# GENERALIZED FRACTIONAL TOTAL COLORINGS OF GRAPHS 

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#### Abstract

Let $\mathcal{P}$ and $\mathcal{Q}$ be additive and hereditary graph properties and let $r, s$ be integers such that $r \geq s$. Then an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of a finite graph $G=(V, E)$ is a mapping $f$, which assigns an $s$-element subset of the set $\{1,2, \ldots, r\}$ to each vertex and each edge, moreover, for any color $i$ all vertices of color $i$ induce a subgraph with property $\mathcal{P}$, all edges of color $i$ induce a subgraph with property $\mathcal{Q}$ and vertices and incident edges have been assigned disjoint sets of colors. The minimum ratio of an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ is called fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number $\chi_{f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=\frac{r}{s}$. We show in this paper that $\chi_{f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}$ of a graph $G$ with $o(V(G))$ vertex orbits and $o(E(G))$ edge orbits can be found as a solution of a linear program with integer coefficients which consists only of $o(V(G))+o(E(G))$ inequalities.


Keywords: fractional coloring, total coloring, automorphism group.
2010 Mathematics Subject Classification: 05C15.

## 1. Introduction

Let $G=(V, E)$ be a finite simple graph with vertex set $V$ and edge set $E$. We denote by $n$ the number of vertices of $G$ and by $e$ the number of edges of $G$. By elements we will mean the vertices and the edges of a graph $G$.

A total coloring of a graph $G$ is a coloring of the vertices and the edges such that each two elements which are adjacent or incident obtain distinct colors. The minimum number of colors of a total coloring of $G$ is called total chromatic number $\chi^{\prime \prime}(G)$ of $G$.

The following conjecture is known as the Total Coloring Conjecture and it was formulated independently in the 1960s by Behzad [1] and Vizing [12]. It has been verified for several special classes of graphs, including for example complete graphs (see $[2,6]$ for surveys).

Conjecture 1. If $G$ is a graph with maximum degree $\Delta(G)$, then $\chi^{\prime \prime}(G) \leq$ $\Delta(G)+2$.

There are several possibilities how to define the fractional version of total colorings. All definitions can be found in [11]. In this paper we will use only two of them.

An $\frac{r}{s}$-fractional total coloring of a graph $G$ is a coloring of the vertices and the edges such that each vertex and each edge has been assigned an $s$-element subset of the set $\{1,2, \ldots, r\}$ and each two adjacent or incident elements receive disjoint sets of colors. The fractional total chromatic number $\chi_{f}^{\prime \prime}(G)$ of $G$ is the infimum ratio $\frac{r}{s}$ of an $\frac{r}{s}$-fractional total coloring of $G$. Kilakos and Reed [10] proved that $\chi_{f}^{\prime \prime \prime}(G) \leq \Delta(G)+2$ for any graph $G$.

In this paper we deal with generalized fractional total colorings of graphs. We denote the class of all finite simple graphs by $\mathcal{I}$. A graph property $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called additive if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$ (we mean the disjoint union of graphs). A property $\mathcal{P}$ is called hereditary if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$ (see [3, 5]).

We use the following standard notations for specific hereditary properties:

$$
\begin{aligned}
\mathcal{O} & =\{G \in \mathcal{I}: E(G)=\emptyset\}, \\
\mathcal{O}^{k} & =\{G \in \mathcal{I}: \chi(G) \leq k\},
\end{aligned}
$$

$\mathcal{D}_{k}=\{G \in \mathcal{I}:$ each subgraph of $G$ contains a vertex of degree at most $k\}$,

$$
\begin{gathered}
\mathcal{T}=\{G \in \mathcal{I}: G \text { is a planar graph }\}, \\
\mathcal{I}_{k}=\left\{G \in \mathcal{I}: G \text { does not contain } K_{k+2}\right\}, \\
\mathcal{O}_{k}=\{G \in \mathcal{I}: \text { each component of } G \text { has at most } k+1 \text { vertices }\}, \\
\mathcal{S}_{k}=\{G \in \mathcal{I}: \Delta(G) \leq k\},
\end{gathered}
$$

where $\chi(G)$ is the chromatic number. In the following we will only consider additive and hereditary graph properties.

