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ON MINIMAL GEODETIC DOMINATION IN GRAPHS

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Abstract

Let G be a connected graph. For two vertices u and v in G, a u–v geodesic is any shortest path joining u and v. The closed geodetic interval $I_G[u, v]$ consists of all vertices of G lying on any u–v geodesic. For $S \subseteq V(G)$, S is a geodetic set in G if $\bigcup_{u,v \in S} I_G[u, v] = V(G)$.

Vertices u and v of G are neighbors if u and v are adjacent. The closed neighborhood $N_G[v]$ of vertex v consists of v and all neighbors of v. For $S \subseteq V(G)$, S is a dominating set in G if $\bigcup_{u \in S} N_G[u] = V(G)$. A geodetic dominating set in G is any geodetic set in G which is at the same time a dominating set in G. A geodetic dominating set in G is a minimal geodetic dominating set if it does not have a proper subset which is itself a geodetic dominating set in G. The maximum cardinality of a minimal geodetic dominating set in G is the upper geodetic domination number of G. This paper initiates the study of minimal geodetic dominating sets and upper geodetic domination numbers of connected graphs.

Keywords: minimal geodetic dominating set, upper geodetic domination number.

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1. Introduction

Throughout this paper we consider only finite connected graphs with no loops or multiple edges. All basic graph theoretic terminologies and notations adapted here are taken from [11].

Let G and H be graphs with disjoint vertex sets. The *join* $G + H$ of G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *composition* (or *lexicographic product*) $G[H]$ of G and H is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Let G be a connected graph. For any two vertices u and v in G, a *u–v geodesic* refers to any shortest path in G joining u and v. The length of a u–v geodesic is called the *distance* between u and v, and is denoted by $d_G(u, v)$. The *eccentricity* $e_G(v)$ of a vertex v is defined by $e_G(v) = \max\{d_G(u, v) : u \in V(G)\}$ and the *diameter* of G is the number $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\}$. The *closed geodetic interval* $I_G[u, v]$ is the set of all vertices lying on any u–v geodesic. For a subset S of the vertex set $V(G)$ of G, the *geodetic closure* of S is the set $I_G[S] = \bigcup_{u,v \in S} I_G[u, v]$. Various concepts inspired by geodetic closures are introduced in [7, 11]. A *geodetic set* in G is any set S of vertices in G satisfying $I_G[S] = V(G)$. The minimum cardinality $g(G)$ of a geodetic set is the *geodetic number* of G. Geodetic sets and geodetic numbers are studied in [1, 2, 3, 4, 5, 6]. A geodetic set S in G is a *minimal geodetic set* if S does not have a proper subset that is itself a geodetic set in G. The maximum cardinality of a minimal geodetic set in G is denoted by $g^+(G)$. Zhang *et al.* investigated a minimal geodetic set in a connected graph in [4].

We also define $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$ and $I_G(S) = \bigcup_{u,v \in S} I_G(u, v)$. We call S a *2–path closure absorbing set* if for each $x \in V(G) \setminus S$, there exist $u, v \in S$ such that $d_G(u, v) = 2$ and $x \in I_G(u, v)$. The minimum cardinality of a 2–path closure absorbing set in G is denoted by $\rho_2(G)$. Since a 2-path closure absorbing set is always a geodetic set, $g(G) \leq \rho_2(G)$ for all connected graphs G. In [6], the geodetic numbers of some classes of graphs are described in terms of 2-path closure absorbing sets. A 2-path closure absorbing set S is a *minimal 2-path closure absorbing set* if S does not contain a proper subset that is itself 2-path closure absorbing. The maximum cardinality of a minimal 2-path closure absorbing set in G is denoted by $\rho_2^+(G)$.

The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *degree*, $deg_G(v)$, of a vertex v refers to the value $|N_G(v)|$, and we define $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. A vertex v is an *extreme vertex* if the induced subgraph $\langle N_G[v]\rangle$ is a complete graph. The symbol $Ext(G)$ denotes the set of all extreme vertices in G. For $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{v\in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. If $N_G[S] = V(G)$, then S is a *dominating set* in G. The minimum cardinality among dominating sets in G is called the *domination number* of G, and is denoted by $\gamma(G)$. A considerable number of studies have been dedicated in obtaining variations of the concept (see [12, 13, 14, 15]). The authors in [9] cited over 75 variations of domination and listed over 1,200 papers related to domination in graphs. An application to electrical power networks is being studied in [10]

A subset S of $V(G)$ is a *geodetic dominating set* in G if S is a geodetic set and at the same time a dominating set in G. The minimum cardinality of a geodetic dominating set is called the *geodetic domination number* of G, and is denoted by $\gamma_g(G)$. The study of geodetic domination was initiated by Escuardo, Gera, Hansberg, Jafari Rad and Volkmann [8] in 2011. Some other interesting results can also be found in [16].

Customarily or as used in several literatures, the symbols like g-set, ρ_2-set, ρ_2^+-set, γ-set, and γ_g-set in a graph G would refer to a geodetic set of cardinality $g(G)$, a 2-path closure absorbing set with cardinality $\rho_2(G)$, a minimal 2-path closure absorbing set with cardinality $\rho_2^+(G)$, a dominating set with cardinality $\gamma(G)$, and a geodetic dominating set with cardinality $\gamma_g(G)$, respectively.

Since a 2-path closure absorbing set is also a geodetic dominating set, $g(G) \leq \gamma_g(G) \leq \rho_2(G)$ for all connected graphs G of order $n \geq 2$. In particular, if $diam(G) = 2$, then $\gamma_g(G) = \rho_2(G)$.

The following is found in [8].

Theorem 1.1 [8]**.** *Let G be a connected graph of order $n \geq 2$. Then*

(i) $\gamma_g(G) = 2$ *if and only if there exists a geodetic set* $S = \{u, v\}$ *such that* $d_G(u, v) \leq 3$.

(ii) $\gamma_g(G) = n$ *if and only if G is the complete graph on n vertices.*

(iii) $\gamma_g(G) = n - 1$ *if and only if there is a vertex v in G such that v is adjacent to every other vertex of G and $G - v$ is the union of at least two complete graphs.*

2. Minimal Geodetic Domination

A geodetic dominating set S in a connected graph G of order $n \geq 2$ is a *minimal geodetic dominating set* in G if S does not have a proper subset which is itself a geodetic dominating set in G. The maximum cardinality of a minimal geodetic dominating set in G is the *upper geodetic domination number* of G, and is denoted by $\gamma_g^+(G)$. A minimal geodetic dominating set with cardinality $\gamma_g^+(G)$ is also called a γ_g^+*-set*.

