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OPTIMAL LOCATING-TOTAL DOMINATING SETS IN STRIPS OF HEIGHT 3

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Abstract

A set C of vertices in a graph $G = (V, E)$ is total dominating in G if all vertices of V are adjacent to a vertex of C. Furthermore, if a total dominating set C in G has the additional property that for any distinct vertices $u, v \in V \setminus C$ the subsets formed by the vertices of C respectively adjacent to u and v are different, then we say that C is a locating-total dominating set in G.

Previously, locating-total dominating sets in strips have been studied by Henning and Jafari Rad (2012). In particular, they have determined the sizes of the smallest locating-total dominating sets in the finite strips of height 2 for all lengths. Moreover, they state as open question the analogous problem for the strips of height 3. In this paper, we answer the proposed question by determining the smallest sizes of locating-total dominating sets in the finite strips of height 3 as well as the smallest density in the infinite strip of height 3.

Keywords: locating-total dominating set, domination, square grid, strip.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For the rest of the paper, assume that G is a simple and undirected graph containing no isolated vertices. Let u and v be vertices in V. If u and v are adjacent to each other, then the edge between u and v is denoted by $\{u, v\}$ or in short by uv. The set of vertices adjacent to u is called the *open neighborhood* of u and is denoted by $N(u)$. The *closed neighborhood* of u is defined as $N[u] = N(u) \cup \{u\}$.

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A nonempty subset of V is called a *code*, of which elements are called *codewords*. A code $C \subseteq V$ is a *dominating set* (or *code*) in G if for each vertex $u \in V \setminus C$ the intersection $N(u) \cap C$ is nonempty. Furthermore, we say that $C \subseteq V$ is a *total dominating set* (or *code*) in G if $N(u) \cap C$ is nonempty for all $u \in V$. In addition, a total dominating set is a *locating-total dominating set* (or *code*) in G if for each vertex $u \in V \setminus C$ the intersection $N(u) \cap C$ is unique. This concept was first introduced by Haynes *et al.* [7]. In what follows, we give a more formal definition for locating-total dominating sets.

Definition 1.1. A code $C \subseteq V$ is a *locating-total dominating set* in G if for each vertex $u \in V$ the intersection $N(u) \cap C$ is nonempty and for all distinct vertices $v_1, v_2 \in V \setminus C$ we have $N(v_1) \cap C \neq N(v_2) \cap C$. If the graph G is finite, then the minimum cardinality of a locating-total dominating set in G is denoted by $\gamma_t^L(G)$ and a locating-total dominating set with $\gamma_t^L(G)$ vertices is called *optimal*.

Previously, locating-total dominating sets have been considered, for example, in [1, 2, 3, 6]. In this paper, we consider locating-total dominating sets in some subgraphs of the infinite square grid. The *infinite square grid* $\mathcal{S}$ is defined as the graph with the vertex set $V = \mathbb{Z}^2$ and the edge set $E = \{(\mathbf{u}, \mathbf{v}) \in V^2 \mid \mathbf{u} - \mathbf{v} \in \{(0, \pm 1), (\pm 1, 0)\}\}$. In other words, two vertices $\mathbf{u}, \mathbf{v} \in V$ are adjacent if they are exactly at Euclidean distance 1 from each other. The *infinite strip* $\mathcal{S}_h$ of height h is the subgraph of the square grid induced by the vertex set $\{1, 2, \ldots, h\} \times \mathbb{Z}$. Part of the infinite strip $\mathcal{S}_3$ is illustrated in Figure 1, where the lines represent the edges of the graph and the intersections of the lines represent the vertices. In order to measure the size of a locating-total dominating set in the infinite strip $\mathcal{S}_h$, we introduce the notion of density. For the formal definition, we first define $Q_n = \{(x, y) \in \mathbb{Z}^2 \mid 1 \leq x \leq h, |y| \leq n\}$.

Then the *density* of a set $C \subseteq V$ is defined as

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap Q_n|}{|Q_n|}.$$

Analogously to finite graphs, a smallest locating-total dominating set in $\mathcal{S}_h$ (regarding density) is called *optimal*. In Section 3, it is shown that the density of an optimal locating-total dominating set in $\mathcal{S}_3$ is equal to 7/18.

The *finite strip* $\mathcal{S}_{h,n}$ of height h and length n is the subgraph of $\mathcal{S}$ induced by the vertex set $\{1, 2, \ldots, h\} \times \{1, 2, \ldots, n\}$. Previously, locating-total domination in finite strips $\mathcal{S}_{h,n}$ with small h have been considered by Henning and Jafari Rad [8]. In particular, they have determined the exact sizes of optimal locating-total dominating sets for all lengths when $h = 2$. More precisely, they have shown that if $n \equiv r \pmod 5$, where $0 \leq r < 5$, then

$$\gamma_t^L(\mathcal{S}_{2,n}) = \begin{cases} 4\left\lfloor \frac{n}{5} \right\rfloor + r & \text{if } r \neq 1, \\ 4\left\lfloor \frac{n}{5} \right\rfloor + 2 & \text{if } r = 1. \end{cases}$$

In the case $h=3$, they have presented a construction stating that for $n \equiv 0 \pmod{11}$

$$\gamma_t^L(\mathcal{S}_{3,n}) \leq \frac{13}{11}n.$$

This construction immediately implies a locating-total dominating set in the infinite strip $\mathcal{S}_3$ with density $13/33$. Furthermore, it is shown that for small values of n, namely $1 \leq n \leq 12$,

$$\gamma_t^L(\mathcal{S}_{3,n}) = \left\lceil \frac{13}{11}n \right\rceil.$$

Finally, they state as an open problem to determine the sizes of optimal locating-total dominating sets in $\mathcal{S}_{3,n}$ when $n \geq 13$.

Recently, an algorithmic approach for determining optimal densities of various domination problems in fasciagraphs and rotagraphs such as infinite strips has been introduced in [4] and [5]. In particular, it is shown in [4, p. 86] that the density of an optimal locating-total dominating set in $\mathcal{S}_3$ is equal to $7/18$. In this paper, we give an analytical proof (without exhaustive computer searches) for the optimal density $7/18$. Furthermore, we determine the sizes of optimal locating-total dominating sets in $\mathcal{S}_{3,n}$ in all the remaining cases (when $n \geq 13$).