The completeness of the property $\mathcal{P}$ is an interesting invariant of a graph property and it is defined as $c(\mathcal{P})=\sup \left\{i: K_{i+1} \in \mathcal{P}\right\}$. Note that $c\left(\mathcal{I}_{k}\right)=$ $c\left(\mathcal{O}_{k}\right)=c\left(\mathcal{D}_{k}\right)=c\left(\mathcal{S}_{k}\right)=k, c(\mathcal{T})=3, c(\mathcal{O})=0$.

Borowiecki and Mihók [5] dealt with graph properties and showed that the set of all additive and hereditary properties is a complete distributive lattice $\left(\mathbb{L}^{a}, \subseteq\right)$, where $\mathcal{O}$ is the smallest element of it and $\mathcal{I}$ is the greatest one. The set
of properties $\mathcal{P} \in \mathbb{L}^{a}$ with $c(\mathcal{P})=k, k \in \mathbb{N}$, is also a complete distributive lattice ( $\mathbb{L}_{k}^{a}, \subseteq$ ) with the smallest element $\mathcal{O}_{k}$ and the greatest element $\mathcal{I}_{k}$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two graph properties. We consider a total coloring of a graph $G$ such that adjacent elements can have the same color assigned but we require that subgraphs of $G$ induced by a set of vertices of the same color must have property $\mathcal{P}$ and subgraphs of $G$ induced by a set of edges of the same color must have property $\mathcal{Q}$ and incident elements cannot have the same color assigned. For example, if $\mathcal{P}=\mathcal{O}$ and $\mathcal{Q}=\mathcal{O}_{1}$, then this is an ordinary total coloring of a graph $G$. Such properties have also been studied by Borowiecki et al. in [4].

An $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of a finite graph $G$ is a mapping $f$, which assigns an $s$-element subset of the set $\{1,2, \ldots, r\}$ to each vertex and each edge $\left(f: V \cup E \rightarrow\binom{\{1,2, \ldots, r\}}{s}\right)$, moreover, for any color $i$ all vertices of color $i$ induce a subgraph with property $\mathcal{P}$, all edges of color $i$ induce a subgraph with property $\mathcal{Q}$ and vertices and incident edges have been assigned disjoint sets of colors. The infimum ratio $\frac{r}{s}$ of an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ is called the fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number $\chi_{f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=\frac{r}{s}$ of $G$.

An automorphism $\pi$ of a graph $G$ is an isomorphism from $G$ to itself. Thus, the vertex bijection $\pi_{V}$ is a permutation on $V(G)$, and the edge bijection $\pi_{E}$ is a permutation on $E(G)$. The action of the automorphism group $\mathcal{A}(G)$ on a graph $G$ partitions $V(G)$ into vertex-orbits. That is, the vertices $u$ and $v$ are in the same orbit if there exists an automorphism $\pi$ such that $\pi(u)=v$. Similary, $\mathcal{A}(G)$ partitions $E(G)$ into edge-orbits. More about automorphism and orbits can be found in [7].

The vertices and edges of each graph can be separated into several disjoint orbits. We denote the number of vertex orbits by $o(V(G))$ and the number of edge orbits by $o(E(G))$. Let $o(G)=o(V(G))+o(E(G))$.

There are two equivalent definitions of fractional colorings and also of generalized $(\mathcal{P}, \mathcal{Q})$-total colorings of a graph. The equivalence has been shown in [9], and so we state both of these definitions here.

Definition 1. Let $G$ be a simple graph. Let $r, s \in \mathbb{N}$ and $r \geq s$. An $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ is a mapping $f: V \cup E \rightarrow(\{1,2, \ldots, r\})$ such that for each color $i$ all vertices of color $i$ induce a subgraph with property $\mathcal{P}$, all edges of color $i$ induce a subgraph with property $\mathcal{Q}$, moreover, each incident vertex and edge have been assigned disjoint sets of colors. The fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number is $\chi_{1, f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=\inf \left\{\frac{r}{s}: G\right.$ has an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring $\}$.

For the second definition we need to define generalized independent sets of graphs. A $(\mathcal{P}, \mathcal{Q})$-independent set is a subset of $V \cup E$ such that the vertices in this set induce a graph with property $\mathcal{P}$, the edges induce a graph with property $\mathcal{Q}$ and, moreover, vertices and edges are not incident.

