 2. There we also show the construction of that linear program by using the orbits of G. In Section 3 we prove the equivalence between the linear program (1) and the reduced linear program from Section 2. An example can be found in Section 4 and finally we mention a result from [8] for the complete graphs in the last section.

2. The Orbits as a Tool for Reducing the Linear Program (1)

Examples of some terms defined below can be found in example in Section 4.

Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be two additive and hereditary graph properties and G be a graph with n vertices and e edges. Take an $\frac{r}{s}$-fractional ($\mathcal{P}$,$\mathcal{Q}$)-total coloring ψ of a graph G, where r is the number of used colors, from which we choose an s-element subset for every element of the graph G. Note that some of the elements can be assigned more than s colors, but as the denominator of the ratio $\frac{r}{s}$ in Definition 1 we consider $\min_{u \in V \cup E} |\psi(u)|$. All elements with the same color form a ($\mathcal{P}$, $\mathcal{Q}$)-independent set.

We denote the number of elements in orbit O_i by $o_i, i = 1, \ldots, o(G)$, and the independent sets of G by $I_j,\ j = 1, \ldots, t$. As in Definition 2, we consider only maximal ($\mathcal{P}$, $\mathcal{Q}$)-independent sets. Let $\mathbf{I}_{\left(b_1, \ldots, b_{o(G)}\right)}$ be the set of all I_j with exactly b_i elements from O_i, where $0 \leq b_i \leq o_i$ for each $i = 1, \ldots, o(G)$. We will say that $I_j \in \mathbf{I}_{(b_1,\ldots,b_{o(G)})}$ is an *independent set of type* $\mathbf{b} = (b_1, \ldots, b_{o(G)})$. Denote the cardinality of $\mathbf{I_b}$ by $m_{\mathbf{b}}$. Denote the set of all $\mathbf{b}$ by M.

In the next, if we have colors assigned to the set I_j, then assign the same colors to each element u from the set I_j. Of course, each element $u \in I_j$ gains colors from every independent set containing u. We will assign colors used exactly on elements from I_j to the independent set I_j. Denote by x_{I_j} the number of colors assigned to the independent set I_j for each $j = 1, \ldots, t$ and then the number of colors used for independent sets from $\mathbf{I_b}$ is $x_{\mathbf{b}} = \sum_{I_j \in \mathbf{I_b}} x_{I_j}$.

Thanks to the transitivity inside the orbits O_i, every $u \in I_j \cap O_i$, $I_j \in \mathbf{I_b}$, belongs to the same number of independent sets from $\mathbf{I_b}$ for each $\mathbf{b}$ and each $i = 1, \ldots, o(G)$ as is proved in the following Lemma 1. Let $p(\mathbf{I_b}, u) = |\{I_j : u \in I_j \in \mathbf{I_b}\}|$ be the number of sets of type $\mathbf{b}$ that contain u.

Lemma 1. Let $u, v \in O_i$, $i = 1, \ldots, o(G)$ and let $\mathbf{b} \in M$. Then $p(\mathbf{I_b}, u) = p(\mathbf{I_b}, v)$.

Proof. Let $u \neq v$, $u, v \in O_i$ for $i = 1, \ldots, o(G)$. Let $I_1(u), \ldots, I_{p(\mathbf{I_b},u)}(u)$ be all independent sets from $\mathbf{I_b}$ containing u, and similarly for v we will consider $I_1(v), \ldots, I_{p(\mathbf{I_b},v)}(v)$. Let φ be an automorphism with $\varphi(u) = v$. Then $\varphi(I_1(u)), \ldots, \varphi(I_{p(\mathbf{I_b},u)}(u))$ are different independent sets from $\mathbf{I_b}$ containing v because each element from the orbit O_j can be moved to an element from the same orbit O_j. Therefore $p(\mathbf{I_b}, v) \geq p(\mathbf{I_b}, u)$. The opposite inequality we get by using φ^{-1}. Thus $p(\mathbf{I_b}, u) = p(\mathbf{I_b}, v)$. ■