In what follows, we outline the structure of the paper. In Section 2, we begin by introducing the concept of share and explain its usage in obtaining lower bounds. Then, in Section 3, we proceed by constructing a locating-total dominating set in the infinite strip $\mathcal{S}_3$ of density $7/18$ and prove that this construction is optimal. Finally, in Section 4, we determine the sizes of optimal locating-total dominating sets in the finite strips $\mathcal{S}_{3,n}$ for all n with the aid of the results concerning an optimal locating-total dominating set in the infinite strip. In particular, we show that

$$\gamma_t^L(\mathcal{S}_{3,n}) = \begin{cases} \left\lceil \frac{7n}{6} \right\rceil & \text{if } n \not\equiv 0 \pmod{6}, \\ \frac{7n}{6} + 1 & \text{if } n \equiv 0 \pmod{6}. \end{cases}$$

2. Lower Bounds Using Share

Let $G = (V, E)$ be a simple and undirected graph with no isolated vertices. Assume also that C is a code in G. The following concept of the share of a codeword has been introduced by Slater in [9]. The *share* of a codeword $c \in C$ is defined as

$$s(C; c) = s(c) = \sum\nolimits_{u \in N[c]} \frac{1}{|N[u] \cap C|}.$$

The notion of share proves to be useful in determining lower bounds of locating-total dominating sets (as explained in the following paragraph).

Assume that $G = (V, E)$ is a finite graph and C is a code in G such that $N[u] \cap C$ is nonempty for all $u \in V$. Then it is easy to conclude that $\sum_{c \in C} s(C; c) = |V|$. Assume further that $s(C; c) \leq \alpha$ for all $c \in C$. Then we have $|V| \leq \alpha |C|$, which immediately implies

$$|C| \geq \frac{1}{\alpha}|V|.$$

Assume then that for any locating-total dominating set C in G we have $s(C; c) \leq \alpha$ for all $c \in C$. By the aforementioned observation, we then obtain the lower bound $|V|/\alpha$ for the size of a locating-total dominating set in G. This reasoning can also be generalized to the case when an infinite graph is considered. In particular, if for any locating-total dominating set C in $\mathcal{S}_h$ we have $s(C; c) \leq \alpha$ for all $c \in C$, then it can be shown that the density of a locating-total dominating set in $\mathcal{S}_h$ is at least $1/\alpha$ (compare to Theorem 3.2).

In the previous paragraph, we assumed that the share of each codeword is bounded above by a fixed value. However, it can be observed that it is actually enough that the shares of codewords are on *average* bounded by a fixed value. Based on this observation and the result stating that the share of a codeword of any locating-total dominating set is on *average* at most 18/7 in $\mathcal{S}_3$, we prove that any locating-total dominating set in $\mathcal{S}_3$ has density at least 7/18 in Section 3 (see Theorem 3.2). Analogous methods are also used for obtaining lower bounds in the finite strips in Section 4.

3. Infinite Strip $\mathcal{S}_3$

In this section, we consider locating-total dominating sets in the infinite strip $\mathcal{S}_3$. We first construct a locating-total dominating set in $\mathcal{S}_3$ with density 7/18 and then show that this construction is optimal, i.e., that there are no locating-total dominating sets in $\mathcal{S}_3$ with smaller density. The lower bound is based on the concept of share combined with an averaging process.

For the construction, we first define the following pattern of vertices:

$$P(k, l) = \bigcup_{i=1}^{3}\{(i, 6k + l)\} \cup \bigcup_{i=1}^{2}\{(i, 6k + l + 2)\} \cup \bigcup_{i=2}^{3}\{(i, 6k + l + 4)\},$$

where k and l are integers. Now we are ready to give the actual construction:

$$\mathcal{C}_\infty = \bigcup_{i=-\infty}^{\infty} P(i, 0).$$

The code $\mathcal{C}_\infty$ is illustrated in Figure 1, where the shaded dots represent the codewords of $\mathcal{C}_\infty$ and the dashed lines divide the code into the patterns $P(i, 0)$. It is straightforward to verify that $\mathcal{C}_\infty$ is a locating-total dominating set in $\mathcal{S}_3$ and that its density is equal to 7/18 (each repeated pattern has 18 vertices and 7 codewords).

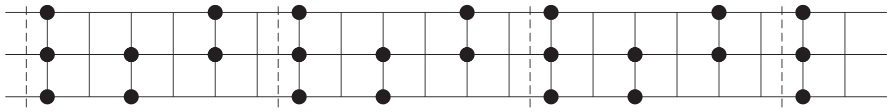


Figure 1. A locating-total dominating set $\mathcal{C}_\infty$ in $\mathcal{S}_3$ with density 7/18 illustrated.

Let us then show that C_∞ is an optimal locating-total dominating set in $\mathcal{S}_3$. Assume first that C is an arbitrary locating-total dominating set in $\mathcal{S}_3$. In what follows, we show that on average the share of a codeword of C is at most 18/7, hence implying the optimal lower bound of 7/18. The averaging process is done by introducing a *shifting scheme* designed to even out the share among the codewords of C. The shifting scheme can also be viewed as a discharging method, which is a terminology more commonly used in the field of graph theory. The *rules* of the shifting scheme are illustrated in Figure 2. In addition to the constellations in the figure, reflections over the lines passing horizontally and vertically through $\mathbf{u}$ can be applied to each rule (in order to obtain new ones). However, assuming $\mathbf{v} \in C$, the shifting scheme is such that if share is shifted from $\mathbf{u}$ to $\mathbf{v}$ according to a certain rule, then no other rule can be applied and the only applicable rule cannot be used in more than one orientation. In the rules, share is shifted as follows:

- In Rule 1, we shift $18/7 - s(\mathbf{v})$ units of share from $\mathbf{u}$ to $\mathbf{v}$.
- In Rule 2, if the squared vertex is a codeword, then 1/42 units of share is shifted from $\mathbf{u}$ to $\mathbf{v}$, else 4/7 units is shifted.
- In Rule 3, if at least one of the squared vertices is a codeword, then 2/14 units of share is shifted from $\mathbf{u}$ to $\mathbf{v}$, else 1/14 units is shifted.
- In Rule 4, we shift 1/14 units of share from $\mathbf{u}$ to $\mathbf{v}$.
- In Rule 5, we shift 1/42 units of share from $\mathbf{u}$ to $\mathbf{v}$.
- In Rule 6, we shift 5/42 units of share from $\mathbf{u}$ to $\mathbf{v}$.
- In Rule 7, we shift 5/28 units of share from $\mathbf{u}$ to $\mathbf{v}$.
- In Rule 8, we shift 5/14 units of share from $\mathbf{u}$ to $\mathbf{v}$.

The modified share of a codeword $\mathbf{c} \in C$, which is obtained after the shifting scheme is applied, is denoted by $\bar{s}(\mathbf{c})$. In the following lemma, we prove that $\bar{s}(\mathbf{c}) \leq 18/7$ for all $\mathbf{c} \in C$.

