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# BIPARTITION POLYNOMIALS, THE ISING MODEL AND DOMINATION IN GRAPHS

Markus Dod[1]

*Faculty Mathematics, Sciences, Computer Science*
*University of Applied Sciences Mittweida*

**e-mail:** mdod@hs-mittweida.de

Tomer Kotek[2]

*Institut für Informationssysteme 184/4*
*Technische Universität Wien, Vienna, Austria*

**e-mail:** kotek@forsyte.at

James Preen

*Mathematics*
*Cape Breton University*
*Sydney, Canada*

**e-mail:** james_preen@capebretonu.ca

and

Peter Tittmann

*Faculty Mathematics, Sciences, Computer Science*
*University of Applied Sciences Mittweida*

**e-mail:** peter@hs-mittweida.de

**Abstract**

This paper introduces a trivariate graph polynomial that is a common generalization of the domination polynomial, the Ising polynomial, the matching polynomial, and the cut polynomial of a graph. This new graph polynomial, called the bipartition polynomial, permits a variety of interesting

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representations, for instance as a sum ranging over all spanning forests. As a consequence, the bipartition polynomial is a powerful tool for proving properties of other graph polynomials and graph invariants. We apply this approach to show that, analogously to the Tutte polynomial, the Ising polynomial introduced by Andrén and Markström in [3], can be represented as a sum over spanning forests.

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## 1. Introduction

A dominating set $W$ of an undirected graph $G$ is a vertex set such that any vertex not belonging to $W$ is a neighbor of a vertex of $W$. One of the earliest approaches to the enumeration and counting of dominating sets is related to the dominating queens problem on a chessboard [1]. For general graphs there exist various exponential time algorithms for counting dominating sets, see e.g. [22]. The domination polynomial of a graph is the ordinary generating function for the number of dominating sets. It was introduced by Arocha and Llano [5].

In this paper we consider finite simple undirected graphs, which we simply call *graphs*. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. The *open neighborhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices that are adjacent to $v$ in $G$. The *closed neighborhood* of $v$ is $N_G(v) \cup \{v\}$. Analogously, we define

$$N_G(W) = \bigcup_{v \in W} N_G(v) \setminus W \text{ and } N_G[W] = N_G(W) \cup W$$

for any vertex subset $W \subseteq V$. For a given vertex subset $W \subseteq V$, let $\partial W$ be the set of all edges of $G$ with exactly one of their end vertices in $W$, i.e.,

$$\partial W = \{\{u, v\} \in E \mid u \in W, v \in V \setminus W\}.$$

A vertex set $W \subseteq V$ is called a *dominating set* of $G$ if $N_G[W] = V$. The *domination polynomial* of a graph $G$ is the ordinary generating function for the number of dominating sets of $G$:

$$D(G, x) = \sum_{\substack{W \subseteq V \\ N_G[W]=V}} x^{|W|}.$$

In [16], it is proved that the domination polynomial is closely related to bipartite spanning subgraphs. It can be used to show that the number $D(G, 1)$

of dominating sets of a graph is odd and to derive other non-trivial properties of this number.

Graphs of different size or with a different number of components may have the same domination polynomial. The reason is that the domination polynomial does not provide any information about edges that are necessary in order to link dominating vertices to those vertices covered by a dominating set. A natural way to extend the definition of the domination polynomial is to allow any vertex subset $W$ but to count the vertices and edges that are necessary to cover $W$. This provides the following definition.

**Definition 1.** Let $G = (V, E)$ be a graph. The *bipartition polynomial* of $G$ is

$$B(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} \sum_{F \subseteq \partial W} y^{|N_{(V,F)}(W)|} z^{|F|}. \tag{1}$$

The smallest pair of non-isomorphic graphs with the same bipartition polynomial is of order 10. We could show by exhaustive computer search that all non-isomorphic trees with up to 15 vertices and all graphs with up to 9 vertices can be distinguished by their bipartition polynomial.

We show that the bipartition polynomial has a spanning tree expansion. This is reminiscent of the tree expansion definition of the Tutte polynomial from Tutte's classical 1954 paper [21]. The Tutte polynomial had arisen independently in statistical mechanics as the partition function of the Potts model. As we will see, a closely related graph polynomial, the bivariate Ising polynomial, is a specialization of the bipartition polynomial. As a consequence, the bivariate Ising polynomial has a spanning tree expansion. The same is true for the domination polynomial, the cut polynomial and, on regular graphs, the independence polynomial.

This paper is organized as follows. Section 2 presents different representations of the bipartition polynomial that will be subsequently employed in order to prove properties of graph invariants. Section 3 gives an overview of graph polynomials and invariants that are encoded in the bipartition polynomial. A recursive representation of the bipartition polynomial is given in Section 4. The paper concludes with remarks and open problems.

