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IMPROVED SUFFICIENT CONDITIONS FOR HAMILTONIAN PROPERTIES

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Abstract

In 1980 Bondy [2] proved that a $(k+s)$-connected graph of order $n \geq 3$ is traceable ($s=-1$) or Hamiltonian ($s=0$) or Hamiltonian-connected ($s=1$) if the degree sum of every set of $k+1$ pairwise nonadjacent vertices is at least $((k+1)(n+s-1)+1)/2$. It is shown in [1] that one can allow exceptional $(k+1)$-sets violating this condition and still implying the considered Hamiltonian property. In this note we generalize this result for $s=-1$ and $s=0$ and graphs that fulfill a certain connectivity condition.

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1. Introduction

Let $G=(V(G),E(G))$ be a finite simple graph. A path or cycle of a graph is called Hamiltonian if it contains each vertex of the graph exactly once. A graph is called traceable if it contains a Hamiltonian path, Hamiltonian if it contains a Hamiltonian cycle, and Hamiltonian-connected if it contains a Hamiltonian path between each pair of vertices. The p-closure $\mathrm{Cl}_p(G)$ is obtained from G by successively joining pairs of nonadjacent vertices with degree sum at least p as long as such pairs do exist.

The following result shows that some Hamiltonian properties are preserved by generating specific closures.

Theorem 1 [3]**.** *Let G be a graph of order n. The graph G is traceable, Hamiltonian, or Hamiltonian-connected if and only if $Cl_{n-1}(G)$ is traceable, $Cl_n(G)$ is Hamiltonian, or $Cl_{n+1}(G)$ is Hamiltonian-connected, respectively.*

The next theorem gives sufficient conditions for Hamiltonian properties depending on the independence number $\alpha(G)$ and the connectivity $\kappa(G)$.

Theorem 2 [4]**.** *If G is a graph of order $n \geq 3$ with independence number $\alpha(G)$ and connectivity $\kappa(G)$ with $\alpha(G) \leq \kappa(G) - s$, then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.*

Bondy proved sufficient conditions for the Hamiltonicity of graphs with respect to independent sets of arbitrary size. We present a short proof of a generalization of this result which also contains the traceable and the Hamiltonian-connected case.

Let $\sigma_k(G)$ be the minimum sum of the degrees of any k independent vertices, that is,

$$\sigma_k(G) = \min \left\{ \sum\nolimits_{v \in S} d(v) : S \subseteq V(G),\, S \text{ is independent},\, |S| = k \right\}.$$

If G does not contain k independent vertices, then we set $\sigma_k(G) = \infty$.

Theorem 3 [2]**.** *Let G be a $(k+s)$-connected graph of order $n \geq 3$. If*

$$\sigma_{k+1}(G) \geq \frac{1}{2}((k+1)(n+s-1)+1),$$

then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.

Proof. Assume that G contains a set of $k+1$ pairwise independent vertices $v_0, \ldots, v_k$ with $d(v_i) + d(v_j) \leq n + s - 1$ for all $0 \leq i < j \leq k$. Then

$$k \cdot \sum\nolimits_{0 \leq i \leq k} d(v_i) = \sum\nolimits_{0 \leq i < j \leq k} d(v_i) + d(v_j) \leq \binom{k+1}{2}(n+s-1) = \frac{1}{2}k(k+1)(n+s-1),$$

which implies $\sigma_{k+1}(G) \leq \sum_{i=0}^{k} d(v_i) \leq \frac{1}{2}(k+1)(n+s-1)$, a contradiction. It follows that there exists a pair of vertices having degree sum at least $n+s$ in each set of $k+1$ independent vertices, that is, the closure $\mathrm{Cl}_{n+s}(G)$ does not contain an independent set of $k+1$ vertices. Therefore, $\alpha(\mathrm{Cl}_{n+s}(G)) \leq k \leq \kappa(\mathrm{Cl}_{n+s}(G)) - s$ since $\mathrm{Cl}_{n+s}(G)$ is $(k+s)$-connected. Then, by Theorem 2, the graph $\mathrm{Cl}_{n+s}(G)$ is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$. Application of Theorem 1 completes the proof. ■

Setting $k = 0$ and $k = 1$ in Theorem 3 results in the sufficient condition of Dirac [5] and Ore [7, 8], respectively.