Lemma 3.1. *Let C be a locating-total dominating set in $\mathcal{S}_3$. Then we have $\bar{s}(\mathbf{c}) \leq 18/7$ for all $\mathbf{c} \in C$.*

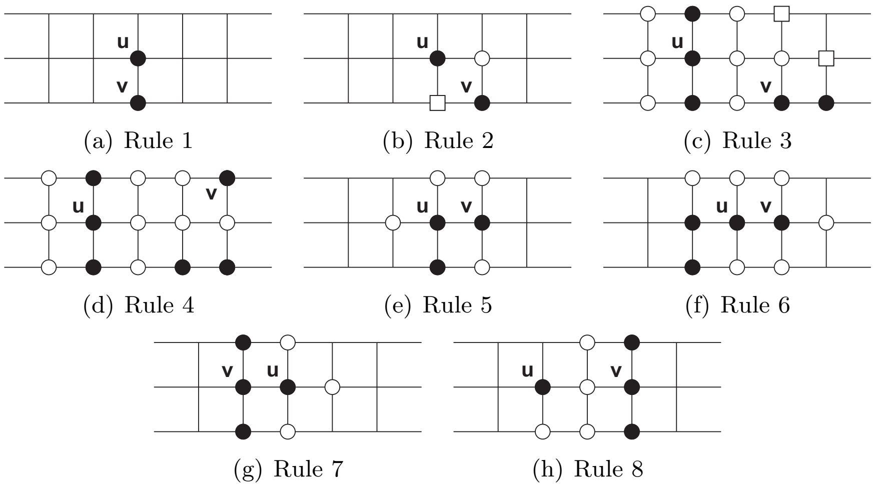


Figure 2. The rules of the shifting scheme illustrated. The shaded dots represent codewords and the light dots represent non-codewords.

Proof. Let C be a locating-total dominating set in $\mathcal{S}_3$ and $\mathbf{c}$ be a codeword in C.

Case 1. Assume first that $\mathbf{c}$ is on the upper or lower row of the strip. Without loss of generality, we may assume that $\mathbf{c} = (1, 0)$. Then it is immediate that share is shifted to $\mathbf{c}$ only according to Rules 1–4 and that no share is shifted from $\mathbf{c}$. Notice also that $s(\mathbf{c}) \leq 1 + 3 \cdot 1/2 = 5/2 \leq 18/7$. Indeed, at most one of the sets $N[\mathbf{u}] \cap C$ with $\mathbf{u} \in N[\mathbf{c}]$ consists of exactly one codeword and the other sets have at least two codewords (since the sets $N[\mathbf{u}] \cap C$ are unique by the definition). The proof now divides into the following symmetrically different cases depending on which of the vertices adjacent to $\mathbf{c}$ are codewords:

Case 1.1. Assume that $(2, 0) \in C$. As $(2, 0) \in C$, no rules other than the first one can be applied to $\mathbf{c}$. Hence, $\mathbf{c}$ receives at most $18/7 - s(\mathbf{c})$ units of share. Therefore, we have $\bar{s}(\mathbf{c}) \leq s(\mathbf{c}) + 18/7 - s(\mathbf{c}) = 18/7$. Thus, from now on, we assume that $(2, 0) \notin C$.

Case 1.2. Assume then that either $(1, -1) \notin C$ and $(1, 1) \in C$, or $(1, -1) \in C$ and $(1, 1) \notin C$. Without loss of generality, we may assume that $(1, -1) \notin C$ and $(1, 1) \in C$. If $(2, -1) \in C$, then for each $\mathbf{u} \in N[\mathbf{c}]$ the intersection $N[\mathbf{u}] \cap C$ contains at least two codewords. Moreover, since $N[(1, -1)] \cap C \neq N[(2, 0)] \cap C$, at least one of these sets consists of at least three codewords. Hence, we have $s(\mathbf{c}) \leq 3 \cdot 1/2 + 1/3 = 11/6$. Thus, since share can be shifted to $\mathbf{c}$ only according to Rule 2 from $(2, -1)$ and $(2, 1)$, we obtain that $\bar{s}(\mathbf{c}) \leq s(\mathbf{c}) + 4/7 + 1/42 \leq 11/6 + 4/7 + 1/42 = 17/7 \leq 18/7$. Assume that $(2, -1) \notin C$ and $(2, 1) \in C$. Now share can be shifted to $\mathbf{c}$ according to Rule 2 (with the squared vertex assumed to be a codeword), and either Rule 3 or Rule 4 (but not both). Thus, since clearly $s(\mathbf{c}) \leq 1+2\cdot 1/2+1/3 = 7/3$, we obtain that $\bar{s}(\mathbf{c}) \leq 7/3+1/42+2/14 = 5/2 \leq 18/7$. Therefore, we may finally assume that neither $(2,-1)$ nor $(2,1)$ is a codeword. Then we observe that $\mathbf{c}$ may receive share only according to either Rule 3 and Rule 4 (but not both). If $1/14$ units of share is shifted to $\mathbf{c}$, then we are immediately done as $\bar{s}(\mathbf{c}) \leq s(\mathbf{c}) + 1/14 \leq 5/2 + 1/14 = 18/7$. Furthermore, if $2/14$ units of share is shifted according to Rule 3, then $(3,0) \in C$, $(1,-2) \in C$ and $s(\mathbf{c}) \leq 4 \cdot 1/2 = 2$. Thus, we are done as $\bar{s}(\mathbf{c}) \leq 2 + 2/14 = 15/7 \leq 18/7$.

Case 1.3. Finally, assume that both $(1,-1) \in C$ and $(1,1) \in C$. Now we have $s(\mathbf{c}) \leq 1+2\cdot 1/2+1/3 = 7/3$. Therefore, since share can be shifted to $\mathbf{c}$ only according to Rule 2 (with the squared vertex assumed to be a codeword) and at most twice, we are immediately done as $\bar{s}(\mathbf{c}) \leq 7/3 + 2 \cdot 1/42 = 50/21 \leq 18/7$.

Case 2. Assume then that $\mathbf{c}$ is on the middle row of the strip. Without loss of generality, we may assume that $\mathbf{c} = (2,0)$. Notice that now share can be shifted to $\mathbf{c}$ only according to Rules 5–8. Again the proof divides into the following symmetrically different cases depending on which of the vertices adjacent to $\mathbf{c}$ are codewords.

Case 2.1. Assume first that $(1,0) \in C$ and that $(2,-1)$, $(3,0)$ and $(2,1)$ do not belong to C. Notice that now no share share is shifted to $\mathbf{c}$. If $(1,1)$ belongs to C, then we have $s(\mathbf{c}) \leq 1 + 3 \cdot 1/2 + 1/3 = 17/6$ and $s((1,0)) \leq 1 + 2 \cdot 1/2 + 1/3 \leq 7/3 = 18/7 - 5/21$. Therefore, as $5/21$ and $1/42$ units of share are respectively shifted from $\mathbf{c}$ to $(1,0)$ and $(1,1)$ according to Rules 1 and 2, we obtain that $\bar{s}(\mathbf{c}) \leq 17/6 - 5/21 - 1/42 = 18/7$. The case with $(1,-1) \in C$ is similar. Thus, we may assume that $(1,-1) \notin C$ and $(1,1) \notin C$. If $(3,1) \in C$, then $4/7$ units of share is shifted from $\mathbf{c}$ to $(3,1)$ according to Rule 2. Hence, since $s(\mathbf{c}) \leq 1 + 4 \cdot 1/2 = 3$, we have $\bar{s}(\mathbf{c}) \leq 3 - 4/7 = 17/7 \leq 18/7$. Thus, we may assume that $(3,1) \notin C$ and also by symmetry that $(3,-1) \notin C$.