Example 2.1. (i) *If $m, n \geq 2$ and U and W are the partite sets of the complete bipartite graph $K_{m,n}$, then the minimal geodetic dominating sets in $K_{m,n}$ are U and W and all sets of the form $S = \{u, v, x, y\}$, where $u, v \in U$ and $x, y \in W$. More precisely,*

$$\gamma_g^+(K_{m,n}) = \begin{cases} 4, & \text{if } m = n = 3, \\ \max\{m, n\}, & \text{otherwise.} \end{cases}$$

(ii) *For $2 \leq n \leq 4$, $\gamma_g^+(P_n) = 2$, and for $n \geq 5$,*

$$\gamma_g^+(P_n) = \begin{cases} 2\left\lfloor \frac{n}{4} \right\rfloor + 1, & \text{if } n \equiv 1 \pmod 4, \\ 2\left\lceil \frac{n}{4} \right\rceil, & \text{otherwise.} \end{cases}$$

Suppose that $n \equiv 1 \pmod 4$, and $P_n = [u_1, u_2, \ldots, u_n]$. Since the set $\{u_1, u_2, u_5, u_6, \ldots, u_{4k-3}, u_{4k-2}, u_{4k+1}\}$ is a minimal geodetic dominating set in P_n, $\gamma_g^+(P_n) \geq 2\left\lfloor \frac{n}{4} \right\rfloor + 1$. Let $S \subseteq V(P_n)$ be a minimal geodetic dominating set in P_n. For every $j = 1, 2, \ldots, n-3$, S contains at most two of the vertices u_j, u_{j+1}, u_{j+2} and u_{j+3}. Thus, $|S| \leq 2\left\lfloor \frac{n}{4} \right\rfloor + 1$. Since S is arbitrary, $\gamma_g^+(P_n) \leq 2\left\lfloor \frac{n}{4} \right\rfloor + 1$. Now, suppose that $n > 4$ but $n \neq 4a+1$ for all positive integers a. Let k be the largest positive integer for which $4k+1 < n$. Since the set of vertices $\{u_1, u_2, u_5, u_6, \ldots, u_{4k+1}, u_n\}$ is a minimal geodetic dominating set in P_n, $\gamma_g^+(P_n) \geq 2\left\lceil \frac{n}{4} \right\rceil$. Using similar arguments, if $S \subseteq V(P_n)$ is a minimal geodetic dominating set in P_n, then $|S| \leq 2\left\lceil \frac{n}{4} \right\rceil$. This means that $\gamma_g^+(P_n) \leq 2\left\lceil \frac{n}{4} \right\rceil$.

Theorem 2.2. *Let G be a connected graph of order $n \geq 2$. Then*

(i) *$\gamma_g^+(G) = 2$ if and only if G is one of the following graphs: P_2, C_4, $\overline{K_2} + H$ where H is connected and either $H = K_{n-2}$ or $\rho_2^+(H) = 2$, G has a g-set $\{u, v\}$ with $u, v \in Ext(G)$ and $d_G(u, v) = 3$.*

(ii) *$\gamma_g^+(G) = n$ if and only if $G = K_n$.*

(iii) *For $n \geq 3$, $\gamma_g^+(G) = n - 1$ if and only if $G = K_1 + \bigcup_{j=1}^{t} K_{r_j}$, where $t \geq 2$.*

Proof. (i) Suppose that $\gamma_g^+(G) = 2$, and let $\{u, v\}$ be a γ_g^+-set in G. Let $S = V(G) \setminus \{u, v\}$. Then $w \in I_G[u, v]$ for all $w \in S$ and $1 \leq d_G(u, v) \leq 3$ by Theorem 1.1. If $d_G(u, v) = 1$, then $S = \emptyset$ and $G = P_2$. Suppose that $d_G(u, v) = 2$. Then $G = \langle\{u, v\}\rangle + \langle S\rangle = \overline{K_2} + \langle S\rangle$. If $|S| = 1$, then $G = P_3 = \overline{K_2} + K_1$. If $|S| = 2$, then either $G = C_4$ or $G = \overline{K_2} + K_2$. Suppose that $|S| \geq 3$. Then $\langle S\rangle$ is connected. If $\langle S\rangle$ is the complete graph K_{n-2}, then $G = \overline{K_2} + K_{n-2}$. Suppose that $H = \langle S\rangle$ is not complete, and let T be a ρ_2^+-set in H. Then T is a γ_g^+-set in G. Thus $|T| = 2$. Hence, $\rho_2^+(H) = 2$. Finally, suppose that $d_G(u, v) = 3$. For each $x \in S$, either $ux \in E(G)$ or $xv \in E(G)$. Suppose that there exist $x, y \in N_G[u]$ with $d_G(x, y) = 2$. Consider $W = N_G(u) \cup \{v\}$. Let $z \in V(G) \setminus W$. If $z = u$, then $[x, z, y]$ is an x–y geodesic in G so that $z \in N_G[W]$ and $z \in I_G[W]$.

Suppose that $z \neq u$. Since $z \in I_G[u,v]$, there exist $a, b \in V(G)$ such that z lies on the u–v geodesic $[u,a,b,v]$. This means that $a \in N_G(u)$ and $z = b$. Thus $z \in N_G[W]$ and $z \in I_G[W]$. Accordingly, W is a geodetic dominating set in G. Let $T \subseteq W$ be a minimal geodetic dominating set in G. Since $v \notin N_G[N_G(u)]$, $v \in T$. Moreover, if $|T \cap N_G(u)| = 1$, then $u \notin I_G[T]$, a contradiction. Thus, $|T \cap N_G(u)| \geq 2$ so that $\gamma_g^+(G) \geq |T| \geq 3$, a contradiction. Therefore, $\langle N_G[u] \rangle$ is complete and $u \in Ext(G)$. Similarly, $v \in Ext(G)$.