## 2. Representations and Properties of the Bipartition Polynomial

Let $G = (V, E)$ be a simple graph with $n$ vertices and $m$ edges. We denote by $b_{ijk}(G)$ the number of triples $(W, T, F)$, where $W$ and $T$ are disjoint vertex sets of $G$ with $|W| = i$ and $|T| = j$, whereas $F$ is an edge subset of $G$ with exactly $k$ edges such that each edge of $F$ links a vertex of $W$ with a vertex of $T$ and each

vertex of $T$ is an end vertex of at least one edge of $F$. Its not hard to see that $b_{ijk}(G)$ are the coefficients of the bipartition polynomial and therefore

$$B(G;x,y,z) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{k=0}^{m} b_{ijk}(G)x^i y^j z^k. \tag{2}$$

Let $G = (V,E)$ be a graph and $F \subseteq E$. As all graphs considered here are assumed to be simple, we can identify an edge $e \in F$ with a two-element subset of $V$. Then $\bigcup F = \bigcup_{e\in F} e$ is the set of end vertices of edges of $F$. The *edge-induced subgraph* $G[F]$ of $G$ is the graph $(\cup F, F)$. Now we can rewrite Definition 1 in the following form:

$$B(G;x,y,z) = \sum_{W\subseteq V} x^{|W|} \sum_{F\subseteq \partial W} y^{|\cup F\setminus W|} z^{|F|}. \tag{3}$$

More interesting is the representation that we obtain by reversing the order of summation. In order to derive this result, we prove first that the bipartition polynomial is multiplicative with respect to components of a graph.

**Lemma 2.** *Let $G = (V,E)$ be a graph with two components $G^1 = (V^1,E^1)$ and $G^2 = (V^2,E^2)$. Then*

$$B(G;x,y,z) = B(G^1;x,y,z)B(G^2;x,y,z).$$

***Proof.*** The representation of $V$ and $E$ as disjoint union $V = V^1 \cup V^2$ and $E = E^1 \cup E^2$, respectively, implies that for all $W' \subseteq V^1$ and $W'' \subseteq V^2$ the set $\partial(W' \cup W'')$ is the disjoint union of $\partial W'$ and $\partial W''$, which yields

$$\begin{aligned}
B(G;x,y,z) &= \sum_{W\subseteq V^1\cup V^2} x^{|W|} \sum_{F\subseteq\partial W} y^{|\cup F\setminus W|} z^{|F|} \\
&= \sum_{W'\subseteq V^1} x^{|W'|} \sum_{W''\subseteq V^2} x^{|W''|} \sum_{F\subseteq\partial(W'\cup W'')} y^{|\cup F\setminus(W'\cup W'')|} z^{|F|} \\
&= \sum_{W'\subseteq V^1} x^{|W'|} \sum_{W''\subseteq V^2} x^{|W''|} \sum_{F'\subseteq\partial W'} y^{|\cup F'\setminus W'|} z^{|F'|} \sum_{F''\subseteq\partial W''} y^{|\cup F''\setminus W''|} z^{|F''|} \\
&= \sum_{W'\subseteq V^1} x^{|W'|} \sum_{F'\subseteq\partial W'} y^{|\cup F'\setminus W'|} z^{|F'|} \sum_{W''\subseteq V^2} x^{|W''|} \sum_{F''\subseteq\partial W''} y^{|\cup F''\setminus W''|} z^{|F''|} \\
&= B(G^1;x,y,z)B(G^2;x,y,z).
\end{aligned}$$

■

For a given spanning subgraph $(V,F)$ of a graph $G = (V,E)$, we denote by $\mathrm{iso}(V,F)$ the number of isolated vertices and by $\mathrm{Comp}(V,F)$ the set of proper components of $(V,F)$, where a component is called *proper* if it contains at least one edge. The next theorem is the motivation for the name of the bipartition polynomial.

**Theorem 3.** *The bipartition polynomial satisfies*

$$B(G;x,y,z) = \sum_{\substack{F\subseteq E\\ (V,F)\ \textit{is bipartite}}} z^{|F|}(1+x)^{\mathrm{iso}(V,F)} \prod_{(V_1\cup V_2,A)\in \mathrm{Comp}(V,F)} \left(x^{|V_1|}y^{|V_2|} + x^{|V_2|}y^{|V_1|}\right),$$

*where $V_1$ and $V_2$ are the bipartition sets of a proper component of $(V,F)$ with edge set $A$.*

***Proof.*** All edge subsets involved in the inner sum of (3) link vertices of $W$ with vertices of $N(W)$. Hence $(V,F)$ is in any case a bipartite graph. Consequently, if we take first the sum ranging over edge subsets, then the sum has to be restricted to edge subsets that form bipartite spanning subgraphs. We need *spanning* subgraphs as to allow the choice of all compatible vertex subsets of $G$. Now consider a spanning subgraph $(V,F)$. If $v \in V$ is an isolated vertex of $(V,F)$ then the subset $F$ appears twice in (3), namely with $v \in W$ and with $v \notin W$, providing the factor $(1+x)$. By Lemma 2, the bipartition polynomial of the subgraph consisting of all isolated vertices is $(1+x)^{\mathrm{iso}(V,F)}$.
A proper component of $(V,F)$ is itself a bipartite graph inducing a bipartition $V_1 \cup V_2$ of its vertex set. If $V_1 \subseteq W$ in (3) then $V_2 \subseteq \cup F \setminus W$, which results in the factor $x^{|V_1|}y^{|V_2|}$ of the generating function. The exchange of the roles of $V_1$ and $V_2$ yields the second term, $x^{|V_2|}y^{|V_1|}$. If $A$ is the vertex set of the proper component then $z^{|A|}(x^{|V_1|}y^{|V_2|} + x^{|V_2|}y^{|V_1|})$ is the bipartition polynomial of this component. Applying Lemma 2 again, we obtain the statement of the theorem. ■