Lemma 2. Let $u \in O_i$ and $\mathbf{b} \in M$. Then

$$p(\mathbf{I_b}, u) = \frac{b_i \cdot m_{\mathbf{b}}}{o_i}.$$

Proof. Consider a table with o_i columns $x \in O_i$, $m_{\mathbf{b}}$ rows $I_j \in \mathbf{I_b}$ and values

$$a_{I_j x} = \begin{cases} 1, & \text{if } x \in I_j, \\ 0, & \text{otherwise.} \end{cases}$$

The number 1 appears in each column $p(\mathbf{I_b}, x) = p(\mathbf{I_b}, u)$ times and in each row b_i times. Therefore $p(\mathbf{I_b}, u) \cdot o_i = b_i \cdot m_{\mathbf{b}}$ from which we get the required equality. ■

This fact is very important for our main result, because it means that we can divide colors/weights assigned to $\mathbf{I_b}$ evenly among all I_j in $\mathbf{I_b}$. The weight (defined in Definition 2) assigned to $\mathbf{I_b}$ is $\frac{x_{\mathbf{b}}}{s}$.

The cardinality of the multiset of all colors used for a fractional total coloring of a graph G is $(n + e)s$. We need ns multicolors for the vertices and es for the edges, but this has to be satisfied for all orbits separately. Let M be the set of $\mathbf{b}$ for which $\mathbf{I_b}$ is non-empty (it means that M is the set of types $\mathbf{b}$ of maximal independent sets). We get the following $o(G)$ inequalities sufficient for determining r for fixed s, what we show in Theorem 1.

$$\begin{aligned} &h_2 : \sum\nolimits_{\mathbf{b}\in M} x_{\mathbf{b}} \to \min \\ &\sum\nolimits_{\mathbf{b}\in M} b_i x_{\mathbf{b}} \geq o_i s, \quad i = 1, \ldots, o(G), \\ &x_{\mathbf{b}} \geq 0,\ \mathbf{b} \in M,\ x_{\mathbf{b}}\text{-integer.} \end{aligned} \tag{2}$$

Let $x'_{\mathbf{b}} = \frac{x_{\mathbf{b}}}{s}$. We can reformulate the problem (2) as follows:

$$\begin{aligned} &h_3 : \sum\nolimits_{\mathbf{b}\in M} x'_{\mathbf{b}} \to \min \\ &\sum\nolimits_{\mathbf{b}\in M} b_i x'_{\mathbf{b}} \geq o_i, \quad i = 1, \ldots, o(G), \\ &x'_{\mathbf{b}} \geq 0,\ \mathbf{b} \in M,\ (x'_{\mathbf{b}}\text{-rational}). \end{aligned} \tag{3}$$

The linear programs (2) and (3) are equivalent in the following sense: Denote by μ the cardinality of the set M and let $\mathbf{b}_k$, $k = 1, \ldots, \mu$, be elements of M. If $(x'_{\mathbf{b}_1}, \ldots, x'_{\mathbf{b}_\mu})$ is an optimal solution of (3) and s is the least common multiple of denominators of all $x'_{\mathbf{b}_k}$, $\mathbf{b}_k \in M$, then $\sum_{\mathbf{b}\in M} x'_{\mathbf{b}} = \frac{r}{s}$ with $r \in \mathbb{N}_0$ and $(x'_{\mathbf{b}_1}s, \ldots, x'_{\mathbf{b}_\mu}s)$ is an optimal solution of (2) for fixed s and $\sum_{\mathbf{b}\in M} x'_{\mathbf{b}} \cdot s = \sum_{\mathbf{b}\in M} x_{\mathbf{b}} = r$. Now we prove that (1) is equivalent to (3) for all graphs.