Observe that $(3,-2) \in C$ and $(3,2) \in C$ since (respectively) $(3,-1)$ and $(3,1)$ are adjacent to a codeword. Furthermore, we know that $(2,-2)$ and $(2,2)$ belong to C since $N[(2,-1)] \cap C$ and $N[(2,1)] \cap C$ differ from $N[(3,0)] \cap C = \{(2,0)\}$. Notice also that $(1,-2)$ or $(1,2)$ (or both) belong to C as $N[(1,-1)] \cap C \neq N[(1,1)] \cap C$. Without loss of generality, we may assume that $(1,2) \in C$. This implies that $18/7 - s((1,0)) \geq 18/7 - 5/2 \geq 1/14$ and $5/14$ units of share is shifted from $\mathbf{c}$ to $(1,0)$ and $(2,2)$ according to Rules 1 and 8, respectively. Thus, we are done as $\bar{s}(\mathbf{c}) \leq s(\mathbf{c}) - 1/14 - 5/14 \leq 3 - 1/14 - 5/14 = 18/7$.

Case 2.2. Assume that $(2,-1) \in C$ and that $(1,0)$, $(3,0)$ and $(2,1)$ do not belong to C. It is immediate that share can be shifted to $\mathbf{c}$ only according to Rules 5 and 6 (and at most once). If $(3,1) \in C$, then we have $s(\mathbf{c}) \leq 1 + 3 \cdot 1/2 + 1/3 = 17/6$ since $N[(3,0)] \cap C$ or $N[(2,1)] \cap C$ has at least 3 codewords. Therefore, since at most $5/42$ units of share is shifted to $\mathbf{c}$ and at least $4/7$

units of share is shifted from $\mathbf{c}$ to $(3,1)$ according to Rule 2, we obtain that $\bar{s}(\mathbf{c}) \leq 17/6 + 5/42 - 4/7 = 50/21 \leq 18/7$. Thus, we may assume that $(3,1) \notin C$ and by symmetry $(1,1) \notin C$. Observe that $(1,-1)$ or $(3,-1)$ belong to C since $N[(1,0)] \cap C \neq N[(3,0)] \cap C$. Without loss of generality, we may assume that $(1,-1) \in C$. If also $(3,-1) \in C$, then $s(\mathbf{c}) \leq 1 + 3 \cdot 1/2 + 1/4 = 11/4$ and $5/28$ units of share is shifted from $\mathbf{c}$ to $(2,-1)$ according to Rule 7. Hence, as no share is now shifted to $\mathbf{c}$, we are done since $\bar{s}(\mathbf{c}) \leq 11/4 - 5/28 = 18/7$. Thus, we may assume that $(3,-1) \notin C$.

Notice that $(1,2)$ and $(3,2)$ belong to C since $(1,1)$ and $(3,1)$ are adjacent to a codeword. Furthermore, $(2,2)$ belongs to C since $N[(2,1)] \cap C \neq N[(3,0)] \cap C = \{(2,0)\}$. Now we have $s(\mathbf{c}) \leq 1 + 3 \cdot 1/2 + 1/3 = 17/6$. Moreover, $1/42$ units of share is shifted to $\mathbf{c}$ from $(2,-1)$ according to Rule 5 and $5/14$ units of share is shifted from $\mathbf{c}$ to $(2,2)$ according to Rule 8. Thus, we are done as $\bar{s}(\mathbf{c}) \leq 17/6 + 1/42 - 5/14 = 5/2 \leq 18/7$.

Case 2.3. Assume that $(1,0) \in C$, $(3,0) \in C$, $(2,-1) \notin C$ and $(2,1) \notin C$. It is immediate that $s((1,0)) \leq 1 + 2 \cdot 1/2 + 1/3 = 7/3 = 18/7 - 5/21$ and similarly $s((3,0)) \leq 7/3$. Hence, at least $5/21$ units of share is shifted from $\mathbf{c}$ to both $(1,0)$ and $(3,0)$ according to Rule 1. We also observe that share can be shifted to $\mathbf{c}$ only according to Rule 8 from the vertices $(2,-2)$ and $(2,2)$. If $(1,1) \in C$, then $s(\mathbf{c}) \leq 1 + 2 \cdot 1/2 + 2 \cdot 1/3 = 8/3$ and share can be shifted to $\mathbf{c}$ only from $(2,-2)$. Therefore, we are immediately done as $\bar{s}(\mathbf{c}) \leq 8/3 + 5/14 - 2 \cdot 5/21 = 107/42 \leq 18/7$. The cases with $(1,-1) \in C$, $(3,-1) \in C$ and $(3,1) \in C$ are similar. Thus, we may assume that $(1,-1)$, $(3,-1)$, $(1,1)$ and $(3,1)$ do not belong to C. Since $N[(2,-1)] \cap C \neq N[(2,1)] \cap C$, we know that $(2,-2) \in C$ or $(2,2) \in C$. If both $(2,-2)$ and $(2,2)$ belong to C, then $s(\mathbf{c}) \leq 4 \cdot 1/2 + 1/3 = 7/3$. Therefore, we are done since $\bar{s}(\mathbf{c}) \leq 7/3 + 2 \cdot 5/14 - 2 \cdot 5/21 = 18/7$. Thus, without loss of generality, we may assume that $(2,-2) \in C$ and $(2,2) \notin C$.

If $(1,-2) \in C$ and $(3,-2) \in C$, then no share is shifted to $\mathbf{c}$ and we are done as $\bar{s}(\mathbf{c}) \leq s(\mathbf{c}) - 2 \cdot 5/21 \leq 1 + 3 \cdot 1/2 + 1/3 - 2 \cdot 5/21 \leq 17/6 - 2 \cdot 5/21 = 33/14 \leq 18/7$. Hence, without loss of generality, we may assume that $(1,-2) \notin C$. Then the vertex $(1,2)$ belongs to C since $N[(1,1)] \cap C \neq N[(1,-1)] \cap C = \{(1,0)\}$. Notice now that $(1,3) \in C$ and $(3,3) \in C$ since $(1,2)$ and $(3,2)$ are adjacent to a codeword, respectively. Furthermore, if $(3,2)$ or $(2,3)$ belongs to C, then $2/14$ units of share is shifted from $\mathbf{c}$ to $(1,2)$ according to Rule 3 and we are done as $\bar{s}(\mathbf{c}) \leq s(\mathbf{c}) + 5/14 - 2 \cdot 5/21 - 2/14 = 17/6 + 5/14 - 2 \cdot 5/21 - 2/14 = 18/7$. Finally, if $(3,2)$ and $(2,3)$ do not belong to C, then $1/14$ units of share is shifted from $\mathbf{c}$ to both $(1,2)$ and $(3,3)$ according to Rules 3 and 4, respectively. Thus, we conclude the case with similar calculations as above.