**Definition 4.** Let $G = (V,E)$ be a graph with $m$ edges. We define a *linear ordering* of the edge set by numbering the edges, i.e., $E = \{e_1,\ldots,e_m\}$. Let $H = (V,F)$ be a spanning forest of $G$. We call an edge $e_j \in E \setminus F$ *externally active* with respect to $H$ if $e_j$ is the largest edge in the unique even cycle of $H + e_j$. Observe that we define an edge to be externally active only in case it closes an *even* cycle, which differs from the original definition as used for the Tutte polynomial. The *external activity* of a spanning forest $H$, denoted by $\mathrm{ext}(H)$, is the number of externally active edges of $H$.

The following statement is a consequence of Theorem 3.

**Corollary 5.** *The bipartition polynomial of a graph $G = (V,E)$ with $n$ vertices and $m$ linearly ordered edges can be represented as a sum over spanning forests of $G$ as*

$$B(G;x,y,z) = \sum_{\substack{H\ \textit{spanning}\\ \textit{forest of } G}} (1+x)^{\mathrm{iso}(H)} z^{n-k(H)} (1+z)^{\mathrm{ext}(H)} \times \prod_{(V_1\cup V_2,F)\in \mathrm{Comp}(H)} \left(x^{|V_1|}y^{|V_2|} + x^{|V_2|}y^{|V_1|}\right),$$

*where $k(H)$ denotes the number of components of $H$.*

**Remark 6.** Observe that a *spanning forest* of $G$ is defined here as a forest that contains all vertices of $G$. This does not imply that all components of a spanning forest are spanning trees of the components of $G$. Thus a spanning forest of $G$ may have more components than $G$.

***Proof.*** Let $(V, F)$ be a spanning bipartite subgraph of $G$ and denote by $\mathcal{C}(V, F)$ the set of components of $(V, F)$. Then we define

$$\pi(V, F) = \{\{V_1, V_2\} \mid (V_1 \cup V_2, A) \in \mathcal{C}(V, F)\}$$

as a nested partition of the vertex set of $G$. Observe that the product

$$(1+x)^{\mathrm{iso}(V,F)} \prod_{(V_1 \cup V_2, A) \in \mathrm{Comp}(V,F)} \left( x^{|V_1|} y^{|V_2|} + x^{|V_2|} y^{|V_1|} \right)$$

in Theorem 3 depends only on the partition $\pi(V, F)$; it is independent of the edge set of a component as long as the induced bipartition is the same. Consequently, we can count first those spanning subgraphs with a minimum number of edges that induce a certain nested partition, that is spanning forests. Let $e = \{u, v\} \in E(G) \setminus F$ be an edge such that $u$ and $v$ belong to the same component of $(V, F)$. If $(V, F \cup \{e\})$ does not contain an odd cycle, then $\pi(V, F \cup \{e\}) = \pi(V, F)$. Every bipartite spanning subgraph of $G$ can be obtained from a spanning forest of $G$ by insertion of edges. In order to create each bipartite spanning subgraph exactly once, we use a linear ordering of the edges and insert an edge $e$ only if it is the maximum edge in the unique cycle generated by inserting $e$ in a spanning forest of $G$. The polynomial $(1+z)^{\mathrm{ext}(H)}$ counts the additional edges, whereas $z^{n-k(H)}$ counts the edges in a spanning forest $H$ with $k(H)$ components. ■

If $f(x, y, z)$ is a polynomial in the variables $x, y, z$, then we denote by

$$[x^i y^j z^k] f(x, y, z)$$

the coefficient of $x^i y^j z^k$ in the expanded form of $f(x, y, z)$. The interpretation of the coefficients of the bipartition polynomial leads to the following observation.

**Theorem 7.** *The bipartition polynomial has the following multiplicative representation:*

$$B(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} \prod_{v \in N_G(W)} \left[ y \left[ (1+z)^{|N_G(v) \cap W|} - 1 \right] + 1 \right]. \tag{4}$$

***Proof.*** Let $W$ be a given subset of $V$ and $k \in \{0, 1, \ldots, |N_G(W)|\}$. Then the coefficient of $x^{|W|}y^k$ is an ordinary generating function in $z$ presenting the number of ways to select an edge subset $F \subseteq \partial W$ such that $|\cup F \setminus W| = k$. A vertex $v \in N_G(W)$ contributes to the exponent $k$ of $y^k$ if and only if at least one the $|N_G(v) \cap W|$ edges that link $v$ with vertices from $W$ belongs to $F$. The generating function for the choice of a nonempty subset of $N_G(v) \cap W$ is $(1+z)^{|N_G(v)\cap W|} - 1$. Now the polynomial $y\left[(1+z)^{|N_G(v)\cap W|} - 1\right] + 1$ represents the two alternatives, namely to select a nonempty subset of $N_G(v) \cap W$ or to choose the empty set, where in the latter case the factor $y$ disappears as $v$ is not connected to $W$. ■

## 3. Encoded Graph Invariants

We consider first some other graph polynomials that can be obtained from the bipartition polynomial by substitution of variables. The first one is the so-called Ising model, that has its roots in statistical mechanics. The following more graph theoretical definition of the Ising polynomial is essentially equivalent to that one given by Andrén and Markström [3]. Uniqueness with respect to the 2-state Potts model partition function, which is contained both in the Ising polynomial and the Tutte polynomial, was studied recently in [9]. For complexity studies of the Ising polynomial, see [15]. Recently, [8] studied graph polynomials arising from partition functions in statistical mechanics under a general framework. The Ising and cut polynomials can also be represented as partition functions [20].