Definition 2. Let $I_{1}, I_{2}, \ldots, I_{t}, \quad t \in \mathbb{N}$, be all (maximal) ( $\left.\mathcal{P}, \mathcal{Q}\right)$-independent sets in $G$. A fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ is a mapping $g$, which assigns a nonnegative weight $g\left(I_{j}\right)$ to each set $I_{j}, \quad j=1, \ldots, t$, such that $\sum_{u \in I_{j}} g\left(I_{j}\right) \geq 1$ for each element $u \in V \cup E$. The fractional $(\mathcal{P}, \mathcal{Q})$-total chromatic number $\chi_{2, f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ of $G$ is the least total weight of a fractional $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$.

As we have mentioned above, Definitions 1 and 2 of the fractional total chromatic numbers $\chi_{1, f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ and $\chi_{2, f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ are equivalent. In the following we will use the common notation of the fractional chromatic number $\chi_{f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$.

In Definition 2 we consider only the maximal $(\mathcal{P}, \mathcal{Q})$-independent sets $I_{j}, j=$ $1, \ldots, t$, because we can take a maximal independent set instead of a non-maximal one and so more elements (vertices and edges) can get the same weight.

For determining $\chi_{f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ according to Definition 2, we have to solve the following linear program:

$$
\begin{align*}
h_{1}: & \sum_{j=1}^{t} f\left(I_{j}\right) \\
& \rightarrow \min  \tag{1}\\
& \sum_{I_{j} \ni u} f\left(I_{j}\right) \geq 1, \quad u \in V \cup E, \\
& f\left(I_{j}\right) \geq 0, \quad j=1, \ldots, t
\end{align*}
$$

Our aim in the next two sections is to prove that we can reduce the number of conditions in this linear program to $o(G)$ and also reduce the number of the variables to $\mu$ (with $\mu \leq t$ ) defined below in Section 2. There we also show the construction of that linear program by using the orbits of $G$. In Section 3 we prove the equivalence between the linear program (1) and the reduced linear program from Section 2. An example can be found in Section 4 and finally we mention a result from [8] for the complete graphs in the last section.

## 2. The Orbits as a Tool for Reducing the Linear Program (1)

Examples of some terms defined below can be found in example in Section 4.
Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_{1}$ be two additive and hereditary graph properties and $G$ be a graph with $n$ vertices and $e$ edges. Take an $\frac{r}{s}$-fractional $(\mathcal{P}, \mathcal{Q})$-total coloring $\psi$ of a graph $G$, where $r$ is the number of used colors, from which we choose an $s$-element subset for every element of the graph $G$. Note that some of the elements can be assigned more than $s$ colors, but as the denominator of the ratio $\frac{r}{s}$ in Definition 1 we consider $\min _{u \in V \cup E}|\psi(u)|$. All elements with the same color form a $(\mathcal{P}, \mathcal{Q})$-independent set.

We denote the number of elements in orbit $O_{i}$ by $o_{i}, i=1, \ldots, o(G)$, and the independent sets of $G$ by $I_{j}, j=1, \ldots, t$. As in Definition 2, we consider only maximal $(\mathcal{P}, \mathcal{Q})$-independent sets. Let $\mathbf{I}_{\left(b_{1}, \ldots, b_{o(G)}\right)}$ be the set of all $I_{j}$ with
exactly $b_{i}$ elements from $O_{i}$, where $0 \leq b_{i} \leq o_{i}$ for each $i=1, \ldots, o(G)$. We will say that $I_{j} \in \mathbf{I}_{\left(b_{1}, \ldots, b_{o(G)}\right)}$ is an independent set of type $\mathbf{b}=\left(b_{1}, \ldots, b_{o(G)}\right)$. Denote the cardinality of $\mathbf{I}_{\mathbf{b}}$ by $m_{\mathbf{b}}$. Denote the set of all $\mathbf{b}$ by $M$.

In the next, if we have colors assigned to the set $I_{j}$, then assign the same colors to each element $u$ from the set $I_{j}$. Of course, each element $u \in I_{j}$ gains colors from every independent set containing $u$. We will assign colors used exactly on elements from $I_{j}$ to the independent set $I_{j}$. Denote by $x_{I_{j}}$ the number of colors assigned to the independent set $I_{j}$ for each $j=1, \ldots, t$ and then the number of colors used for independent sets from $\mathbf{I}_{\mathbf{b}}$ is $x_{\mathbf{b}}=\sum_{I_{j} \in \mathbf{I}_{\mathbf{b}}} x_{I_{j}}$.