Conversely, if G is P_2 or C_4 or $\overline{K_2} + K_{n-2}$, then $\gamma_g^+(G) = 2$. Suppose that $G = \overline{K_2} + H$, where H is connected and noncomplete with $\rho_2^+(H) = 2$. Then $diam(G) = 2$ and $T = V(\overline{K_2})$ is a minimal geodetic dominating set in G. Put $T = \{u, v\}$, and let Z be a minimal geodetic dominating set in G distinct from T. Then $|Z \cap T| \leq 1$. Suppose that $Z \cap T = \{u\}$. Since $ux \in E(G)$ for all $x \in V(H)$, $v \in I_G[Z \setminus \{u\}]$ and $V(H) \subseteq I_G[Z \setminus \{u\}]$ so that $Z \setminus \{u\}$ is a geodetic dominating set in G, a contradiction. Thus $Z \subseteq V(H)$ and, consequently, Z is a minimal 2-path closure absorbing set in H. Thus, $2 \leq |Z| \leq \rho_2^+(H) = 2$ so that $|Z| = 2$. Since Z is arbitrary, $\gamma_g^+(G) = |Z| = 2$. Now, let G have a g-set $\{u, v\}$ with $d_G(u,v) = 3$ and where the induced subgraphs $\langle N_G[u] \rangle$ and $\langle N_G[v] \rangle$ are complete. Then $Ext(G) = \{u, v\}$, which is the unique γ_g^+-set in G. The conclusion follows.

(ii) If $G = K_n$, then $\gamma_g(G) = n$, by Theorem 1.1. Hence $\gamma_g^+(G) = n$. Suppose that $\gamma_g^+(G) = n$. Then each proper subset of $V(G)$ is not a geodetic dominating set in G. Let $v \in V(G)$, and set $S = V(G) \setminus \{v\}$. Then $v \notin N_G[S]$ or $v \notin I_G[S]$. If $v \notin N_G[S]$, then v is an isolated vertex, a contradiction. Thus $v \notin I_G[S]$ so that $v \in Ext(G)$. Since v is arbitrary, $V(G) = Ext(G)$ and $G = K_n$.

(iii) If $G = K_1 + \bigcup_{j=1}^{t} K_{r_j}$ for some $t \geq 2$, then $\gamma_g(G) = n-1$, by Theorem 1.1. By statement (ii), $\gamma_g^+(G) < n$. Thus $\gamma_g^+(G) = n-1$. Suppose that $\gamma_g^+(G) = n-1$. Let $S = V(G) \setminus \{v\}$, where $v \in V(G)$, be a γ_g^+-set in G. We claim that $uv \in E(G)$ for all $u \in S$. Since v is not an endvertex, there exist $x, y \in S$ such that $[x, v, y]$ is an x–y geodesic in G. Suppose that, in the contrary, there exists $u \in S$ with $d_G(u,v) = 2$. Let $[u, w, v]$ be a u–v geodesic in G. If $x = w$ or $y = w$, then $S \setminus \{w\}$ is a geodetic dominating set in G, which is impossible. Suppose that $x \neq w$ and $y \neq w$. If $uy \notin E(G)$, then $S \setminus \{w\}$ is a geodetic dominating set in G. If $uy \in E(G)$ and $wy \notin E(G)$, then $S \setminus \{u\}$ is a geodetic dominating set in G. If $uy, wy \in E(G)$ and $ux \in E(G)$, then $S \setminus \{u\}$ is a geodetic dominating set in G. If $uy, wy \in E(G)$ and $ux \notin E(G)$, then $S \setminus \{w\}$ is a geodetic dominating set in G. Any of the above cases yields a contradiction. This proves the claim. Therefore, $G = K_1 + H$ for some graph H. Next, we show that $H = \bigcup_{j=1}^{t} K_{r_j}$, where $t \geq 2$. Suppose that H has a component K which is not a complete graph. Then K, consequently G, has a geodesic $[x, y, z]$ of length 2. Then $S \setminus \{y\}$ is a geodetic dominating set in G, a contradiction. Therefore, $H = \bigcup_{j=1}^{t} K_{r_j}$. Since

G is not a complete graph, $t \geq 2$. ■

Now follows a Nordhaus-Gaddum-type result. Let the symbol Ξ denote the infinite collection of all connected graphs G such that $\overline{G}$ is also connected.

Theorem 2.3. *For all $G \in \Xi$ of order $n \geq 4$,*

$$4 \leq \gamma_g^+(G) + \gamma_g^+(\overline{G}) \leq 2n - 4.$$

In particular, $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 4$ if and only if $n = 4$.

Proof. Let $G \in \Xi$ be of order $n \geq 4$. Note that if G is either K_n or $K_1 + \bigcup_{j=1}^{t} K_{r_j}$ with $t \geq 2$, then $\overline{G}$ is disconnected, a contradiction. In view of Theorem 2.2,

$$\gamma_g^+(G) + \gamma_g^+(\overline{G}) \leq (n-2) + (n-2) = 2n - 4.$$

The inequality at the left side is obvious.

In particular, if $n = 4$, then $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 4$. Conversely, suppose that $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 4$. Necessarily, $\gamma_g^+(G) = 2$ and $\gamma_g^+(\overline{G}) = 2$. By Theorem 2.2, G has a g-set $\{u, v\}$ with $u, v \in Ext(G)$ and $d_G(u, v) = 3$. Similarly, $\overline{G}$ has a g-set $\{x, y\}$ with $x, y \in Ext(\overline{G})$ and $d_{\overline{G}}(x, y) = 3$. Assume that $x \in N_G[u]$. Suppose that $x = u$. Note that $N_{\overline{G}}(x) = N_G[v]$, and $\langle N_G[v] \rangle$ is not complete in $\overline{G}$. This means that $x \notin Ext(\overline{G})$, a contradiction. Suppose that $xu \in E(G)$. Since $xy \notin E(\overline{G})$, $xy \in E(G)$. If $vy \notin E(G)$, then $xv, vy \in E(\overline{G})$ so that $[x, v, y]$ is a geodesic in $\overline{G}$, a contradiction. Thus $[u, x, y, v]$ is a u–v geodesic in G. Suppose that $n \geq 5$, and let $z \in V(G)$ be distinct from u, x, y and v. Assume $xz \in E(\overline{G})$. Since $x \in Ext(\overline{G})$ and $xv \in E(\overline{G})$, $zv \in E(\overline{G})$ and, consequently, $zu \in E(G)$. Since $u \in Ext(G)$, $xz \in E(G)$, a contradiction. Therefore, $n = 4$. ■

Corollary 2.4. *If $G \in \Xi$ is of order $n \geq 4$, then $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 4$ if and only if $G = P_4$.*

Theorem 2.3 implies that if $G \in \Xi$ of order $n \geq 5$, then

$$5 \leq \gamma_g^+(G) + \gamma_g^+(\overline{G}) \leq 2n - 4.$$

Since $\gamma_g^+(C_5) = \gamma_g^+(\overline{C_5}) = 3$, this upper bound is sharp. Consider the graph G obtained from the cycle $C_4 = [v_1, v_2, v_3, v_4, v_1]$ by adding to C_4 two vertices x and y and the edges xv_1, xv_4, yv_2 and yv_3. For this G, $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 5$, showing that the lower bound is sharp.