### 3.1. The Ising polynomial

**Definition 8.** The *Ising polynomial* of a graph $G = (V, E)$ with $n$ vertices and $m$ edges is defined by

$$Z(G; x, y) = x^n y^m \sum_{W \subseteq V} x^{-|W|} y^{-|\partial W|}. \tag{5}$$

**Remark 9.** Observe that the Ising polynomial is given in [3] as a Laurent polynomial. Let $\hat{Z}(G; x, y)$ be the Ising polynomial given in [3]. Then the two polynomials can be transformed by

$$\hat{Z}(G; x, y) = x^{-n} y^{-m} Z(G; x^2, y^2).$$

**Theorem 10.** *The Ising polynomial of a graph $G = (V, E)$ with $n$ vertices and $m$ edges is given by*

$$Z(G; x, y) = x^n y^m B\left(G; \frac{1}{x}, 1, \frac{1}{y} - 1\right).$$

***Proof.*** By Theorem 7 we can easily verify, that

$$B(G;x,1,z-1)=\sum_{W\subseteq V} x^{|W|}z^{|\partial W|}.$$

Comparing this polynomial with

$$Z(G;x,y)=x^n y^m \sum_{W\subseteq V} x^{-|W|}y^{-|\partial W|}$$

yields the statement. ■

The proof also shows that the *degree generating function* of a graph $G=(V,E)$ is

$$\sum_{v\in V} t^{\deg v}=\left[x^1\right] B(G;x,1,t-1).$$

Combining Corollary 5 and Theorem 10, we obtain the following result.

**Theorem 11.** *The Ising polynomial of a graph $G=(V,E)$ with $n$ vertices and $m$ edges is given by*

$$Z(G;x,y)=\sum_{\substack{H \text{ spanning}\\ \text{forest of } G}} y^{m-n+k(H)}(1+x)^{\mathrm{iso}(H)}(1-y)^{n-k(H)} \prod_{(S\cup T,F)\in \mathrm{Comp}(H)} \left(x^{|S|}+x^{|T|}\right).$$

### 3.2. The independence polynomial

An *independent set* of a graph $G(V,E)$ is a vertex set $W\subseteq V$ such that $u,v\in W$ implies $\{u,v\}\notin E$. The *independence polynomial* [4, 12, 14] of $G$ is defined by

$$I(G;t)=\sum_{\substack{W\subseteq V\\ W \text{ ind. in } G}} t^{|W|}.$$

Andrén and Markström [3] showed that the independence polynomial can be obtained from the Ising polynomial of a $r$-regular graph. The next theorem gives a direct way to obtain the independence polynomial from the bipartition polynomial in $r$-regular graphs.

**Theorem 12.** *Let $G=(V,E)$ be a simple $r$-regular graph. Then the independence polynomial of $G$ is given by*

$$I(G,t)=\lim_{x\to 0} B\left(G;tx^r,1,\frac{1}{x}-1\right).$$

***Proof.*** If $W$ is an independent set of $G$, then $|\partial W| = r|W|$, whereas for any dependent set $W$ the relation $|\partial W| < r|W|$ follows. Now consider the polynomial

$$B\left(G; tx^r, 1, \frac{1}{x} - 1\right) = \sum_{W \subseteq V} t^{|W|} x^{r|W| - |\partial W|},$$

which we may verify again by Theorem 7. Any term $t^j x^0$ in this polynomial corresponds to an independent set of size $j$. Dependent sets are counted by $t^j x^k$ with $k > 0$, hence

$$I(G,t) = \left[x^0\right] B\left(G; tx^r, 1, \frac{1}{x} - 1\right).$$

■

### 3.3. The domination polynomial

We can obtain the (univariate) domination polynomial from the bipartition polynomial of a graph $G$.

**Theorem 13.** *The domination polynomial of a graph $G$ with $n$ vertices can be obtained from the bipartition polynomial by*

$$D(G,x) = [y^n]\, B(G; xy, 1-y, -1). \tag{6}$$

***Proof.*** We use Theorem 7 and getting:

$$\begin{aligned} \mathrm{B}(G; xy, 1-y, -1) &= \sum_{W \subseteq V} (xy)^{|W|} \prod_{v \in N_G(W)} (-(1-y)+1) \\ &= \sum_{W \subseteq V} x^{|W|} y^{|W| + |N_G(W)|}. \end{aligned}$$

This gives us the theorem. ■

The *domination number* $\gamma(G)$ of a graph $G$ is the cardinality of a minimum dominating set of $G$. We obtain from Equation (6) for a graph of order $n$,

$$\gamma(G) = \min\left\{k \in \mathbb{N} \mid [x^k y^n] B(G; xy, 1-y, -1) > 0\right\}.$$

A vertex subset $W$ in a graph $G = (V, E)$ is non-dominating if and only if $V \setminus W$ contains the closed neighborhood of at least one vertex of $G$. By applying the principle of inclusion–exclusion, we conclude that the domination polynomial satisfies the equation

$$D(G,x) = \sum_{W \subseteq V} (-1)^{|W|} (1+x)^{|V| - |N[W]|}, \tag{7}$$

compare also [7].