3. The Equivalence Between the Linear Programs (1) and (3)

Theorem 1. *Let G be a simple graph, $\mathcal{P}$, $\mathcal{Q}$ be two additive and hereditary graph properties. Then for each optimal solution $\boldsymbol{f}(\boldsymbol{I})=(f(I_1), f(I_2), \ldots, f(I_t))$ of the linear program* (1) *there exists a feasible solution $\boldsymbol{x}' = (x'_{\boldsymbol{b}_1}, \ldots, x'_{\boldsymbol{b}_\mu})$ of the linear program* (3) *with $h_1(\boldsymbol{f}(\boldsymbol{I})) = h_3(\boldsymbol{x}')$ and vice versa for each optimal solution $\boldsymbol{x}' = (x'_{\boldsymbol{b}_1}, \ldots, x'_{\boldsymbol{b}_\mu})$ of the linear program* (3) *there exists a feasible solution $\boldsymbol{f}(\boldsymbol{I}) = (f(I_1), f(I_2), \ldots, f(I_t))$ with $h_3(\boldsymbol{x}') = h_1(\boldsymbol{f}(\boldsymbol{I}))$.*

Proof. Let $I_1, \ldots, I_t$, $t \in \mathbb{N}$, be all $(\mathcal{P}, \mathcal{Q})$-independent sets of G. Suppose that there exists an optimal solution $\boldsymbol{f}(\boldsymbol{I})=(f(I_1), f(I_2), \ldots, f(I_t))$ of problem (1).

Set $x'_{\mathbf{b}} = \sum_{I_j\in \mathbf{I_b}} f(I_j)$, $\mathbf{b} \in M$. According to the assumptions it holds that

$$\sum\nolimits_{I_j\ni u} f(I_j) \geq 1,\ u \in V \cup E \qquad \text{and} \qquad f(I_j) \geq 0,$$

from which we obtain

$$\sum_{u\in O_i}\sum_{I_j\ni u} f(I_j)\geq o_i \quad \text{for every} \quad i=1,\dots,o(G).$$

Then the following inequality is satisfied for each orbit $O_i,\ i=1,\dots,o(G)$,

$$\sum_{\mathbf{b}\in M} b_i x'_{\mathbf{b}}=\sum_{\mathbf{b}\in M} b_i\sum_{I_j\in \mathbf{I_b}} f(I_j)=\sum_{u\in O_i}\sum_{I_j\ni u} f(I_j)\geq o_i.$$

The last equality holds because on both sides there is the sum of all weights of the independent sets over all elements in the orbit O_i.

Now we show that the second part of this theorem is true as well. Let $\mathbf{x}'=(x'_{\mathbf{b}_1},\dots,x'_{\mathbf{b}_\mu})$ be an optimal solution of linear program (3). Let

$$f(I_j)=\begin{cases}\frac{x'_{\mathbf{b}}}{m_{\mathbf{b}}}, & \text{if } I_j\in I_{\mathbf{b}},\\ 0, & \text{otherwise.}\end{cases}$$

We know that for each orbit O_i, it holds that

$$\sum_{\mathbf{b}\in M} b_i x'_{\mathbf{b}}\geq o_i,\quad i=1,\dots,o(G).$$

Consider an element u from an orbit O_i.

$$\begin{aligned}\sum_{I_j\ni u} f(I_j)&=\sum_{\mathbf{b}\in M}\sum_{I_j\ni u: I_j\in I_{\mathbf{b}}} f(I_j)\\ &=\sum_{\mathbf{b}\in M}\sum_{I_j\ni u: I_j\in I_{\mathbf{b}}}\frac{x'_{\mathbf{b}}}{m_{\mathbf{b}}}\\ &=\sum_{\mathbf{b}\in M}\frac{x'_{\mathbf{b}}}{m_{\mathbf{b}}}\cdot p(\mathbf{I_b},u)=\sum_{\mathbf{b}\in M}\frac{x'_{\mathbf{b}}}{m_{\mathbf{b}}}\cdot\frac{b_i m_{\mathbf{b}}}{o_i}\geq 1.\end{aligned}$$

The values of the objective functions of problems (1) and (3) are equal:

$$h_1(\boldsymbol{f(I)})=\sum_{\mathbf{b}\in M}\sum_{I_j\in \mathbf{I_b}} f(I_j)=\sum_{\mathbf{b}\in M} x'_{\mathbf{b}}=h_3(\mathbf{x}').$$

■

4. EXAMPLE $\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(K_{1,1,2})$

In the following example we explain the reason for considering only the maximal independent sets of G. To determine $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ we need to find all types $\mathbf{b}\in M$.