Case 2.4. Assume that $(2,-1) \in C$, $(2,1) \in C$, $(1,0) \notin C$ and $(3,0) \notin C$. Since $N[(1,0)] \cap C \neq N[(3,0)] \cap C$, at least one of the vertices $(1,-1)$, $(3,-1)$, $(1,1)$ and $(3,1)$ is a codeword. Without loss of generality, we may assume that

$(1,-1) \in C$. If $(1,1)$, $(3,-1)$ or $(3,1)$ belongs to C, then we respectively have $s(\mathbf{c}) \le 1+4\cdot 1/3 = 7/3$, $s(\mathbf{c}) \le 3\cdot 1/2+1/3+1/4 = 25/12$ or $s(\mathbf{c}) \le 2\cdot 1/2+3\cdot 1/3 = 2$. In each case, share can be shifted to $\mathbf{c}$ only according to Rule 5 and at most twice. Therefore, we are done as $\bar{s}(\mathbf{c}) \le s(\mathbf{c})+2\cdot 1/42 \le 7/3+2\cdot 1/42 = 50/21 \le 18/7$. Thus, we may assume that $(1,1)$, $(3,-1)$ and $(3,1)$ do not belong to C. Now share can be shifted to $\mathbf{c}$ only according to Rule 5 and at most once.

Assume then that $(2,2) \in C$. Now we have $s(\mathbf{c}) \le 1+1/2+3\cdot 1/3 = 5/2$. This further implies that $\bar{s}(\mathbf{c}) \le 5/2 + 1/42 = 53/21 \le 18/7$. Hence, we may assume that $(2,2) \notin C$. Then we have $s(\mathbf{c}) \le 1+2\cdot 1/2+2\cdot 1/3 = 8/3$ and share is shifted from $\mathbf{c}$ to $(2,1)$ according to Rule 6. Thus, we have $\bar{s}(\mathbf{c}) \le s(\mathbf{c}) + 1/42 - 5/42 = 8/3 + 1/42 - 5/42 = 18/7$ and we are done.

Case 2.5. Assume that $(1,0) \in C$, $(2,1) \in C$, $(2,-1) \notin C$ and $(3,0) \notin C$. Notice first that no share is shifted to $\mathbf{c}$ as $(1,0) \in C$ and $(3,0) \notin C$. Observe then that $s((1,0)) \le 1 + 2\cdot 1/2 + 1/3 = 7/3 = 18/7 - 5/21$. Hence, at least $5/21$ units of share is shifted from $\mathbf{c}$ to $(1,0)$ according to Rule 1. If now $(1,1) \in C$ or $(3,1) \in C$, then $s(\mathbf{c}) \le 1 + 2\cdot 1/2 + 2\cdot 1/3 = 8/3$ and we are immediately done since $\bar{s}(\mathbf{c}) \le 8/3 - 5/21 = 17/7 \le 18/7$. Thus, we may assume that $(1,1) \notin C$ and $(3,1) \notin C$. Then $1/42$ units of share is shifted from $\mathbf{c}$ to $(2,1)$ according to Rule 5. Thus, we are done as $\bar{s}(\mathbf{c}) \le s(\mathbf{c}) - 5/21 - 1/42 \le 1+3\cdot 1/2+1/3-5/21-1/42 = 18/7$.

Case 2.6. Assume that $(1,0) \in C$, $(3,0) \in C$, $(2,1) \in C$ and $(2,-1) \notin C$. Notice first that the values $s((1,0))$ and $s((3,0))$ are both at most $1+2\cdot 1/2+1/4 = 9/4 = 18/7 - 9/28$. Hence, at least $9/28$ units of share is shifted from $\mathbf{c}$ to both $(1,0)$ and $(3,0)$ according to Rule 1. Observe also that share can be shifted to $\mathbf{c}$ only according to Rules 7 and 8 from $(2,1)$ and $(2,-2)$, respectively. If share is shifted from $(2,-2) \in C$, then we have $s(\mathbf{c}) \le 4\cdot 1/2 + 1/4 = 9/4$. Therefore, we are done as $\bar{s}(\mathbf{c}) \le s(\mathbf{c}) + 5/28 + 5/14 - 2\cdot 9/28 \le 9/4 + 5/28 + 5/14 - 2\cdot 9/28 = 15/7 \le 18/7$. Hence, we may assume that share is shifted only according to Rule 7. Thus, we have $\bar{s}(\mathbf{c}) \le s(\mathbf{c}) + 5/28 - 2\cdot 9/28 \le 1 + 3\cdot 1/2 + 1/4 + 5/28 - 2\cdot 9/28 = 16/7 \le 18/7$.

Case 2.7. Assume that $(2,-1) \in C$, $(1,0) \in C$, $(2,1) \in C$ and $(3,0) \notin C$. It is immediate that no share is shifted to $\mathbf{c}$ as $(1,0) \in C$ and $(3,0) \notin C$. Furthermore, we have $s(\mathbf{c}) \le 1 + 3\cdot 1/2 + 1/4 = 11/4$ and $s((1,0)) \le 3\cdot 1/2 + 1/4 = 7/4 = 18/7 - 23/28$. Thus, since $23/28$ units of share is shifted from $\mathbf{c}$ to $(1,0)$ according to Rule 1, we have $\bar{s}(\mathbf{c}) \le 11/4 - 23/28 = 27/14 \le 18/7$.

Case 2.8. Assume that $(2,-1) \in C$, $(1,0) \in C$, $(3,0) \in C$ and $(2,1) \in C$. Observe that now share can be shifted to $\mathbf{c}$ only according to Rule 7 and at most twice. Thus, since $s(\mathbf{c}) \le 4\cdot 1/2 + 1/5 = 11/5$, we are immediately done as $\bar{s}(\mathbf{c}) \le 11/5 + 2\cdot 5/28 \le 179/70 \le 18/7$. ∎

In the following theorem, using the result of the previous lemma, we prove that the locating-total dominating set C_∞ in $\mathcal{S}_3$ with density $7/18$ is optimal.