In [1] it was asked whether it would be possible to allow $(k+1)$-sets of pairwise independent vertices in Theorem 3 violating the degree condition and still implying traceability, Hamiltonicity, or Hamiltonian-connectedness.

Let $\overline{n}^s_{k+1}$ be the number of such sets, that is,

$$\overline{n}^s_{k+1} = \left|\left\{\{v_0, \ldots, v_k\} \text{ indep.} : \sum\nolimits_{i=0}^{k} d(v_i) < \frac{1}{2}((k+1)(n+s-1)+1)\right\}\right|.$$

The following upper bounds for $\overline{n}^s_{k+1}$ have been proved.

Theorem 4 [1]**.** *Let G be a $(k+s)$-connected graph of order at least 3 and connectivity $\kappa(G) \geq 2$. If*

$$\overline{n}^s_{k+1}(G) < \binom{\kappa(G) - s + 1}{k+1},$$

then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.

If the minimum degree of the graph is sufficiently large, then this result can be improved.

Theorem 5 [1]**.** *Let G be a $(k+s)$-connected graph, $k+s \geq 2$, of order $n \geq 3$ and minimum degree $\delta(G) \geq \frac{n+2(s+1)}{3}$. If*

$$\overline{n}^s_{k+1}(G) < \binom{\delta(G) - s + 1}{k+1},$$

then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.

The upper bounds for $\overline{n}^s_{k+1}$ in Theorems 4 and 5 are tight for odd $k+s$ (see [1]).

In this note we generalize the result of Theorem 5 for $s = -1$ and $s = 0$ and graphs that fulfill a certain connectivity condition.

2. Results

A cycle C is called a D_λ-cycle of a graph G if every component of $G - V(C)$ has less than λ vertices.

We use the following result on D_λ-cycles for the proof of Theorem 7.

Lemma 6 [6]**.** *Let G be a λ-connected graph of order $n \geq 3$ and minimum degree δ. If*

$$\delta \geq \frac{n + (\lambda - 1)\lambda}{\lambda + 1},$$

then G has a D_λ-cycle.

Theorem 7. *Let G be a λ-connected graph of order $n \geq 3$ and minimum degree $\delta = \delta(G) \geq \frac{n+(\lambda-1)\lambda}{\lambda+1}$. If*

$$\overline{n}^0_{k+1}(G) < \binom{\delta - \lambda + 3}{k+1} + (\lambda - 2)\binom{\delta - \lambda + 2}{k},$$

then G is Hamiltonian.

Proof. If $\lambda = 1$, then G is Hamiltonian by Dirac's condition $\delta \geq \frac{n}{2}$ ($k = s = 0$ in Theorem 3). Therefore, let $\lambda \geq 2$.

Let $G' \cong \mathrm{Cl}_n(G)$ be the n-closure of G. Assume that G' is not Hamiltonian. Due to Lemma 6, there is a D_λ-cycle in G' since $\delta' = \delta(G') \geq \delta \geq \frac{n+(\lambda-1)\lambda}{\lambda+1}$. Let C be a longest D_λ-cycle in G'. By the assumption, there exists a vertex $x \in V(G') \setminus V(C)$ which has $t \leq \lambda - 2$ neighbors in its component of $G' - V(C)$ and at least $\delta' - t$ neighbors on C. Let $v_1, \ldots, v_{\delta'-t}$ be neighbors of x on C and v_i^+ be the successor of v_i on C with respect to a given orientation of the vertices of C. If there were adjacent vertices v_i^+ and v_j^+ or adjacent vertices x and v_i^+, then this would imply the existence of a cycle being longer than C. Therefore, $\{x, v_1^+, \ldots, v_{\delta'-t}^+\}$ is an independent set in G'. By the same reason none of the other at least t vertices of the component of $G' - V(C)$ containing x is adjacent to a vertex v_i^+, $1 \leq i \leq \delta' - t$. Therefore, the number of $(k+1)$-sets of independent vertices of G' is at least $f(t) = \binom{\delta'-t+1}{k+1} + t\binom{\delta'-t}{k}$. In $\mathrm{Cl}_n(G)$ the degree sum of every pair of nonadjacent vertices is at most $n - 1$ and hence, as in the proof of Theorem 3, the degree sum of every $(k+1)$-set of independent vertices is at most $\frac{1}{2}(k+1)(n-1)$. Hence $\overline{n}^0_{k+1}(G') \geq f(t)$. Since $f(t) - f(t+1) = (t+1)\binom{\delta'-t-1}{k-1} \geq 0$, the function $f(t)$ is decreasing and hence has its minimum for the largest possible t, that is, for $t = \lambda - 2$. Therefore,