Consider a complete tripartite graph $K_{1,1,2}$. Let the vertex set be $V=\{v_1,v_2,v_3,v_4\}$ and the edge set $E=\{e_1,e_2,e_3,e_4,e_5\}$, $e_1=\{v_1,v_3\}, e_2=\{v_2,v_3\}$, $e_3=\{v_1,v_4\}, e_4=\{v_2,v_4\}, e_5=\{v_1,v_2\}$. This graph has two vertex orbits

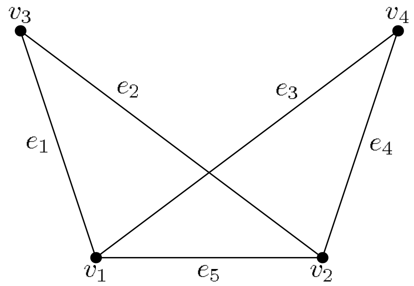

Figure 1. $K_{1,1,2}$.

$O_1 = \{v_1, v_2\}$, $O_2 = \{v_3, v_4\}$ and two edge orbits $O_3 = \{e_5\}$, $O_4 = \{e_1, e_2, e_3, e_4\}$ (see Figure 1).

We want to determine $\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(K_{1,1,2})$. Let us remind that $\mathcal{O}_1$ means that components formed by all vertices of the same color cannot contain three and more vertices and $\mathcal{I}_1$ means that edges with the same color cannot contain a triangle. Sets of maximal independent sets look as follows: $\mathbf{I}_{(0,0,0,4)}, \mathbf{I}_{(0,0,1,2)}, \mathbf{I}_{(1,0,0,2)}, \mathbf{I}_{(0,1,1,1)}, \mathbf{I}_{(0,1,0,2)}, \mathbf{I}_{(1,1,0,1)}, \mathbf{I}_{(2,0,0,0)}, \mathbf{I}_{(0,2,1,0)}$.

Note that there are two kinds of independent sets in $\mathbf{I}_{(0,0,1,2)}$. One of them is represented by $\{e_5, e_1, e_4\}$ and the second one by $\{e_5, e_1, e_3\}$.

We know that

$$m_{(0,0,0,4)} = 1,\ m_{(0,0,1,2)} = 4,\ m_{(1,0,0,2)} = 2,\ m_{(0,1,1,1)} = 4,$$
$$m_{(0,1,0,2)} = 2,\ m_{(1,1,0,1)} = 4,\ m_{(2,0,0,0)} = 1,\ m_{(0,2,1,0)} = 1.$$

We can see that $M = \{(0,0,0,4), (0,0,1,2), \ldots, (0,2,1,0)\}$. It means that we have $|M| = \mu = 8$ and we use the following variables:

$$x'_{(0,0,0,4)} = x'_1,\ x'_{(0,0,1,2)} = x'_2,\ x'_{(1,0,0,2)} = x'_3,\ x'_{(0,1,1,1)} = x'_4,$$
$$x'_{(0,1,0,2)} = x'_5,\ x'_{(1,1,0,1)} = x'_6,\ x'_{(2,0,0,0)} = x'_7,\ x'_{(0,2,1,0)} = x'_8.$$

Our graph has 4 orbits. It means that the linaer program has 4 inequalities:

$$\begin{aligned}
\textstyle\sum_{i=1}^{8} x'_i &\ \to\ \min \\
x'_3 + x'_6 + 2x'_7 &\ \geq\ 2, \\
x'_4 + x'_5 + x'_6 + 2x'_8 &\ \geq\ 2, \\
x'_2 + x'_4 + x'_8 &\ \geq\ 1, \\
4x'_1 + 2x'_2 + 2x'_3 + x'_4 + 2x'_5 + x'_6 &\ \geq\ 4, \\
x'_1, \ldots, x'_8 &\ \geq\ 0.
\end{aligned}$$