Thanks to the transitivity inside the orbits $O_{i}$, every $u \in I_{j} \cap O_{i}, I_{j} \in$ $\mathbf{I}_{\mathbf{b}}$, belongs to the same number of independent sets from $\mathbf{I}_{\mathbf{b}}$ for each $\mathbf{b}$ and each $i=1, \ldots, o(G)$ as is proved in the following Lemma 1. Let $p\left(\mathbf{I}_{\mathbf{b}}, u\right)=$ $\left|\left\{I_{j}: u \in I_{j} \in \mathbf{I}_{\mathbf{b}}\right\}\right|$ be the number of sets of type $\mathbf{b}$ that contain $u$.

Lemma 1. Let $u, v \in O_{i}, i=1, \ldots, o(G)$ and let $\mathbf{b} \in M$. Then $p\left(\mathbf{I}_{\mathbf{b}}, u\right)=$ $p\left(\mathbf{I}_{\mathbf{b}}, v\right)$.

Proof. Let $u \neq v, u, v \in O_{i}$ for $i=1, \ldots, o(G)$. Let $I_{1}(u), \ldots, I_{p\left(\mathbf{I}_{\mathbf{b}}, u\right)}(u)$ be all independent sets from $\mathbf{I}_{\mathbf{b}}$ containing $u$, and similarly for $v$ we will consider $I_{1}(v), \ldots, I_{p\left(\mathbf{I}_{\mathbf{b}}, v\right)}(v)$. Let $\varphi$ be an automorphism with $\varphi(u)=v$. Then $\varphi\left(I_{1}(u)\right), \ldots, \varphi\left(I_{p\left(\mathbf{I}_{\mathbf{b}}, u\right)}(u)\right)$ are different independent sets from $\mathbf{I}_{\mathbf{b}}$ containing $v$ because each element from the orbit $O_{j}$ can be moved to an element from the same orbit $O_{j}$. Therefore $p\left(\mathbf{I}_{\mathbf{b}}, v\right) \geq p\left(\mathbf{I}_{\mathbf{b}}, u\right)$. The opposite inequality we get by using $\varphi^{-1}$. Thus $p\left(\mathbf{I}_{\mathbf{b}}, u\right)=p\left(\mathbf{I}_{\mathbf{b}}, v\right)$.

Lemma 2. Let $u \in O_{i}$ and $\mathbf{b} \in M$. Then

$$
p\left(\mathbf{I}_{\mathbf{b}}, u\right)=\frac{b_{i} \cdot m_{\mathbf{b}}}{o_{i}} .
$$

Proof. Consider a table with $o_{i}$ columns $x \in O_{i}, m_{\mathbf{b}}$ rows $I_{j} \in \mathbf{I}_{\mathbf{b}}$ and values

$$
a_{I_{j} x}= \begin{cases}1, & \text { if } x \in I_{j} \\ 0, & \text { otherwise }\end{cases}
$$

The number 1 appears in each column $p\left(\mathbf{I}_{\mathbf{b}}, x\right)=p\left(\mathbf{I}_{\mathbf{b}}, u\right)$ times and in each row $b_{i}$ times. Therefore $p\left(\mathbf{I}_{\mathbf{b}}, u\right) \cdot o_{i}=b_{i} \cdot m_{\mathbf{b}}$ from which we get the required equality.

This fact is very important for our main result, because it means that we can divide colors/weights assigned to $\mathbf{I}_{\mathbf{b}}$ evenly among all $I_{j}$ in $\mathbf{I}_{\mathbf{b}}$. The weight (defined in Definition 2) assigned to $\mathbf{I}_{\mathbf{b}}$ is $\frac{x_{\mathbf{b}}}{s}$.