Theorem 3.2. *If C is a locating-total dominating set in the infinite strip $\mathcal{S}_3$, then the density*

$$D(C) \geq \frac{7}{18}.$$

Proof. Assume that C is a locating-total dominating set in $\mathcal{S}_3$. Notice that each vertex $\mathbf{u} \in Q_{n-1}$ with $|N[\mathbf{u}] \cap C| = i$ contributes the summand $1/i$ to $s(\mathbf{c})$ for each of the i codewords $\mathbf{c} \in N[\mathbf{u}]$. Therefore, we have

$$\sum\nolimits_{\mathbf{c} \in C \cap Q_n} s(\mathbf{c}) \geq |Q_{n-1}|. \tag{1}$$

Furthermore, we have

$$\sum\nolimits_{\mathbf{c} \in C \cap Q_n} s(\mathbf{c}) \leq \sum\nolimits_{\mathbf{c} \in C \cap Q_n} \bar{s}(\mathbf{c}) + \frac{18}{7}|Q_{n+3} \setminus Q_n|. \tag{2}$$

Indeed, shifting shares inside Q_n does not affect the sum and each codeword in $Q_{n+3} \setminus Q_n$ can receive at most $4/7 \leq 18/7$ units of share from the codewords in Q_n (as each codeword in $Q_{n+3} \setminus Q_n$ can receive share only according to Rules 2–8 and at most once). Notice also that codewords in Q_n cannot shift share to codewords outside Q_{n+3}. Therefore, combining the equations (1) and (2) with the fact that $\bar{s}(\mathbf{c}) \leq 18/7$ for any $\mathbf{c} \in C$, we obtain that

$$\frac{|C \cap Q_n|}{|Q_n|} \geq \frac{7}{18} \cdot \frac{|Q_{n-1}|}{|Q_n|} - \frac{|Q_{n+3} \setminus Q_n|}{|Q_n|}.$$

Since $|Q_k| = 3(2k+1)$ for any positive integer k, it is straightforward to conclude from the previous inequality that the density $D(C) \geq 7/18$. ■

4. Finite Strips $\mathcal{S}_{3,n}$

In this section, we determine the sizes of optimal locating-total dominating sets in the finite strips $\mathcal{S}_{3,n}$ of height 3 for all lengths n. We begin by presenting the constructions of locating-total dominating sets and then proceed by proving that these constructions are optimal in Theorems 4.2 and 4.3.

The constructions are based on the optimal locating-total dominating set C_∞ in the infinite strip $\mathcal{S}_3$. More precisely, the middle part of the finite strip is taken care of by repeating the pattern $P(k,l)$ and then suitable constellations are chosen for the beginning and the end of the strip. The constructions depend on the residue of n modulo 6. For general $n \geq 6$, we obtain the following cases.

- If $n \equiv 0 \pmod 6$, i.e., $n = 6m$ for some positive integer m, then

$$C_n = \bigcup_{i=1}^{3} \{(i,2)\} \cup \bigcup_{i=0}^{m-2} P(i,4) \cup \bigcup_{i=1}^{3} \{(i,n-2)\} \cup \bigcup_{i=1}^{2} \{(i,n)\}.$$

- If $n \equiv 1 \pmod 6$, i.e., $n = 6m+1$ for some positive integer m, then

$$C_n = \bigcup_{i=1}^{3} \{(i,2)\} \cup \bigcup_{i=0}^{m-2} P(i,4) \cup \bigcup_{i=1}^{3} \{(i,n-3)\} \cup \bigcup_{i=1}^{3} \{(i,n-1)\}.$$

- If $n \equiv 2 \pmod 6$, i.e., $n = 6m+2$ for some positive integer m, then

$$\begin{aligned} C_n &= \bigcup_{i=1}^{2} \{(i,1)\} \cup \bigcup_{i=2}^{3} \{(i,3)\} \cup \bigcup_{i=0}^{m-2} P(i,5) \cup \bigcup_{i=1}^{3} \{(i,n-3)\} \\ &\cup \bigcup_{i=1}^{3} \{(i,n-1)\}. \end{aligned}$$

- If $n \equiv 3 \pmod 6$, i.e., $n = 6m+3$ for some positive integer m, then

$$C_n = \bigcup_{i=1}^{2} \{(i,1)\} \cup \bigcup_{i=2}^{3} \{(i,3)\} \cup \bigcup_{i=0}^{m-1} P(i,5).$$

- If $n \equiv 4 \pmod 6$, i.e., $n = 6m+4$ for some positive integer m, then

$$C_n = \bigcup_{i=1}^{2} \{(i,1)\} \cup \bigcup_{i=2}^{3} \{(i,3)\} \cup \bigcup_{i=0}^{m-1} P(i,5) \cup \{(2,n)\}.$$

- If $n \equiv 5 \pmod 6$, i.e., $n = 6m+5$ for some positive integer m, then

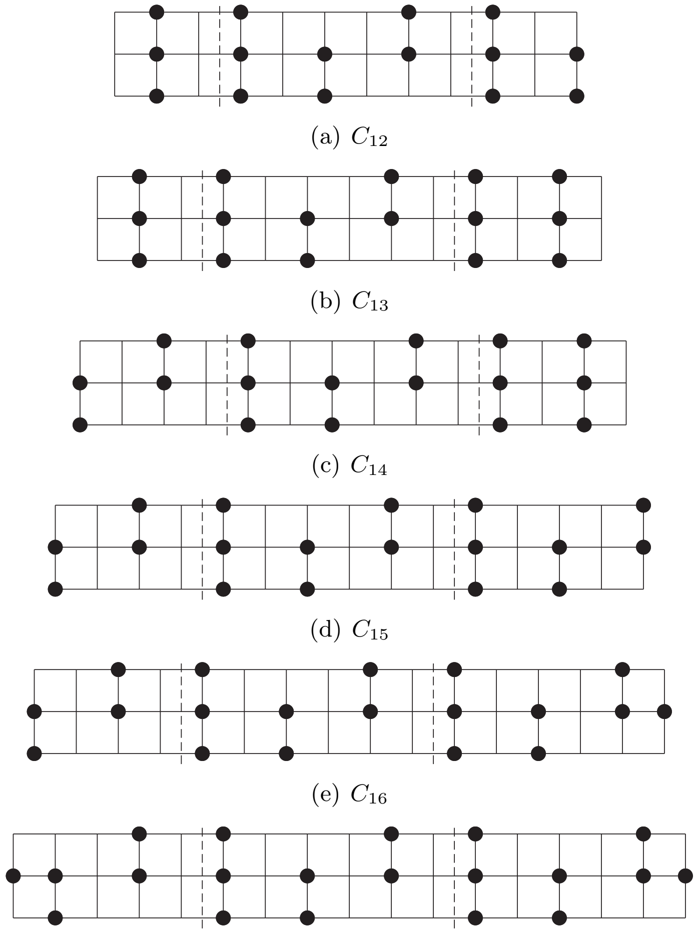
$$C_n = \{(2,1)\} \cup \bigcup_{i=1}^{2} \{(i,2)\} \cup \bigcup_{i=2}^{3} \{(i,4)\} \cup \bigcup_{i=0}^{m-1} P(i,6) \cup \{(2,n)\}.$$

The previous constructions are illustrated in Figure 3 for $12 \leq n \leq 17$. Observe that in each illustrated code at least one repetition of the pattern $P(k,l)$ occurs and the pattern is represented as the constellation between the dashed lines.