Firstly, we show that $\sum_{i=1}^{8} x'_i \geq 3$. Therefore the solution listed below will be one of the optimal solutions. Sum $\frac{1}{2}$ of the first and the third inequality and $\frac{1}{4}$ of the second, and the fourth one. We get the following statement

$$\sum_{i=1}^{8} x'_i \geq x'_1 + x'_2 + x'_3 + x'_4 + \frac{3}{4}x'_5 + x'_6 + x'_7 + x'_8 \geq 3.$$

Take the optimal solution $\mathbf{x}' = (1, 0, 0, 0, 0, 0, 1, 1)$ with $\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(K_{1,1,2}) = 3$. This optimal solution does not contain any fraction, therefore $\mathbf{x} = \mathbf{x}' = (1, 0, 0, 0, 0, 0, 1, 1)$. We will assign one color to $\mathbf{I}_{(0,0,0,4)}$, another one to $\mathbf{I}_{(2,0,0,0)}$ and the last color to $\mathbf{I}_{(0,2,1,0)}$. All of these sets of the independent sets consist of only one maximal independent set, so we will not divide colors among more members of $\mathbf{I_b}$. See a coloring in Figure 2.

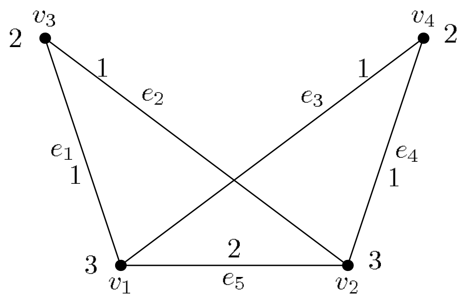

Figure 2. Optimal coloring of $K_{1,1,2}$.

Another optimal solution is $\mathbf{x}' = (\frac{1}{2}, 0, 0, 1, 0, 1, \frac{1}{2}, 0)$. We will assign colors to the independent sets from $\mathbf{I}_{0,0,0,4}$, $\mathbf{I}_{0,1,1,1}$, $\mathbf{I}_{1,1,0,1}$ and $\mathbf{I}_{2,0,0,0}$.

We have to divide colors assigned to e.g., $\mathbf{I}_{(0,1,1,1)}$ among 4 independent sets, but we have only 1 color. We need one color for each I_j from all used $\mathbf{I_b}$. Therefore we have to take the least common multiple as s and also we have to multiply our number of colors and each non-zero x'_i by this number. In this case $\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(K_{1,1,2}) = \frac{12}{4}$ and a coloring (according to Definition 1) can be seen in Figure 3.

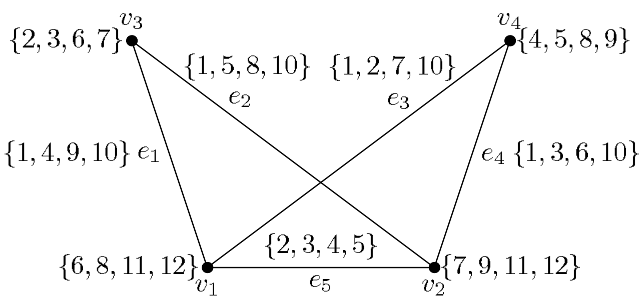

Figure 3. Optimal coloring of $K_{1,1,2}$.

Consider the optimal solution $\mathbf{x}' = (\frac{1}{2}, 0, 0, 1, 0, 1, \frac{1}{2}, 0)$ again and now assign the weights according to Definition 2 to each element (vertex and edge) of $K_{1,1,2}$. Every $I_j \in \mathbf{I_b}$ gains the weight $\frac{x'_{\mathbf{b}}}{m_{\mathbf{b}}}$. Figure 4 shows the weights that each vertex and each edge gains. The condition that every element must have a total weight (the sum of weights of all independent sets containing the element) of at least 1 is satisfied for all vertices and edges of $K_{1,1,2}$.