3. Realization Problems

For nontrivial connected graphs G,

$$2 \leq \gamma_g(G) \leq \gamma_g^+(G) \leq \rho_2^+(G).$$

In particular, $\gamma_g^+(K_{n,n}) = \rho_2^+(K_{n,n})$ for $n \geq 4$.

Theorem 3.1. *For every pair of positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_g^+(G) = a$ and $\rho_2^+(G) = b$.*

Proof. If $a = b$, then we pick $G = K_{a,a}$. Suppose that $b = a+1$. Obtain the graph G from $P_3 = [v_1, v_2, v_3]$ by adding $(a-1)$ pendant edges v_3x_j, $j = 1, 2, \ldots, a-1$. Then $\gamma_g^+(G) = a$ and $\rho_2^+(G) = b$, which are determined by the sets $\{v_1, x_1, x_2, \ldots, x_{a-1}\}$ and $\{v_1, v_2, x_1, x_2, \ldots, x_{a-1}\}$, respectively.

Suppose that $b = a+k$, where $k \geq 2$. Write $V(K_k) = \{u_1, u_2, \ldots, u_k\}$. Obtain G by joining $P_2 = [v_1, v_2]$ and $K_k + \overline{K_{a-1}}$ using new k edges v_2u_j, $j = 1, 2, \ldots, k$. Note that $Ext(G) = \{v_1\} \cup V(\overline{K_{a-1}})$, and is a γ_g^+-set in G. Thus, $\gamma_g^+(G) = a$. On the other hand, if $k \geq 2$, then $\rho_2^+(G) = a + k = b$ and is determined by the set $V(K_k) \cup Ext(G)$. ■

Corollary 3.2. *For every pair of positive integers a and b with $2 \leq a < b$, the smallest possible order of a connected graph G for which $\gamma_g^+(G) = a$ and $\rho_2^+(G) = b$ is $b+1$.*

Theorem 3.3. *For all positive integers a, b, c with $2 \leq a \leq b < c$ and $c \geq b+2$, there exists a connected graph G such that $\gamma_g(G) = a$, $\gamma_g^+(G) = b$ and $|V(G)| = c$.*

Proof. Suppose that $c = b+2$. Write $a = 2+k$ and $b = r+k$, $r \geq 2$ and $k = 0, 1, 2, \ldots$. If $k = 0$, then we take $G = K_{2,r}$. In this case, $\gamma_g(G) = 2$ and $\gamma_g^+(G) = r$. Suppose that $k \geq 1$. Consider the graph G as in Figure 1.

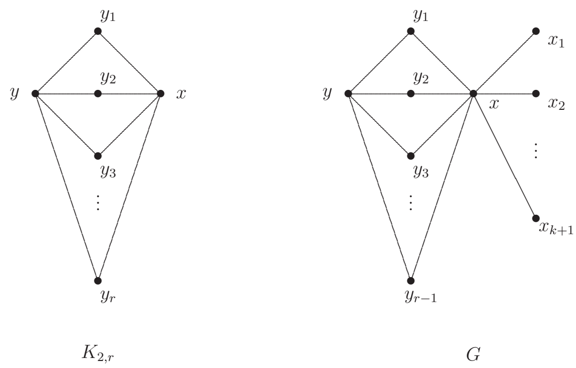

Figure 1

If $r = 2$, G is obtained by adjoining to path $[y, y_1, x]$ $(k+1)$ pendant edges xx_j, $j = 1, 2, \ldots, k+1$. Then $Ext(G) = \{y, x_1, x_2, \ldots, x_{k+1}\}$ is the unique minimal geodetic dominating set in G. Here we have $\gamma_g(G) = \gamma_g^+(G) = 2 + k$. Now, suppose that $r \geq 3$. G is obtained from $K_{2,r-1}$ (with partite sets $U = \{x, y\}$ and $W = \{y_1, y_2, \ldots, y_{r-1}\}$) by adding to $K_{2,r-1}$ $(k+1)$ pendant edges xx_j, $j = 1, 2, \ldots, k+1$. The minimal geodetic dominating sets in G are $\{y, x_1, x_2, \ldots, x_{k+1}\}$ and $W \cup \{x_1, x_2, \ldots, x_{k+1}\}$. Thus $\gamma_g(G) = 2 + k$ and $\gamma_g^+(G) = r + k$.

Suppose that $c = b + 3$. Write $a = 2 + k$ and $b = r + k$, $k = 0, 1, \ldots$ and $r \geq 2$. Suppose that $k = 0$. Consider the graph $G = G_1$ as in Figure 2.

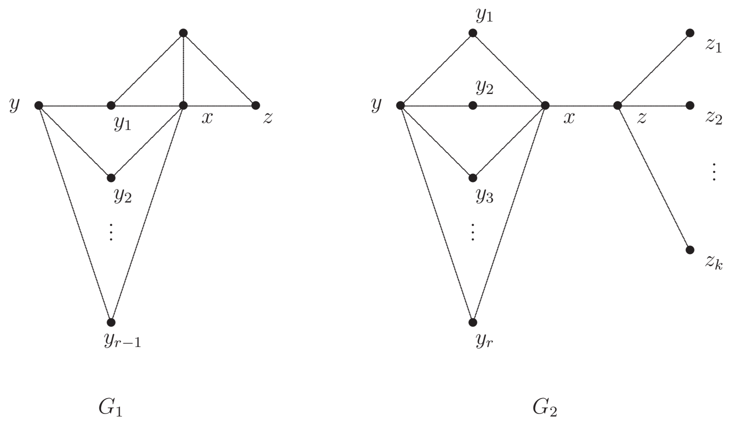

Figure 2

If $r = 2$, then $\{y, z\}$ is the unique minimal geodetic dominating set in G so that $\gamma_g(G) = \gamma_g^+(G) = 2$. If $r \geq 3$, then $\gamma_g(G) = 2$ and $\gamma_g^+(G) = (r-1) + 1 = r$, the latter being determined by $\{z, y_1, y_2, \ldots, y_{r-1}\}$. Suppose that $k \geq 1$. Obtain G as the graph G_2 in Figure 2 by taking the union of $K_{1,k+1}$ (with partite sets $\{z\}$ and $\{x, z_1, z_2, \ldots, z_k\}$) and $K_{2,r}$ (with partite sets $\{x, y\}$ and $\{y_1, y_2, \ldots, y_r\}$). Note that $\{z_1, z_2, \ldots, z_k\}$ is always contained in a geodetic dominating set in G. Thus $\gamma_g(G) = 2 + k$ and $\gamma_g^+(G) = r + k$.