**Proposition 14.** *The domination polynomial of a graph $G$ of order $n$ satisfies*

$$D(G,x) = (1+x)^n B\left(G; \frac{-1}{1+x}, \frac{x}{1+x}, -1\right).$$

***Proof.*** We substitute the variable in the product representation (4) of the bipartition polynomial according to the statement of the proposition, which yields

$$\begin{aligned}
&(1+x)^n \sum_{W\subseteq V} \left(\frac{-1}{1+x}\right)^{|W|} \prod_{v\in N_G(W)} \left[\frac{x}{1+x}\left(0^{|N_G(v)\cap W|}-1\right)+1\right] \\
&= \sum_{W\subseteq V} (-1)^{|W|}(1+x)^{n-|W|} \prod_{v\in N_G(W)} \frac{1}{1+x} \\
&= \sum_{W\subseteq V} (-1)^{|W|}(1+x)^{n-|N_G[W]|},
\end{aligned}$$

which is just the representation (7) of the domination polynomial. ■

The next theorem was presented in [16]. The bipartition polynomial makes the proof much simpler than the one given in [16].

**Theorem 15.** *The domination polynomial of any graph $G=(V,E)$ satisfies*

$$\begin{aligned}
D(G,x) = &\sum_{\substack{F\subseteq E \\ (V,F) \text{ is bipartite}}} x^{\mathrm{iso}(V,F)} \\
&\times \prod_{(Y\cup Z,A)\in \mathrm{Comp}(V,F)} (-1)^{|A|}\left[(-1)^{|Y|}x^{|Z|}+(-1)^{|Z|}x^{|Y|}\right].
\end{aligned}$$

***Proof.*** Evaluating $(1+x)^n B\left(G; \frac{-1}{1+x}, \frac{x}{1+x}, -1\right)$ as a sum over bipartite spanning subgraphs of $G$ according to Theorem 3 gives

$$\begin{aligned}
D(G,x) = (1+x)^n &\sum_{\substack{F\subseteq E \\ (V,F) \text{ is bipartite}}} (-1)^{|F|}\left(\frac{x}{1+x}\right)^{\mathrm{iso}(V,F)} \\
&\times \prod_{(V_1\cup V_2,A)\in \mathrm{Comp}(V,F)} \frac{(-1)^{|V_1|}x^{|V_2|}+(-1)^{|V_2|}x^{|V_1|}}{(1+x)^{|V_1\cup V_2|}} \\
= &\sum_{\substack{F\subseteq E \\ (V,F) \text{ is bipartite}}} (-1)^{|F|}x^{\mathrm{iso}(V,F)} \prod_{(V_1\cup V_2,A)\in \mathrm{Comp}(V,F)} \left[(-1)^{|V_1|}x^{|V_2|}+(-1)^{|V_2|}x^{|V_1|}\right],
\end{aligned}$$

which has been shown in [16] as Equation (3). ■

The number $d(G)$ of dominating sets of a graph $G$ of order $n$ is obtained by substituting $x = 1$ in the domination polynomial; we obtain by Proposition 14

$$d(G) = 2^n B\left(G; -\frac{1}{2}, \frac{1}{2}, -1\right). \tag{8}$$

Let $\mathcal{F}_0(G)$ be the set of spanning forests of external activity 0 that contain only trees of even order and, possibly, isolated vertices. We call a tree $T$ of even order *essential* if there exists a vertex $v$ in $T$ such that the number of vertices that have even distance from $v$ in $T$ is odd. Observe that this definition does not depend on the choice of the vertex $v$ from which we measure the distance. We denote by $\mathrm{ess}(H)$ the number of essential trees of a forest $H$. Substituting Equation (8) in Corollary 5, we obtain the following statement.

**Theorem 16.** *The number of dominating sets of a graph $G$ of order $n$ satisfies*

$$d(G) = \sum_{H \in \mathcal{F}_0(G)} 2^{k(H)-\mathrm{iso}(H)} (-1)^{n-k(H)+\mathrm{ess}(H)}.$$

There is exactly one term in this sum that is odd, which corresponds to the edgeless forest. Hence the number of dominating sets of any graph is odd. This fact has been proven by Brouwer in [6].

A similar result can be obtained for the domination polynomial.