The previous general construction provides locating-total dominating sets for strips $\mathcal{S}_{3,n}$ when $n \geq 6$. This is actually enough since the exact values of $\gamma_t^L(\mathcal{S}_{3,n})$ have previously been determined for $1 \leq n \leq 12$ in [8]. However, for completeness, we also present the constructions for $1 \leq n \leq 5$: $C_1 = \{(1,1),(2,1)\}$, $C_2 = \{(1,2),(2,2),(3,2)\}$, $C_3 = C_1 \cup \{(2,3),(3,3)\}$, $C_4 = C_3 \cup \{(2,4)\}$ and $C_5 = C_2 \cup \{(1,4),(2,4),(3,4)\}$. In the following theorem, we show that the previous constructions are locating-total dominating sets in the finite strips and determine the exact sizes of the sets.

Theorem 4.1. *For any positive integer n, the code C_n is a locating-total dominating set in $\mathcal{S}_{3,n}$. Hence, we obtain that*

$$\gamma_t^L(\mathcal{S}_{3,n}) \leq |C_n| = \begin{cases} \left\lceil \frac{7n}{6} \right\rceil & \text{if } n \not\equiv 0 \pmod 6, \\ \frac{7n}{6} + 1 & \text{if } n \equiv 0 \pmod 6. \end{cases}$$



(a) C_{12}

(b) C_{13}

(c) C_{14}

(d) C_{15}

(e) C_{16}

(f) C_{17}

Figure 3. The locating-total dominating sets C_n in $\mathcal{S}_3$ illustrated for $12 \leq n \leq 17$.

Proof. First of all, it is straightforward to verify that for $1 \leq n \leq 5$ the sets C_n satisfy the claims of the theorem. Recall that the set C_∞, which is formed by repeating the pattern $P(k, l)$, is locating-total dominating in $\mathcal{S}_3$. Observe that the codes C_n in the finite strips $\mathcal{S}_{3,n}$ are also formed by repeating the pattern $P(k, l)$ combined with some suitable constellations in the beginning and the end of the strip. Thus, by verifying that both ends of the strip are taken care of, it can straightforwardly be seen that for all $n \geq 6$ the set C_n is locating-total dominating in $\mathcal{S}_{3,n}$.

For determining the cardinality of C_n, assume first that $n = 6m + k$ where m is a positive integer and k is an integer such that $1 \leq k \leq 5$. Then we clearly have $|C_n| = 7m + k + 1$ and the claim immediately follows. If $n = 6m$ with m

being a positive integer, then $|C_n| = 7m + 1$ and we are again done. ■

In what follows, we show that the locating-total dominating sets C_n in $\mathcal{S}_{3,n}$ are optimal. We first consider the case when $n \not\equiv 0 \,(\text{mod } 6)$.

Theorem 4.2. *If n is a positive integer and $n \not\equiv 0 \,(\text{mod } 6)$, then we have*

$$\gamma_t^L(\mathcal{S}_{3,n}) = \left\lceil \frac{7n}{6} \right\rceil .$$

Proof. Let n be a positive integer such that $n \not\equiv 0 \,(\text{mod } 6)$. Assume that C is a locating-total dominating set in $\mathcal{S}_{3,n}$. For the claim, it is enough to show that $|C| \geq (7n)/6$. Assume to the contrary that $|C| < (7n)/6$. Define

$$C' = \{(x, n + 1 - y) \mid (x, y) \in C\}.$$

In other words, C' is obtained from C by reading C from the end of the strip to the beginning. Obviously, C' is also a locating-total dominating set in $\mathcal{S}_{3,n}$ and $|C| = |C'|$. Consider then the set

$$B = C \cup (C' + (0, n)) = C \cup \{(x, y + n) \mid (x, y) \in C'\}.$$

It is clear that B is a locating-total dominating set in $\mathcal{S}_{3,2n}$ if $N[(x_1, n)] \cap B \neq N[(x_2, n + 1)] \cap B$ for all $(x_1, n) \notin B$ and $(x_2, n + 1) \notin B$. Assume that (x, n) is a vertex of $\mathcal{S}_{3,2n}$, i.e., $x \in \{1, 2, 3\}$. Notice that by the construction of B the vertex $(x, n) \in B$ if and only if $(x, n + 1) \in B$. Therefore, if (x, n) does not belong to B, then $(x, n+1) \notin B$ and $N[(x, n)] \cap B = N[(x, n)] \cap C$. Hence, as C is a locating-total dominating set in $\mathcal{S}_{3,n}$, B is a locating-total dominating set in $\mathcal{S}_{3,2n}$.

Analogously to the construction of B, we can form a locating-total dominating set B_∞ in the infinite strip $\mathcal{S}_3$ by alternating the sets C and C'. Since $|C| = |C'| < (7n)/6$, the density of B_∞ is smaller than

$$\frac{(7n)/6}{3n} = \frac{7}{18}.$$

This implies a contradiction with Theorem 3.2, which states that the density of a locating-total dominating set in $\mathcal{S}_3$ is always at least $7/18$. ■

Let us then consider the case when $n \equiv 0 \,(\text{mod } 6)$.

Theorem 4.3. *If n is a positive integer and $n \equiv 0 \,(\text{mod } 6)$, then we have*

$$\gamma_t^L(\mathcal{S}_{3,n}) = \frac{7n}{6} + 1.$$

Proof. Let n be a positive integer and $n \equiv 0 \pmod 6$. Assume that C is a locating-total dominating set (or code) in $\mathcal{S}_{3,n}$. In what follows, we first prove that on average the share of a codeword of C is less than $18/7$ and then show that this implies $|C| > (7/18)|V| = (7n)/6$ as described in Section 2.

Let $\mathbf{u}_1 = (2, k_1)$ and $\mathbf{u}_2 = (2, k_2)$ be such codewords of C that $(2, i) \notin C$ for any $1 \leq i < k_1$ and $k_2 < i \leq n$, i.e., $\mathbf{u}_1$ is the first and $\mathbf{u}_2$ is the last codeword in the middle row of the strip. Based on the shifting scheme introduced in Section 3, we define a slightly modified scheme to even out the share among the codewords in the finite strip $\mathcal{S}_{3,n}$ as follows: share is shifted as in the original scheme except for the codewords $\mathbf{u}_1$ and $\mathbf{u}_2$ *from* which share is shifted only according to Rule 1. The share of a codeword $\mathbf{c} \in C$ obtained after the shifting scheme is applied is denoted by $\bar{s}'(\mathbf{c})$.

We first observe that $\bar{s}'(\mathbf{c}) \leq 18/7$ for any codeword $\mathbf{c} \in C$ other than $\mathbf{u}_1$ and $\mathbf{u}_2$.

Case 1. Assume first that $\mathbf{c} = (x, y) \in C$ is such that $k_1 < y < k_2$. Now no share is shifted from $\mathbf{c}$ to a codeword $\mathbf{c}' = (x', y')$ with $y' < k_1$ or $y' > k_2$ since $\mathbf{u}_1 = (2, k_1)$ and $\mathbf{u}_2 = (2, k_2)$ are codewords. Therefore, share is shifted from and to $\mathbf{c}$ as in the case of the infinite strip $\mathcal{S}_3$ (with the except that share is not shifted to $\mathbf{c}$ from $\mathbf{u}_1$ or $\mathbf{u}_2$). Thus, by the proof of Theorem 3.2, we have $\bar{s}'(\mathbf{c}) \leq 18/7$.