Finally, suppose that $c = b + d$, where $d \geq 4$. Write $a = 2 + k$ and $b = r + k$, where $r \geq 2$ and $k = 0, 1, 2, \ldots$, and put $l = c - b - 3$. For $k = 0$, we obtain G as the graph G_1^* in Figure 3 by joining $K_{l+1} + \overline{K_2}$ and $K_{r-1,2}$ using the common vertices x and y_1. Note that $Ext(G) = \{z\}$, and $S = \{z, y\}$ is a minimal geodetic dominating in G and every minimal geodetic dominating set that contains y coincides S. Thus, aside from S, the other minimal geodetic dominating set in G is $\{z, y_1, y_2, \ldots, y_{r-1}\}$. Consequently, $\gamma_g(G) = 2 = a$ and $\gamma_g^+(G) = r = b$. Now, suppose that $k \geq 1$. Consider G as the graph G_2^* in Figure 3 obtained from G_1^* in Figure 3 by adjoining pendant edges zz_j. $Ext(G) = \{z_1, z_2, \ldots, z_k\}$ and if S is a γ_g^+-set that contains y, then $|S| = k + 2$. In this case, $\gamma_g(G) = k + 2 = a$ and $\gamma_g^+(G) = r + k = b$. ∎

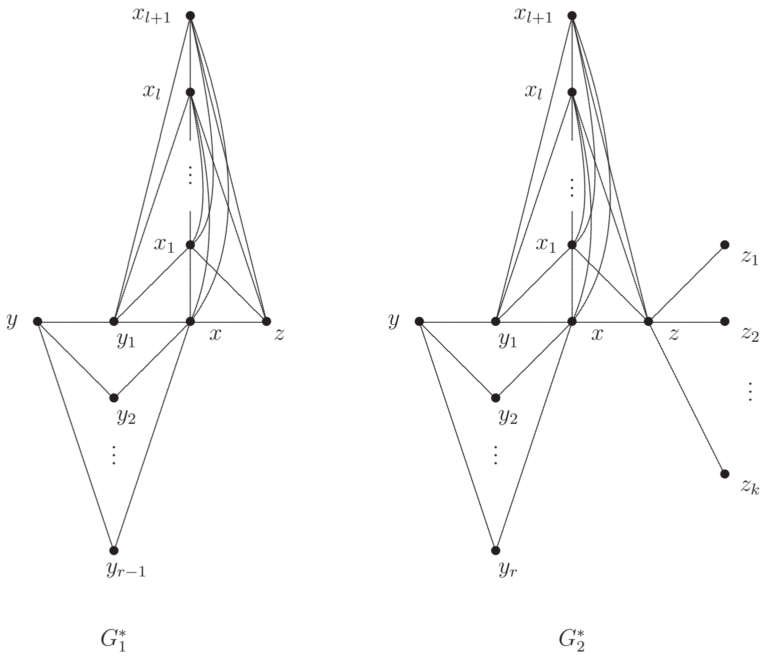

Figure 3

The next corollary follows from Theorem 2.2 and the existence proof of Theorem 3.3.

Corollary 3.4. *For every pair of positive integers a and b with $2 \leq a < b$, the minimum order of a connected graph G for which $\gamma_g(G) = a$ and $\gamma_g^+(G) = b$ is $b + 2$.*

4. ρ_3-sets

Let G be a connected graph of order $n \geq 2$ and $S \subseteq V(G)$. S is said to be a ρ_3-*set* in G if for every $w \in V(G) \setminus S$ there exist $u, v \in S$ such that $d_G(u, v) \leq 3$ and $w \in I_G[u, v]$. We denote by $\rho_3(G)$ the minimum cardinality of a ρ_3-set in G. Since every 2-path closure absorbing set is a ρ_3-set, $\rho_3(G) \leq \rho_2(G)$. In particular, if $diam(G) = 2$, then $\rho_2(G) = \rho_3(G)$. Since a ρ_3-set is a geodetic dominating set, $\gamma_g(G) \leq \rho_3(G)$. If $diam(G) \leq 3$, then $\gamma_g(G) = \rho_3(G)$. However, in general, $\gamma_g(G)$ and $\rho_3(G)$ are not necessarily equal.

Graph G is said to be K_3-*free* (resp. C_4-*free*) if G does not contain K_3 (resp. C_4) as a subgraph.

Theorem 4.1. *Let G be a connected graph of order $n \geq 2$, and let $S \subseteq V(G)$.*
(i) *If S is a ρ_3-set in G, then for all $v \in S$, $\min\{d_G(u, v) : u \in S\} \leq 3$.*

(ii) *If G is K_3-free and C_4-free and S is a geodetic dominating set in G, then S is a ρ_3-set in G.*

Proof. The conclusions in statements (i) and (ii) are trivially satisfied for cases where $n = 2$, $n = 3$ and $n = 4$. Assume that $n \geq 5$, and let $S \subseteq V(G)$. To prove statement (i), suppose that S is a ρ_3-set in G, and suppose that there is $v \in S$ such that $d_G(v) = \min\{d_G(u, v) : u \in S\} \geq 4$. Let $w \in S$ be such that $d_G(w, v) = d_G(v)$. Then there exists $u \in V(G) \setminus S$ lying on a w–v geodesic with $d_G(u, v) = 2$. Since S is dominating in G, there exists $z \in S$ such that $uz \in E(G)$. Observe that $d_G(z, v) \leq 3$, a contradiction. Therefore, $d_G(v) \leq 3$ for all $v \in S$.