**Proposition 17.** *Let $G = (V, E)$ be a graph of order $n$ and $\mathcal{H}_0(G)$ the set of spanning forests of external activity 0 of $G$. The domination polynomial of $G$ satisfies*

$$D(G, x) = \sum_{H \in \mathcal{H}_0(G)} (-1)^{n-k(H)} x^{\mathrm{iso}(V,F)} \prod_{(V_1 \cup V_2, A) \in \mathrm{Comp}(H)} \left[(-1)^{|V_1|} x^{|V_2|} + (-1)^{|V_2|} x^{|V_1|}\right].$$

***Proof.*** Using Proposition 14 and Corollary 5, we obtain

$$\begin{aligned}
&(1+x)^n B\left(G; \frac{-1}{1+x}, \frac{x}{1+x}, -1\right) \\
&\quad = (1+x)^n \sum_{\substack{H \text{ spanning} \\ \text{forest of } G}} \left(\frac{x}{1+x}\right)^{\mathrm{iso}(H)} (-1)^{n-k(H)} 0^{\mathrm{ext}(H)} \\
&\quad \times \prod_{(V_1 \cup V_2, F) \in \mathrm{Comp}(H)} \frac{(-1)^{|V_1|} x^{|V_2|} + (-1)^{|V_2|} x^{|V_1|}}{(1+x)^{|V_1 \cup V_2|}} \\
&\quad = \sum_{H \in \mathcal{H}_0(G)} x^{\mathrm{iso}(H)} (-1)^{n-k(H)} \\
&\quad \times \prod_{(V_1 \cup V_2, F) \in \mathrm{Comp}(H)} \left[(-1)^{|V_1|} x^{|V_2|} + (-1)^{|V_2|} x^{|V_1|}\right],
\end{aligned}$$

which provides the statement. ∎

### 3.4. The cut polynomial

Let $G = (V, E)$ be a graph and $W \subseteq V$. A *cut* in $G$ is an edge subset of the form

$$(W, V \setminus W) = \{\{u, v\} \mid u \in W, v \in V \setminus W\}$$

and $C(G, x)$ is the ordinary generating function for the number of cuts in $G$.

Observe that $\partial W = (W, V \setminus W)$. If $G$ is connected and $\partial W \neq \emptyset$, then $G - \partial W$, i.e., the graph obtained from $G$ by the removal of all edges from $\partial W$, has at least two components.

**Theorem 18.** *Let $G$ be a graph and $C(G, x)$ the cut polynomial of the graph. Then*

$$C(G, z) = \frac{1}{2^{k(G)}} B(G; 1, 1, z - 1).$$

***Proof.*** By Theorem 7,

$$\begin{aligned} B(G; 1, 1, z - 1) &= \sum_{W \subseteq V} \prod_{v \in N_G(W)} z^{|N_G(v) \cap W|} \\ &= \sum_{W \subseteq V} z^{\sum_{v \in N_G(W)} |N_G(v) \cap W|} = \sum_{W \subseteq V} z^{|\partial W|}. \end{aligned}$$

Here each cut is counted $2^{k(G)}$ times as a set $W$ and its complement induce the same cut. ■

The *edge connectivity* $\lambda(G)$ of a connected graph $G$ is the minimum number of edges of $G$ that has to be removed in order to make $G$ disconnected. For a disconnected graph $G$, we define $\lambda(G) = 0$. The edge connectivity of a graph is alternatively defined by

$$\lambda(G) = \min\{|\partial W| \mid W \subset V, W \neq \emptyset\}.$$

A graph $G$ is said to be *$k$-edge connected* if $\lambda(G) \geq k$.

We obtain the following statements as an immediate consequence of Theorem 18.

**Corollary 19.** *Let $k \geq 2$ be an integer. A graph $G$ is $k$-edge connected if and only if $[z^j]B(G; 1, 1, z - 1) = 0$ for all $j = 1, \ldots, k - 1$.*

**Corollary 20.** *The edge connectivity of a graph $G$ of order $n$ is*

$$\lambda(G) = n - \deg\left[z^n \left(B\left(G; 1, 1, \frac{1}{z} - 1\right) - 2\right)\right].$$

**Corollary 21.** *A graph $G = (V, E)$ is bipartite if and only if* $\deg B(G; 1, 1, z - 1) = |E|$.

A tree is the only connected graph for which any edge subset is a cut. This together with Theorem 18 gives the following statement.

**Proposition 22.** *A connected graph of order $n$ is a tree if and only if*

$$B(G; 1, 1, z-1) = 2(1+z)^{n-1}.$$

**Proposition 23.** *The cut polynomial of a graph $G$ of order $n$ satisfies*

$$C(G, z) = \sum_{\substack{H \text{ spanning} \\ \text{forest of } G}} 2^{k(H)-k(G)}(z-1)^{n-k(H)} z^{\mathrm{ext}(H)}.$$

***Proof.*** The statement is easily obtained by substituting the variables of the bipartition polynomial as represented in Corollary 5 according to Theorem 18. ■

### 3.5. The matching polynomial

Let $G = (V, E)$ be a graph with $n$ vertices. We denote by $m_k(G)$ the number of matchings of cardinality $k$ of $G$. The *matching polynomial* of $G$ is defined by

$$M(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m_k(G) x^k.$$

**Remark 24.** Observe that the matching polynomial of a graph, introduced in [13], is defined in other contexts as

$$M_G(x) = x^n M(G, -x^{-2}).$$

For more information on matching polynomials, see [10].