Case 2. If $\mathbf{c} = (x, y) \in C$ is such that $y = k_1$ or $y = k_2$, then $\mathbf{c}$ belongs to the lower or upper row of the strip. Hence, we have $s(\mathbf{c}) \leq 1 + 3 \cdot 1/2 = 5/2 \leq 18/7$ and share is shifted to $\mathbf{c}$ only according to Rule 1 from either $\mathbf{u}_1$ or $\mathbf{u}_2$. Therefore, we are done as $\bar{s}'(\mathbf{c}) \leq s(\mathbf{c}) + 18/7 - s(\mathbf{c}) = 18/7$.

Case 3. If $\mathbf{c} = (x, y) \in C$ is such that $y < k_1$ or $y > k_2$, then no share is shifted to $\mathbf{c}$ since share is shifted from $\mathbf{u}_1$ and $\mathbf{u}_2$ only according to Rule 1. Hence, as $\mathbf{c}$ is in the bottom or in the top row of the strip, we have $\bar{s}'(\mathbf{c}) \leq s(\mathbf{c}) \leq 1 + 3 \cdot 1/2 = 5/2 \leq 18/7$.

In what follows, we show that $\bar{s}'(\mathbf{u}_1) < 18/7$ or there exists a codeword $\mathbf{c} = (x, y)$ such that $y < k_1$ and $(\bar{s}'(\mathbf{c}) + \bar{s}'(\mathbf{u}_1))/2 < 18/7$. In both cases, we obtain that on average the share of a codeword of C is less than $18/7$. (Notice that an analogous result also holds for the codeword $\mathbf{u}_2$.) The proof now divides into the following cases.

Case 1. Assume first that $\mathbf{u}_1 = (2, 1)$. If now $(1, 1) \in C$, then we have $s((1, 1)) \leq 1 + 2 \cdot 1/2 = 2 = 18/7 - 4/7$. Hence, at least $4/7$ units of share is shifted from $\mathbf{u}_1$ to $(1, 1)$ according to Rule 1. Observe that $\mathbf{u}_1$ can receive share only according to Rules 5–8 and at most once. Therefore, at most $5/14$ units of share is shifted to $\mathbf{u}_1$ and we obtain that $\bar{s}'(\mathbf{u}_1) \leq s(\mathbf{u}_1) + 5/14 - 4/7 \leq 1 + 3 \cdot 1/2 + 5/14 - 4/7 = 16/7 < 18/7$, hence implying the claim. Thus, by symmetry, we may assume that $(1, 1) \notin C$ and $(3, 1) \notin C$. This implies that

$(1,2) \in C$ or $(3,2) \in C$. Therefore, $\mathbf{u}_1$ can receive share only according to Rule 5 and we are done since $\bar{s}'(\mathbf{u}_1) \leq s(\mathbf{u}_1) + 1/42 \leq 5/2 + 1/42 = 53/21 < 18/7$.

Case 2. Assume then that $(2,1) \notin C$ and $(1,1) \in C$. (The case with $(2,1) \notin C$ and $(3,1) \in C$ is symmetrical.) Notice first that share can be shifted to $\mathbf{u}_1$ only according to Rules 5–8 and at most once. Observe also that $\bar{s}'((1,1)) \leq s((1,1)) \leq 1 + 2 \cdot 1/2 = 2$. If $\mathbf{u}_1$ receives share according to Rules 7 or 8, then both $\mathbf{u}_1 + (1,0)$ and $\mathbf{u}_1 + (-1,0)$ are codewords. Since the shares of $\mathbf{u}_1 + (1,0)$ and $\mathbf{u}_1 + (-1,0)$ are at most $1 + 2 \cdot 1/2 + 1/3 = 7/3 = 18/7 - 5/21$, then at least $5/21$ units of share is shifted from $\mathbf{u}_1$ to both $\mathbf{u}_1 + (1,0)$ and $\mathbf{u}_1 + (-1,0)$ according to Rule 1. Thus, as at most $5/14$ units of share is shifted to $\mathbf{u}_1$, we obtain that $\bar{s}'(\mathbf{u}_1) \leq s(\mathbf{u}_1) + 5/14 - 2 \cdot 5/21 \leq 1 + 3 \cdot 1/2 + 1/3 + 5/14 - 2 \cdot 5/21 = 19/7$. Therefore, we are done since $(\bar{s}'((1,1)) + \bar{s}'(\mathbf{u}_1))/2 \leq 33/14 < 18/7$.

If $\mathbf{u}_1$ do not receive share according to Rules 7 or 8, then at most $5/42$ units of share is shifted to $\mathbf{u}_1$. Hence, we have $\bar{s}'(\mathbf{u}_1) \leq s(\mathbf{u}_1) + 5/42 \leq 1 + 4 \cdot 1/2 + 5/42 = 131/42$. Thus, we are done as $(\bar{s}'((1,1)) + s(\mathbf{u}_1))/2 \leq 215/84 < 18/7$.

Case 3. Finally, assume that $(1,1)$, $(2,1)$ and $(3,1)$ do not belong to C. Therefore, since $(1,1)$, $(2,1)$ and $(3,1)$ are adjacent to a codeword, we obtain that $(1,2) \in C$, $(2,2) \in C$ and $(3,2) \in C$. Now we clearly have $\mathbf{u}_1 = (2,2)$. Observe that $s((1,2)) \leq 1 + 2 \cdot 1/2 + 1/3 = 7/3 = 18/7 - 5/21$ and by symmetry that also $s((3,2)) \leq 7/3$. Hence, at least $5/21$ units of share is shifted from $\mathbf{u}_1$ to both $(1,2)$ and $(3,2)$ according to Rule 1. If $\mathbf{u}_1$ receives share according to Rule 8, then $N[(1,3)] \cap C = \{(1,2)\}$ or $N[(3,3)] \cap C = \{(3,2)\}$. This leads to a contradiction as $N[(1,1)] \cap C = \{(1,2)\}$ and $N[(3,1)] \cap C = \{(3,2)\}$. Hence, at most $5/28$ units of share is shifted to $\mathbf{u}_1$. Thus, we are done as $\bar{s}'(\mathbf{u}_1) \leq s(\mathbf{u}_1) + 5/28 - 2 \cdot 5/21 \leq 1 + 3 \cdot 1/2 + 1/3 + 5/28 - 2 \cdot 5/21 = 71/28 < 18/7$.

In conclusion we have shown that on average the share of a codeword of C is less than $18/7$. By Section 2, we then have

$$|V| = \sum\nolimits_{\mathbf{c} \in C} s(\mathbf{c}) < \frac{18}{7}|C|.$$

Thus, we obtain that $|C| > (7/18)|V| = (7n)/6$ and the claim immediately follows. ■

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