Next, we prove statement (ii). Suppose that G is K_3-free and C_4-free, and let $v \in V(G) \setminus S$. If S is a dominating set in G, then there exists $x \in S$ such that $xv \in E(G)$. If S is a geodetic set, then v is not an endvertex of G. Pick $u \in N_G(v)$ with $u \neq x$. Since G is K_3-free, $[x, v, u]$ is an x–u geodesic in G. If $u \in S$, then x and u are the desired vertices in S for v. Suppose that $u \notin S$. Pick $y \in S$ such that $uy \in E(G)$. Sine G is K_3-free and C_4-free, $xy, vy \notin E(G)$. Consequently, $[x, v, u, y]$ is an x–y geodesic in G with $d_G(x, y) = 3$. ■

If G is a connected graph of order $n \geq 2$ which is K_3-free and C_4-free, then $\rho_3(G) = \gamma_g(G)$. In particular, if T is a tree of order $n \geq 2$, then $\rho_3(T) = \gamma_g(T)$.

Theorem 4.2. *Let G be a connected K_3-free graph of order $n \geq 2$. Then*

$$\rho_3(G) \leq \gamma_g^+(G).$$

Proof. Suppose that $Ext(G) \neq \emptyset$. Put $Ext(G) = \{x_1, x_2, \ldots, x_k\}$ for some positive integer k. For each $j = 1, 2, \ldots, k$, define $S_j = \{x_1, x_2, \ldots, x_j\}$. If $N_G[S_k] \neq V(G)$, choose $x_{k+1} \in V(G) \setminus N_G[S_k]$, and put $S_{k+1} = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$. If $N_G[S_{k+1}] \neq V(G)$, then choose $x_{k+2} \in V(G) \setminus N_G[S_{k+1}]$, and put $S_{k+2} = \{x_1, x_2, \ldots, x_{k+1}, x_{k+2}\}$. Continuing in this way, there is a smallest positive integer m such that $N_G[S_m] = V(G)$. If $Ext(G) = \emptyset$, then construct $S_m = \{x_1, x_2, \ldots, x_m\}$ by choosing any $x_1 \in V(G)$ and put $S_1 = \{x_1\}$, and for $j \geq 2$, $x_j \in V(G) \setminus N_G[S_{j-1}]$, where $S_{j-1} = \{x_1, x_2, \ldots, x_{j-1}\}$. In any case, we claim that $S = S_m$ is a minimal geodetic dominating set and at the same time a ρ_3-set in G. Clearly, S is a dominating set in G. Let $u \in V(G) \setminus S$. Then there exists $w \in S$ such that $uw \in E(G)$. Since $u \notin Ext(G)$ and G is K_3-free, there exists $v \in V(G)$ such that $[v, u, w]$ is a v–w geodesic in G. Suppose that $v \notin S$. There exists $z \in S$ such that $zv \in E(G)$. Since G is K_3-free, $uz \notin E(G)$. Also, by the construction of S, $zw \notin E(G)$. Thus, $[w, u, v, z]$ is a w–z geodesic in G. Here, $d_G(w, z) \leq 3$ and $u \in I_G[w, z] \subseteq I_G[S]$. Since u is arbitrary, S is a ρ_3-set and a geodetic dominating set in G. Now let $S^* = S \setminus \{x_j\}$, $j = 1, 2, \ldots, m$. We will show that S^* is not a dominating set in G. Suppose that $Ext(G) \neq \emptyset$. If $j \leq k$, then $x_j \in Ext(G)$ and S^* is not a geodetic set in G. Suppose that $j > k$. Since $x_j \notin Ext(G)$, there exist

$u, v \in V(G)$ such that $[u, x_j, v]$ is a u–v geodesic in G. Since $x_j \in S$, $u, v \notin S$. In fact, $x \notin S$ for all $x \in N_G[x_j]$. Thus $x_j \notin N_G[S^*]$, and S^* is not a dominating set in G. The case where $Ext(G) = \emptyset$ is handled similarly. Since j is arbitrary, S is a minimal geodetic dominating set in G. Therefore, $\rho_3(G) \leq |S| \leq \gamma_g^+(G)$. ■

It is easy to verify that $\rho_3(P_5) = 3 = \gamma_g^+(P_5)$. Hence the bound given in Theorem 4.2 is sharp.

5. Join and Composition of Graphs

For connected graphs G and H, if $S \subseteq V(G)$ is a 2-path closure absorbing set in G, then S is a geodetic dominating set in $G + H$.

Theorem 5.1. *For noncomplete connected graphs G, $\gamma_g^+(G + K_n) = \rho_2^+(G)$.*

Proof. First, we claim that if $S \subseteq V(G + K_n)$ is a geodetic dominating set in $G + K_n$, then $A = S \cap V(G)$ is a 2-path closure absorbing set in G. Let $S \subseteq V(G + K_n)$ be a geodetic dominating set in $G + K_n$. Let $x \in V(G) \setminus A$, and let $u, v \in S$ such that $x \in I_G[u, v]$. Necessarily, $u, v \in V(G)$. Since $diam(G + K_n)$ is 2, $[u, x, v]$ is a u–v geodesic in G. Thus $d_G(u, v) = 2$ and A is a 2-path closure absorbing set in G.

Now let $S \subseteq V(G + K_n)$ be a minimal geodetic dominating set in $G + K_n$. The above result implies that $A = S \cap V(G)$ is a 2-path closure absorbing set in G, and consequently, A is a geodetic dominating set in $G + K_n$. Since S is a minimal geodetic dominating set, $S = A$ so that S is a minimal 2-path closure absorbing set in G. Since S is arbitrary, $\gamma_g^+(G + K_n) \leq \rho_2^+(G)$.