**Theorem 25.** *Let $G$ be a graph and $b_{ijk}(G)$ the coefficients of the bipartition polynomial as defined in Equation* (2)*. The matching polynomial of $G$ satisfies*

$$M(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{x}{2}\right)^k \sum_{i=1}^{k} (-1)^{k-i} \binom{n-k-i}{k-i} b_{ikk}(G).$$

***Proof.*** Let $G$ be a graph of order $n$ and let $k$ be a given positive integer. We use the abbreviations $b_i = b_{ikk}(G)$ and $p = n - k$. Let $D_i(G)$ be the set of all bipartite subgraphs $H = (X \cup Y, F)$ of $G$ with $|X| = i$, $|Y| = k$, $|F| = k$, such that all vertices of $Y$ have degree 1 in $H$. Hence the cardinality of $D_i(G)$ is $b_i$. Observe that we consider $X \cup Y$ as an *ordered* bipartition, which implies that bipartite subgraphs that are identical except that the sets $X$ and $Y$ are

exchanged are counted twice in $b_i$. Let $C_i(G)$ be the subset of $D_i(G)$ consisting of those (ordered) bipartite subgraphs of $G$ that do not have any isolated vertices in $X$ and define $c_i = |C_i(G)|$. As the end vertices of any edge in a matching can be arbitrarily assigned to $X$ or $Y$, we have $2^k m_k = c_k$. Each bipartite subgraph in $D_i(G)$ that contains exactly $i-j$ isolated vertices in $X$ is composed of a subgraph $H$ from $C_j(G)$ and a selection of $i-j$ vertices out of the $n-k-j$ vertices that do not belong to $H$. Consequently, we obtain

$$b_i = \sum_{j=1}^{i} \binom{n-k-j}{i-j} c_j = \sum_{j=1}^{i} \binom{p-j}{i-j} c_j. \tag{9}$$

The theorem states that

$$m_k = \frac{1}{2^k} \sum_{i=1}^{k} (-1)^{k-i} \binom{n-k-i}{k-i} b_{ikk}(G)$$

and hence

$$c_k = \sum_{i=1}^{k} (-1)^{k-i} \binom{p-i}{k-i} b_i.$$

Replacing $k$ by $j$ and substituting $c_j$ in (9) yields

$$\begin{aligned} b_i &= \sum_{j=1}^{i} \binom{p-j}{i-j} \sum_{l=1}^{j} (-1)^{j-l} \binom{p-l}{j-l} b_l \\ &= \sum_{l} b_l \sum_{j} \binom{p-l}{j-l} \binom{p-j}{i-j} (-1)^{j-l}. \end{aligned}$$

Thus it remains to prove that

$$\sum_{j} \binom{p-l}{j-l} \binom{p-j}{i-j} (-1)^{j-l} = \delta_{il}.$$

Rearranging the binomial coefficients gives

$$\sum_{j} \binom{p-l}{j-l} \binom{p-j}{i-j} (-1)^{j-l} = \binom{p-l}{p-i} \sum_{j} \binom{i-l}{i-j} (-1)^{j-l}.$$

If $i = l$, then the last sum has only one non-vanishing term, which is 1.

Otherwise, if $i \neq l$, then the binomial coefficient or the sum vanishes, which completes the proof. ∎

## 4. A Recursive Representation

**Lemma 26.** *Let $G = (V, E)$ be a connected bipartite graph with $V = S \cup T$, where $S$ and $T$ are the bipartition sets of $G$. Then*

$$\sum_{J \subseteq E} (-1)^{|J|} B(G - J; x, y, z) = \begin{cases} 1 + x, \text{ if } E = \emptyset, \\ z^{|E|} \left( x^{|S|} y^{|T|} + x^{|T|} y^{|S|} \right), \text{ otherwise.} \end{cases}$$

***Proof.*** In order to simplify the presentation of the proof, we introduce the notation

$$f(G) = \sum_{J \subseteq E} (-1)^{|J|} B(G - J; x, y, z).$$

If the edge set of $G$ is empty then $G$ consists of a single vertex, otherwise $G$ would be disconnected. Thus the sum contains only one term yielding $f(G) = B(G; x, y, z) = 1 + x$.

Now assume that $G$ is a connected bipartite graph with at least one edge. We rearrange the sum, use the definition of the bipartition polynomial, and change the order of summation:

$$\begin{aligned} f(G) &= \sum_{J \subseteq E} (-1)^{|E|-|J|} B((V, J); x, y, z) \\ &= \sum_{J \subseteq E} (-1)^{|E|-|J|} \sum_{W \subseteq V} x^{|W|} \sum_{F \subseteq \partial W \cap J} y^{\left|N_{(V,F)}(W)\right|} z^{|F|} \\ &= \sum_{W \subseteq V} x^{|W|} \sum_{J \subseteq E} (-1)^{|E|-|J|} \sum_{F \subseteq \partial W \cap J} y^{\left|N_{(V,F)}(W)\right|} z^{|F|} \\ &= \sum_{W \subseteq V} x^{|W|} \sum_{F \subseteq \partial W} z^{|F|} y^{\left|N_{(V,F)}(W)\right|} \sum_{J \supseteq F} (-1)^{|E|-|J|} \end{aligned}$$