The cardinality of the multiset of all colors used for a fractional total coloring of a graph $G$ is $(n+e) s$. We need $n s$ multicolors for the vertices and es for
the edges, but this has to be satisfied for all orbits separately. Let $M$ be the set of $\mathbf{b}$ for which $\mathbf{I}_{\mathbf{b}}$ is non-empty (it means that $M$ is the set of types $\mathbf{b}$ of maximal independent sets). We get the following $o(G)$ inequalities sufficient for determining $r$ for fixed $s$, what we show in Theorem 1.

$$
\begin{align*}
& h_{2}: \sum_{\mathbf{b} \in M} x_{\mathbf{b}} \rightarrow \min \\
& \quad \sum_{\mathbf{b} \in M} b_{i} x_{\mathbf{b}} \geq o_{i} s, \quad i=1, \ldots, o(G),  \tag{2}\\
& \quad x_{\mathbf{b}} \geq 0, \quad \mathbf{b} \in M, x_{\mathbf{b}} \text {-integer. }
\end{align*}
$$

Let $x_{\mathbf{b}}^{\prime}=\frac{x_{\mathbf{b}}}{s}$. We can reformulate the problem (2) as follows:

$$
\begin{align*}
h_{3} & : \sum_{\mathbf{b} \in M} x_{\mathbf{b}}^{\prime} \rightarrow \min \\
& \sum_{\mathbf{b} \in M} b_{i} x_{\mathbf{b}}^{\prime} \geq o_{i}, \quad i=1, \ldots, o(G),  \tag{3}\\
& x_{\mathbf{b}}^{\prime} \geq 0, \mathbf{b} \in M,\left(x_{\mathbf{b}}^{\prime} \text {-rational }\right) .
\end{align*}
$$

The linear programs (2) and (3) are equivalent in the following sense: Denote by $\mu$ the cardinality of the set $M$ and let $\mathbf{b}_{k}, k=1, \ldots, \mu$, be elements of M. If ( $x_{\mathbf{b}_{1}}^{\prime}, \ldots, x_{\mathbf{b}_{\mu}}^{\prime}$ ) is an optimal solution of (3) and $s$ is the least common multiple of denominators of all $x_{\mathbf{b}_{k}}^{\prime}$, $\mathbf{b}_{k} \in M$, then $\sum_{\mathbf{b} \in M} x_{\mathbf{b}}^{\prime}=\frac{r}{s}$ with $r \in \mathbb{N}_{0}$ and $\left(x_{\mathbf{b}_{1}}^{\prime} s, \ldots, x_{\mathbf{b}_{\mu}}^{\prime} s\right)$ is an optimal solution of (2) for fixed $s$ and $\sum_{\mathbf{b} \in M} x_{\mathbf{b}}^{\prime} \cdot s=$ $\sum_{\mathbf{b} \in M} x_{\mathbf{b}}=r$. Now we prove that (1) is equivalent to (3) for all graphs.

## 3. The Equivalence Between the Linear Programs (1) and (3)

Theorem 1. Let $G$ be a simple graph, $\mathcal{P}, \mathcal{Q}$ be two additive and hereditary graph properties. Then for each optimal solution $\boldsymbol{f}(\boldsymbol{I})=\left(f\left(I_{1}\right), f\left(I_{2}\right), \ldots, f\left(I_{t}\right)\right)$ of the linear program (1) there exists a feasible solution $\boldsymbol{x}^{\prime}=\left(x_{b_{1}}^{\prime}, \ldots, x_{b_{\mu}}^{\prime}\right)$ of the linear program (3) with $h_{1}(\boldsymbol{f}(\boldsymbol{I}))=h_{3}\left(\boldsymbol{x}^{\prime}\right)$ and vice versa for each optimal solution $\boldsymbol{x}^{\prime}=\left(x_{\boldsymbol{b}_{1}}^{\prime}, \ldots, x_{\boldsymbol{b}_{\mu}}^{\prime}\right)$ of the linear program (3) there exists a feasible solution $\boldsymbol{f}(\boldsymbol{I})=$ $\left(f\left(I_{1}\right), f\left(I_{2}\right), \ldots, f\left(I_{t}\right)\right)$ with $h_{3}\left(\boldsymbol{x}^{\prime}\right)=h_{1}(\boldsymbol{f}(\boldsymbol{I}))$.

Proof. Let $I_{1}, \ldots, I_{t}, \quad t \in \mathbb{N}$, be all $(\mathcal{P}, \mathcal{Q})$-independent sets of $G$. Suppose that there exists an optimal solution $\boldsymbol{f}(\boldsymbol{I})=\left(f\left(I_{1}\right), f\left(I_{2}\right), \ldots, f\left(I_{t}\right)\right)$ of problem (1).