$$\begin{aligned}\overline{n}^0_{k+1}(G) \geq \overline{n}^0_{k+1}(G') \geq f(\lambda - 2) &\geq \tbinom{\delta'-\lambda+3}{k+1} + (\lambda - 2)\tbinom{\delta'-\lambda+2}{k} \\ &\geq \tbinom{\delta-\lambda+3}{k+1} + (\lambda - 2)\tbinom{\delta-\lambda+2}{k},\end{aligned}$$

which is a contradiction to the condition of the theorem.

Hence G' is Hamiltonian, which implies the Hamiltonicity of G according to Theorem 1. ■

Theorem 8. *Let G be a $(\lambda-1)$-connected graph of order $n = n(G) \geq 2$ and minimum degree $\delta = \delta(G) \geq \frac{n+(\lambda-2)\lambda}{\lambda+1}$. If*

$$\overline{n}_{k+1}^{-1}(G) < \binom{\delta-\lambda+4}{k+1} + (\lambda-2)\binom{\delta-\lambda+3}{k},$$

then G is traceable.

Proof. Let $H \cong G + K_1$. We show that H fulfills all conditions of Theorem 7.

The graph H is λ-connected and has order $n(H) \geq 3$ and minimum degree $\delta(H) = \delta(G) + 1 \geq \frac{n(H)+(\lambda-1)\lambda}{\lambda+1}$. The $(k+1)$-sets of independent vertices in G and H coincide since the vertex of K_1 is adjacent to all other vertices. Moreover, each $(k+1)$-set of independent vertices in H having a degree sum less than $\frac{1}{2}((k+1)(n(H)-1)+1)$ has a degree sum less than $\frac{1}{2}((k+1)(n(G)-2)+1)$ since $d_H(u) = d_G(u) + 1$ for every vertex $u \in V(G)$ and $n(H) = n(G) + 1$. This implies

$$\begin{aligned}\overline{n}_{k+1}^{0}(H) = \overline{n}_{k+1}^{-1}(G) &< \tbinom{\delta(G)-\lambda+4}{k+1} + (\lambda-2)\tbinom{\delta(G)-\lambda+3}{k} \\ &= \tbinom{\delta(H)-\lambda+3}{k+1} + (\lambda-2)\tbinom{\delta(H)-\lambda+2}{k}.\end{aligned}$$

Therefore, H is Hamiltonian according to Theorem 7 implying the traceability of G. ■

Setting $\lambda = 2$ in Theorems 7 and 8 implies the result of Theorem 5 for the cases $s = 0$ and $s = -1$. Moreover, it turns out that the condition $k + s \geq 2$ in Theorem 5 can be omitted in these cases.

The preceding results can be summarized as follows.

Corollary 9. *Let G be a $(\lambda+s)$-connected graph of order $n \geq 3+s$ and minimum degree $\delta = \delta(G) \geq \frac{n+(\lambda+s-1)\lambda}{\lambda+1}$. If*

$$\overline{n}_{k+1}^{s}(G) < \binom{\delta-\lambda-s+3}{k+1} + (\lambda-2)\binom{\delta-\lambda-s+2}{k},$$

then G is traceable for $s = -1$ and Hamiltonian for $s = 0$.

Up to now, we did not succeed in proving the statement of Corollary 9 for $s = 1$, that is, for Hamiltonian-connectedness.

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