Conversely, let $S \subseteq V(G)$ be a ρ_2^+-set in G. Then S is a geodetic dominating set in $G + K_n$. That S is, in fact, a minimal geodetic dominating set in $G + K_n$ follows from the claim above. This yields $\rho_2^+(G) \leq \gamma_g^+(G + K_n)$. ■

Theorem 5.2. *For all noncomplete connected graphs G and H,*

$$\gamma_g^+(G + H) = \max\{4, \rho_2^+(G), \rho_2^+(H)\}.$$

Proof. Let $S \subseteq V(G + H)$ be a minimal geodetic dominating set in $G + H$. If $S \subseteq V(G)$, then S is a minimal 2-path closure absorbing set in G since $diam(G + H) = 2$. This means that $|S| \leq \rho_2^+(G)$. Similarly, if $S \subseteq V(H)$, then $|S| \leq \rho_2^+(H)$. Suppose that $A = S \cap V(G) \neq \emptyset$ and $B = S \cap V(H) \neq \emptyset$. Then $|A| \geq 2$ and $|B| \geq 2$, and $V(H) \subseteq I_{G+H}[A]$ and $V(G) \subseteq I_{G+H}[B]$. The minimality of S implies that $|A| = |B| = 2$ and $|S| = 4$. Hence $\gamma_g^+(G + H) \leq \max\{4, \rho_2^+(G), \rho_2^+(H)\}$.

To prove the other inequality, note that if $S \subseteq V(G)$, then S is a minimal geodetic dominating set in $G + H$ if and only if S is a minimal 2-path closure absorbing set in G. This means that $\max\{\rho_2^+(G), \rho_2^+(H)\} \leq \gamma_g^+(G+H)$. Since G and H are noncomplete, we can pick $u, v \in V(G)$ and $x, y \in V(H)$ such that $d_G(u,v) = 2$ and $d_H(u,v) = 2$. Then $\{u, v, x, y\}$ is a minimal geodetic dominating set in $G+H$. This means that $4 \leq \gamma_g^+(G+H)$. This completely establishes the desired inequality. ■

Next, we investigate the minimal geodetic domination in the composition of graphs $G + K_n$.

For $A \subseteq V(G)$, we define $A^g = A \cap I_G(A)$, and for $S \subseteq V(G[H])$, $S_G = \{u \in V(G) : (u,v) \in S$ for some $v \in V(H)\}$.

It is known (see [16]) that if $S \subseteq V(G[H])$ is a geodetic dominating set in $G[H]$, then S_G is a geodetic dominating set in G.

Theorem 5.3. [16] *Let G be a noncomplete connected graph and $n \geq 2$. Then $S \subseteq V(G[K_n])$ is a geodetic dominating set in $G[K_n]$ if and only if $S = [(A \setminus A^g) \times V(K_n)] \cup T$, where $A = S_G$ and $T_G = A^g$.*

Corollary 5.4. *For all noncomplete connected graphs G and $n \geq 2$,*

$$\gamma_g^+(G[K_n]) \geq \max\{n|A| - (n-1)|A^g| : A \text{ is a minimal geodetic dominating set in } G\}.$$

Proof. Let

$$\alpha = \max\{n|A| - (n-1)|A^g| : A \text{ is a minimal geodetic dominating set in } G\}.$$

Let $A \subseteq V(G)$ be a minimal geodetic dominating set in G, and let $S = [(A \setminus A^g) \times V(K_n)] \cup [A^g \times \{v\}]$, where $v \in V(K_n)$. By Theorem 5.3, S is a geodetic dominating set in $G[K_n]$. Suppose that there exists $S^* \subseteq S$ such that S^* is a geodetic dominating set in $G[K_n]$. By Theorem 5.3, $S^* = [(B \setminus B^g) \times V(K_n)] \cup T$, where B is a geodetic dominating set in G and $T_G = B^g$. Since $S^* \subseteq S$, $B \subseteq A$. Since A is a minimal geodetic dominating set in G, $A = B$. Therefore, $S = S^*$ and S is a minimal geodetic dominating set in $G[K_n]$. Thus, $\gamma_g^+(G[K_n] \geq |S| = n|A| - (n-1)|A^g|$. Since A is arbitrary, $\gamma_g^+(G[K_n]) \geq \alpha$. ■

Lemma 5.5. *Let G be a noncomplete connected graph and $n \geq 2$.*

(i) *If $S \subseteq V(G)$ is a geodetic dominating set (respectively, minimal geodetic dominating set) in G, then $\{u\} \times S$ is a geodetic dominating set (resp. minimal geodetic dominating set) in $K_n[G]$ for all $u \in V(K_n)$.*

(ii) *If $S \subseteq V(G)$ is a geodetic set (resp. minimal geodetic set but not dominating) in G, then $\{(w,z)\} \cup (\{u\} \times S)$ is a geodetic dominating set (respectively, minimal geodetic dominating set) in G for all $z \in V(G)$ and for all distinct $w, u \in V(K_n)$.*

Proof. Let S be a geodetic dominating set in G and $u \in V(K_n)$. Let $(x, y) \in V(K_n[G]) \setminus (\{u\} \times S)$. Suppose that $x \neq u$. Then $(x, y)(u, v) \in E(K_n[G])$ for all $v \in S$. Thus, $(x, y) \in N_{K_n[G]}[\{u\} \times S]$. Choose $v_1, v_2 \in S$ such that $d_G(v_1, v_2) \geq 2$. Then $(x, y) \in I_{K_n[G]}[(u, v_1), (u, v_2)] \subseteq I_{K_n[G]}[\{u\} \times S]$. Suppose that $x = u$. Then $y \notin S$. Since S is a geodetic dominating set in G, $y \in N_G[S] \cap I_G[S]$. Thus, $(x, y) \in N_{K_n[G]}[\{u\} \times S]$ and $(x, y) \in I_{K_n[G]}[\{u\} \times S]$. This proves that $\{u\} \times S$ is a geodetic dominating set in $K_n[G]$. Finally, let $\{u\} \times T \subseteq \{u\} \times S$ be a geodetic dominating set in $K_n[G]$. Then $T \subseteq S$ and T is a geodetic dominating set in G. If S is a minimal geodetic dominating set in G, then $T = S$, and this proves statement (i).