Set $x_{\mathbf{b}}^{\prime}=\sum_{I_{j} \in \mathbf{I}_{\mathbf{b}}} f\left(I_{j}\right), \mathbf{b} \in M$. According to the assumptions it holds that

$$
\sum_{I_{j} \ni u} f\left(I_{j}\right) \geq 1, u \in V \cup E \quad \text { and } \quad f\left(I_{j}\right) \geq 0
$$

from which we obtain

$$
\sum_{u \in O_{i}} \sum_{I_{j} \ni u} f\left(I_{j}\right) \geq o_{i} \quad \text { for every } i=1, \ldots, o(G)
$$

Then the following inequality is satisfied for each orbit $O_{i}, i=1, \ldots, o(G)$,

$$
\sum_{\mathbf{b} \in M} b_{i} x_{\mathbf{b}}^{\prime}=\sum_{\mathbf{b} \in M} b_{i} \sum_{I_{j} \in \mathbf{I}_{\mathbf{b}}} f\left(I_{j}\right)=\sum_{u \in O_{i}} \sum_{I_{j} \ni u} f\left(I_{j}\right) \geq o_{i} .
$$

The last equality holds because on both sides there is the sum of all weights of the independent sets over all elements in the orbit $O_{i}$.

Now we show that the second part of this theorem is true as well. Let $\mathbf{x}^{\prime}=\left(x_{\mathbf{b}_{1}}^{\prime}, \ldots, x_{\mathbf{b}_{\mu}}^{\prime}\right)$ be an optimal solution of linear program (3). Let

$$
f\left(I_{j}\right)= \begin{cases}\frac{x_{\mathbf{b}}^{\prime}}{m_{\mathbf{b}}}, & \text { if } I_{j} \in I_{\mathbf{b}} \\ 0, & \text { otherwise }\end{cases}
$$

We know that for each orbit $O_{i}$, it holds that

$$
\sum_{\mathbf{b} \in M} b_{i} x_{\mathbf{b}}^{\prime} \geq o_{i}, \quad i=1, \ldots, o(G)
$$

Consider an element $u$ from an orbit $O_{i}$.

$$
\begin{aligned}
\sum_{I_{j} \ni u} f\left(I_{j}\right) & =\sum_{\mathbf{b} \in M} \sum_{I_{j} \ni u: I_{j} \in I_{\mathbf{b}}} f\left(I_{j}\right) \\
& =\sum_{\mathbf{b} \in M} \sum_{I_{j} \ni u: I_{j} \in I_{\mathbf{b}}} \frac{x_{\mathbf{b}}^{\prime}}{m_{\mathbf{b}}} \\
& =\sum_{\mathbf{b} \in M} \frac{x_{\mathbf{b}}^{\prime}}{m_{\mathbf{b}}} \cdot p\left(\mathbf{I}_{\mathbf{b}}, u\right)=\sum_{\mathbf{b} \in M} \frac{x_{\mathbf{b}}^{\prime}}{m_{\mathbf{b}}} \cdot \frac{b_{i} m_{\mathbf{b}}}{o_{i}} \geq 1
\end{aligned}
$$

The values of the objective functions of problems (1) and (3) are equal:

$$
h_{1}(f(I))=\sum_{\mathbf{b} \in M} \sum_{I_{j} \in \mathbf{I}_{\mathbf{b}}} f\left(I_{j}\right)=\sum_{\mathbf{b} \in M} x_{\mathbf{b}}^{\prime}=h_{3}\left(\mathbf{x}^{\prime}\right) .
$$

## 4. Example $\chi_{f, \mathcal{O}_{1}, \mathcal{I}_{1}}^{\prime \prime}\left(K_{1,1,2}\right)$

In the following example we explain the reason for considering only the maximal independent sets of $G$. To determine $\chi_{f, \mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ we need to find all types $\mathbf{b} \in M$.