Here the notation $J \supseteq F$ means that $J$ is a superset of $F$ but clearly also a subset of $E$. We distinguish three choices for the vertex subset $W \subseteq V$ that is used in the first sum, namely $W = S$, $W = T$, and $W$ different from $S$ and $T$. Observe that $\partial S = \partial T = E$ as $G$ is connected and bipartite. Consequently, we obtain

$$\begin{aligned} f(G) = {} & x^{|S|} \sum_{F \subseteq E} z^{|F|} y^{\left|N_{(V,F)}(S)\right|} \sum_{J \supseteq F} (-1)^{|E|-|J|} \\ & + x^{|T|} \sum_{F \subseteq E} z^{|F|} y^{\left|N_{(V,F)}(T)\right|} \sum_{J \supseteq F} (-1)^{|E|-|J|} \\ & + \sum_{\substack{W \subseteq V \\ W \neq S \\ W \neq T}} x^{|W|} \sum_{F \subseteq \partial W} z^{|F|} y^{\left|N_{(V,F)}(W)\right|} \sum_{J \supseteq F} (-1)^{|E|-|J|}. \end{aligned}$$

Now the sum $\sum_{J\supseteq F}(-1)^{|E|-|J|}$ vanishes for all $F \neq E$; in case $F = E$, it equals 1. The open neighborhood of $S$ in $G = (V, E)$ is $T$ and analogous $N_G(T) = S$, which gives

$$\begin{aligned} f(G) &= x^{|S|}z^{|F|}y^{|T|} + x^{|T|}z^{|F|}y^{|S|} \\ &\quad + \sum_{\substack{W\subseteq V \\ W\neq S \\ W\neq T}} x^{|W|}z^{|E|}[E = \partial W]y^{|N_{(V,E)}(W)|}. \end{aligned}$$

However, the condition $E = \partial W$ is only satisfied for $W = S$ or $W = T$. As these two cases are excluded by the range of summation, the last sum vanishes. ■

If $H$ is a graph, then we denote by $E(H)$ its edge set and by $\mathcal{C}(H)$ the set of all components of $H$. The following theorem provides a recursive representation for the bipartition polynomial.

**Theorem 27.** *Let $G = (V, E)$ be a graph. Then*

$$B(G; x, y, z) = \sum_{\substack{F\subseteq E \\ (V,F) \text{ is bipartite}}} \prod_{H\in\mathcal{C}((V,F))} \sum_{J\subseteq E(H)} (-1)^{|J|}B(H - J; x, y, z).$$

***Proof.*** We rewrite the product representation given in Theorem 3 in the following form

$$\begin{aligned} B(G; x, y, z) &= \sum_{\substack{F\subseteq E \\ (V,F) \text{ is bipartite}}} (1 + x)^{\mathrm{iso}(V,F)} \prod_{(V_1\cup V_2,A)\in\mathrm{Comp}(V,F)} z^{|A|}\left(x^{|V_1|}y^{|V_2|} + x^{|V_2|}y^{|V_1|}\right) \\ &= \sum_{\substack{F\subseteq E \\ (V,F) \text{ is bipartite}}} \prod_{H\in\mathcal{C}((V,F))} g(H), \end{aligned}$$

where we denote the bipartition of the vertex set of $H$ by $S \cup T$ and define

$$g(H) = \begin{cases} 1 + x, \text{ if } E(H) = \emptyset, \\ z^{|E(H)|}\left(x^{|S|}y^{|T|} + x^{|T|}y^{|S|}\right), \text{ otherwise.} \end{cases}$$

Substituting the function $g(H)$ according to Lemma 26 yields the desired result. ■

Theorem 27 shows that the bipartition polynomial of a non-bipartite graph is determined by the set of bipartition polynomials of all its bipartite spanning subgraphs.

## 5. Conclusion and Open Questions

The bipartition polynomial is a powerful tool for proving properties of graph polynomials and graph invariants that are related to dominating sets, matchings, independent sets, or the cut structure of graphs. However, in order to cover also coloring problems, an extension of its definition is required that includes information about the edges which are completely in $W$ and completely outside of $W$. Thus, we cannot expect to find a close relation between the bipartition polynomial and the Tutte-rank-Potts family of polynomials.

There remain some interesting open questions for future research with respect to the bipartition polynomial. The bipartition polynomial distinguishes non-isomorphic graphs quite well. However, it seems to be difficult to characterize non-isomorphic graphs that have the same bipartition polynomial.

**Problem 28.** Which properties of two non-isomorphic graphs cannot be distinguished by the bipartition polynomial?

**Problem 29.** Are there two non-isomorphic trees with the same bipartition polynomial?

**Problem 30.** Is the bipartition polynomial reconstructible from its polynomial deck?

We found some nice representations for the bipartition polynomial. However, the following question is still open.

**Problem 31.** Is there a recurrence relation for the bipartition polynomial with respect to elementary vertex and edge operations?

There exists some results for the calculation of domination polynomials of graph products [17], e.g. for the Cartesian product and the strong product. But for the bipartition polynomial not such a result exists.

**Problem 32.** How can we calculate the bipartition polynomial for graph products?

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