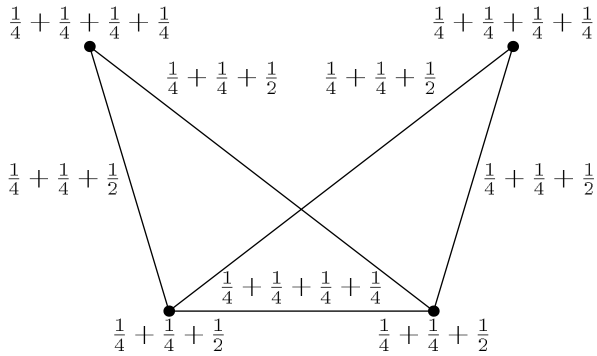

Figure 4. Optimal weighting of $K_{1,1,2}$.

In this case the linear program (1) has 9 inequalities and 19 variables. Note that it is not trivial to find all orbits and all maximal independent sets of a graph, but it is easier than to find all independent sets including the non-maximal ones and then to solve the linear program (1) with so many inequalities and variables. The linear program (3) will not have the reduced number of inequalities if each element (vertex and edge) of a graph forms one orbit. This occurs when the automorphism group consists of the identity only.

5. Complete Graphs

Finally, complete graphs have only one vertex orbit and one edge orbit and so we get the following linear program with two inequalities for K_n:

$$\begin{aligned} h_3 : \sum_{\mathbf{b} \in M} x'_{(b_1,b_2)} &\to \min \\ \sum_{\mathbf{b} \in M} b_1 x'_{(b_1,b_2)} &\geq n, \\ \sum_{\mathbf{b} \in M} b_2 x'_{(b_1,b_2)} &\geq e, \\ x'_{(b_1,b_2)} \geq 0,\ \mathbf{b} &\in M. \end{aligned}$$

This result has also been solved in another way and it is published in [8].

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REFERENCES

[1] M. Behzad, Graphs and their chromatic numbers, Ph.D. Thesis, (Michigan State University, 1965).

[2] M. Behzad, *The total chromatic number of a graph, a survey*, in: Proc. Conf. Oxford, 1969, Combinatorial Mathematics and its Applications, (Academic Press, London, 1971) 1–8.

[3] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50. doi:10.7151/dmgt.1037

[4] M. Borowiecki, A. Kemnitz, M. Marangio and P. Mihók, *Generalized total colorings of graphs*, Discuss. Math. Graph Theory **31**(2011) 209–222. doi:10.7151/dmgt.1540

[5] M. Borowiecki and P. Mihók, *Hereditary properties of graphs*, in: V.R. Kulli (Ed.), Advances in Graph Theory (Vishwa International Publication, Gulbarga, 1991) 41–68.

[6] A.G. Chetwynd, *Total colourings*, in: Graphs Colourings, R. Nelson and R.J. Wilson (Eds.), Pitman Research Notes in Mathematics **218** (London, 1990) 65–77.

[7] J.L. Gross and J. Yellen, Graph Theory and Its Applications, (CRC Press, New York 2006) 58–72.

[8] G. Karafová, *Generalized fractional total coloring of complete graphs*, Discuss. Math. Graph Theory **33** (2013) 665–676. doi:10.7151/dmgt.1697

[9] A. Kemnitz, M. Marangio, P. Mihók, J. Oravcová and R. Soták, *Generalized fractional and circular total colorings of graphs*, (2010), preprint.

[10] K. Kilakos and B. Reed, *Fractionally colouring total graphs*, Combinatorica **13** (1993) 435–440. doi:10.1007/BF01303515

[11] E.R. Scheinerman and D.H. Ullman, Fractional Graph Theory (John Wiley and Sons, New York, 1997).

[12] V.G. Vizing, *Some unsolved problems in graph theory*, Russian Math. Surveys **23** (1968) 125–141. doi:10.1070/RM1968v023n06ABEH001252

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