Consider a complete tripartite graph $K_{1,1,2}$. Let the vertex set be $V=$ $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and the edge set $E=\left\{e_{1}, e_{2}, e_{3}, e_{4}, e_{5}\right\}, e_{1}=\left\{v_{1}, v_{3}\right\}, e_{2}=\left\{v_{2}, v_{3}\right\}$, $e_{3}=\left\{v_{1}, v_{4}\right\}, e_{4}=\left\{v_{2}, v_{4}\right\}, e_{5}=\left\{v_{1}, v_{2}\right\}$. This graph has two vertex orbits


Figure 1. $K_{1,1,2}$.
$O_{1}=\left\{v_{1}, v_{2}\right\}, O_{2}=\left\{v_{3}, v_{4}\right\}$ and two edge orbits $O_{3}=\left\{e_{5}\right\}, O_{4}=\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ (see Figure 1).

We want to determine $\chi_{f, \mathcal{O}_{1}, \mathcal{I}_{1}}^{\prime \prime}\left(K_{1,1,2}\right)$. Let us remind that $\mathcal{O}_{1}$ means that components formed by all vertices of the same color cannot contain three and more vertices and $\mathcal{I}_{1}$ means that edges with the same color cannot contain a triangle. Sets of maximal independent sets look as follows: $\mathbf{I}_{(0,0,0,4)}, \mathbf{I}_{(0,0,1,2)}, \mathbf{I}_{(1,0,0,2)}$, $\mathbf{I}_{(0,1,1,1)}, \mathbf{I}_{(0,1,0,2)}, \mathbf{I}_{(1,1,0,1)}, \mathbf{I}_{(2,0,0,0)}, \mathbf{I}_{(0,2,1,0)}$.

Note that there are two kinds of independent sets in $\mathbf{I}_{(0,0,1,2)}$. One of them is represented by $\left\{e_{5}, e_{1}, e_{4}\right\}$ and the second one by $\left\{e_{5}, e_{1}, e_{3}\right\}$.

We know that

$$
\begin{aligned}
& m_{(0,0,0,4)}=1, m_{(0,0,1,2)}=4, m_{(1,0,0,2)}=2, m_{(0,1,1,1)}=4, \\
& m_{(0,1,0,2)}=2, m_{(1,1,0,1)}=4, m_{(2,0,0,0)}=1, m_{(0,2,1,0)}=1 .
\end{aligned}
$$

We can see that $M=\{(0,0,0,4),(0,0,1,2), \ldots,(0,2,1,0)\}$. It means that we have $|M|=\mu=8$ and we use the following variables:

$$
\begin{aligned}
& x_{(0,0,0,4)}^{\prime}=x_{1}^{\prime}, x_{(0,0,1,2)}^{\prime}=x_{2}^{\prime}, x_{(1,0,0,2)}^{\prime}=x_{3}^{\prime}, x_{(0,1,1,1)}^{\prime}=x_{4}^{\prime}, \\
& x_{(0,1,0,2)}^{\prime}=x_{5}^{\prime}, x_{(1,1,0,1)}^{\prime}=x_{6}^{\prime}, x_{(2,0,0,0)}^{\prime}=x_{7}^{\prime}, x_{(0,2,1,0)}^{\prime}=x_{8}^{\prime} .
\end{aligned}
$$

Our graph has 4 orbits. It means that the linaer program has 4 inequalities:

$$
\begin{aligned}
\sum_{i=1}^{8} x_{i}^{\prime} & \rightarrow \min \\
x_{3}^{\prime}+x_{6}^{\prime}+2 x_{7}^{\prime} & \geq 2, \\
x_{4}^{\prime}+x_{5}^{\prime}+x_{6}^{\prime}+2 x_{8}^{\prime} & \geq 2, \\
x_{2}^{\prime}+x_{4}^{\prime}+x_{8}^{\prime} & \geq 1, \\
4 x_{1}^{\prime}+2 x_{2}^{\prime}+2 x_{3}^{\prime}+x_{4}^{\prime}+2 x_{5}^{\prime}+x_{6}^{\prime} & \geq 4, \\
x_{1}^{\prime} \ldots, x_{8}^{\prime} & \geq 0 .
\end{aligned}
$$

Firstly, we show that $\sum_{i=1}^{8} x_{i}^{\prime} \geq 3$. Therefore the solution listed below will be one of the optimal solutions. Sum $\frac{1}{2}$ of the first and the third inequality and $\frac{1}{4}$ of the second, and the fourth one. We get the following statement

$$
\sum_{i=1}^{8} x_{i}^{\prime} \geq x_{1}^{\prime}+x_{2}^{\prime}+x_{3}^{\prime}+x_{4}^{\prime}+\frac{3}{4} x_{5}^{\prime}+x_{6}^{\prime}+x_{7}^{\prime}+x_{8}^{\prime} \geq 3
$$

Take the optimal solution $\mathbf{x}^{\prime}=(1,0,0,0,0,0,1,1)$ with $\chi_{f, \mathcal{O}_{1}, \mathcal{I}_{1}}^{\prime \prime}\left(K_{1,1,2}\right)=3$. This optimal solution does not contain any fraction, therefore $\mathbf{x}=\mathbf{x}^{\prime}=(1,0,0,0,0,0,1$, 1). We will assign one color to $\mathbf{I}_{(0,0,0,4)}$, another one to $\mathbf{I}_{(2,0,0,0)}$ and the last color to $\mathbf{I}_{(0,2,1,0)}$. All of these sets of the independent sets consist of only one maximal independent set, so we will not divide colors among more members of $\mathbf{I}_{\mathbf{b}}$. See a coloring in Figure 2.


Figure 2. Optimal coloring of $K_{1,1,2}$.
Another optimal solution is $\mathbf{x}^{\prime}=\left(\frac{1}{2}, 0,0,1,0,1, \frac{1}{2}, 0\right)$. We will assign colors to the independent sets from $\mathbf{I}_{0,0,0,4}, \mathbf{I}_{0,1,1,1}, \mathbf{I}_{1,1,0,1}$ and $\mathbf{I}_{2,0,0,0}$.

We have to divide colors assigned to e.g., $\mathbf{I}_{(0,1,1,1)}$ among 4 independent sets, but we have only 1 color. We need one color for each $I_{j}$ from all used $\mathbf{I}_{\mathbf{b}}$. Therefore we have to take the least common multiple as $s$ and also we have to multiply our number of colors and each non-zero $x_{i}^{\prime}$ by this number. In this case $\chi_{f, \mathcal{O}_{1}, \mathcal{I}_{1}}^{\prime \prime}\left(K_{1,1,2}\right)=\frac{12}{4}$ and a coloring (according to Definition 1) can be seen in Figure 3.


Figure 3. Optimal coloring of $K_{1,1,2}$.
Consider the optimal solution $\mathbf{x}^{\prime}=\left(\frac{1}{2}, 0,0,1,0,1, \frac{1}{2}, 0\right)$ again and now assign the weights according to Definition 2 to each element (vertex and edge) of $K_{1,1,2}$. Every $I_{j} \in \mathbf{I}_{\mathbf{b}}$ gains the weight $\frac{x_{\mathbf{b}}^{\prime}}{m_{\mathbf{b}}}$. Figure 4 shows the weights that each vertex and each edge gains. The condition that every element must have a total weight
(the sum of weights of all independent sets containing the element) of at least 1 is satisfied for all vertices and edges of $K_{1,1,2}$.


Figure 4. Optimal weighting of $K_{1,1,2}$.
In this case the linear program (1) has 9 inequalities and 19 variables. Note that it is not trivial to find all orbits and all maximal independent sets of a graph, but it is easier than to find all independent sets including the non-maximal ones and then to solve the linear program (1) with so many inequalities and variables. The linear program (3) will not have the reduced number of inequalities if each element (vertex and edge) of a graph forms one orbit. This occurs when the automorphism group consists of the identity only.

## 5. Complete Graphs

Finally, complete graphs have only one vertex orbit and one edge orbit and so we get the following linear program with two inequalities for $K_{n}$ :

$$
\begin{gathered}
h_{3}: \sum_{\mathbf{b} \in M} x_{\left(b_{1}, b_{2}\right)}^{\prime} \rightarrow \min \\
\sum_{\mathbf{b} \in M} b_{1} x_{\left(b_{1}, b_{2}\right)}^{\prime} \geq n \\
\sum_{\mathbf{b} \in M} b_{2} x_{\left(b_{1}, b_{2}\right)}^{\prime} \geq e \\
x_{\left(b_{1}, b_{2}\right)}^{\prime} \geq 0, \mathbf{b} \in M
\end{gathered}
$$

This result has also been solved in another way and it is published in [8].

## Acknowledgement

This work was supported by Science and Technology Assistance Agency under the contract No. APVV-0023-10 and by Slovak VEGA Grant 1/0652/12.

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