To prove statement (ii), let $C = \{(w, z)\} \cup (\{u\} \times S)$, where $S \subseteq V(G)$ is a geodetic set in G, $z \in V(G)$ and $u, w \in V(K_n)$ with $u \neq w$. Let $(a, b) \in V(K_p[G]) \setminus C$. Suppose that $a = u$. Then $b \notin S$ and $aw \in E(K_n)$ so that $(a, b)(w, z) \in E(K_p[G])$. Since S is a geodetic set in G, there exist $x, y \in S$ such that $b \in I_G[x, y]$. Then $(u, x), (u, y) \in C$ and $(a, b) \in I_{K_p[G]}[((u, x), (u, y)]$. Suppose that $a \neq u$. Then $au \in E(K_n)$ and $(a, b) \in N_{K_p[G]}[\{u\} \times S]$. Choose $x, y \in S$ such that $d_G(x, y) \geq 2$. Then $(u, x), (u, y) \in C$ and $(a, b) \in I_{K_p[G]}[(u, x), (u, y)]$. In any case, $(a, b) \in N_{K_p[G]}[C]$ and $(a, b) \in I_{K_p[G]}[C]$. Since (a, b) is arbitrary, C is a geodetic dominating set in $K_p[G]$. Suppose that S is a minimal geodetic set in G but not dominating. Let $(a, b) \in C$, and put $C^* = C \setminus \{(a, b)\}$. If $a = u$, then $b \in S$ and $S \setminus \{b\}$ is not a geodetic set in G. This case means that C^* is not a geodetic set in $K_p[G]$. On the other hand, if $a \neq u$, then $(a, b) = (w, z)$ and C^* is not a dominating set in $K_p[G]$. Therefore, C is a minimal geodetic dominating set in $K_p[G]$. ∎

Theorem 5.6. *Let G be a noncomplete connected graph and $n \geq 2$, and let $C \subseteq V(K_n[G])$. Then C is a minimal geodetic dominating set in $K_n[G]$ if and only if one of the following is true:*

(i) *$C = \{u\} \times S$ for some minimal geodetic dominating set in G and $u \in V(K_n)$;*

(ii) *$C = \{(w, z)\} \cup (\{u\} \times S)$ for some nondominating but minimal geodetic set S in G, for some $z \in V(G)$ and distinct $w, u \in V(K_n)$;*

(iii) *$C = \{(u_1, v_1), (u_1, v_2), (u_2, w_1), (u_2, w_2)\}$ for some distinct $u_1, u_2 \in V(K_n)$ and some $v_1, w_1, v_2, w_2 \in V(G)$ with $d_G(v_1, v_2) \geq 2$ and $d_G(w_1, w_2) \geq 2$.*

Proof. By Lemma 5.5, if property (i) or property (ii) holds, then C is a minimal geodetic dominating set in $K_n[G]$. It can also be readily verified that if property (ii) holds, then C is a minimal geodetic dominating set.

Let $C \subseteq V(K_n[G])$ be a minimal geodetic dominating set in $K_n[G]$. Then C contains distinct vertices (u, v) and (u, y). For if it were false and $(u, v) \in C$, then for all $y \in V(G) \setminus \{v\}$, $(u, y) \notin I_{K_n[G]}[C]$, a contradiction. Moreover, since G is noncomplete, we may choose v and y such that $d_G(v, y) \geq 2$. Suppose that $C = \{u\} \times S$ for some $S \subseteq V(G)$. Let $z \in V(G) \setminus S$. Since $(u, z) \in N_{K_n[G]}[C]$,

$z \in N_G[S]$. Similarly, $z \in I_G[S]$. Accordingly, S is a geodetic dominating set in G. Let $T \subseteq S$ be a geodetic dominating set in G. Then $\{u\} \times T \subseteq C$ and is a geodetic dominating set in $K_n[G]$ by Lemma 5.5. By the definition of C, $T = S$ and S is a minimal geodetic dominating set in G. This establishes property (i).

Now suppose that $C \neq \{u\} \times S$ for any $S \subseteq V(G)$. Let $S = \{t \in V(G) : (u,t) \in C\}$. Note that $(a,b) \in N_{K_n[G]}[(u,v),(u,y)] \cap I_{K_n[G]}[(u,v),(u,y)]$ for all $a \neq u$ and all $b \in V(G)$. Since C is a minimal geodetic dominating set in $K_n[G]$, $\{u\} \times S$ is not a geodetic dominating set in $K_p[G]$. Thus, $N_G[S] \neq V(G)$ or $I_G[S] \neq V(G)$. Suppose that $I_G[S] = V(G)$. Then S is not a dominating set in G. Let $b \in V(G) \setminus N_G[S]$. There exists $(w,z) \in C$ such that $(u,b)(w,z) \in E(K_p[G])$. Necessarily, $w \neq u$. Since $\{(w,z)\} \cup (\{u\} \times S)$ is a geodetic dominating set in $K_p[G]$, $C = \{(w,z)\} \cup (\{u\} \times S)$. In view of Lemma 5.5, S is a minimal geodetic set in G, and property (ii) is established. Finally suppose that $I_G[S] \neq V(G)$, and let $b \in V(G) \setminus I_G[S]$. Then there exists $w \in V(K_n)$ distinct from u and some $z,r \in V(G)$ with $d_G(z,r) \geq 2$ such that $(u,b) \in I_{K_n[G]}[(w,z),(w,r)]$. Since $\{(u,v),(u,y),(w,z),(w,r\} \subseteq C$ is a geodetic dominating set, $C = \{(u,v),(u,y),(w,z),(w,r)\}$. ■

Corollary 5.7. *Let G be a noncomplete connected graph with $g^+(G) < \gamma_g^+(G)$ and $n \geq 2$. Then*

$$\gamma_g^+(K_n[G]) = \max\{4, \gamma_g^+(G)\}.$$

6. γ_g^+-subgraph

A graph H is a γ_g^+*-subgraph* if there exists a connected graph G containing H as an induced subgraph such that $V(H)$ is a γ_g^+-set in G.

The idea of the following result is taken from [4].

Theorem 6.1. *Let H be a connected graph. Then H is a γ_g^+-subgraph if and only if either H is complete or H has no vertex v with $e_H(v) = 1$.*

Proof. Let there be a connected graph G containing H as an induced subgraph and such that $V(H)$ is a γ_g^+-set in G. Suppose that H is noncomplete and suppose that $v \in V(H)$ with $e_H(v) = 1$. We claim that $S = V(H) \setminus \{v\}$ is a geodetic dominating set in G. Let $w \in V(G) \setminus S$. Suppose that $w = v$. Since H is noncomplete, there exist $a,b \in V(H)$ such that $d_G(a,b) = d_H(a,b) = 2$. Necessarily, $a \neq v$ and $b \neq v$ so that $a,b \in S$. Since $av, bv \in E(H) \subseteq E(G)$, $w \in I_G[a,b] \subseteq I_G[S]$ and $w \in N_G[a] \subseteq N_G[S]$. Suppose that $w \neq v$. Since $V(H)$ is a geodetic set in G, there exist $a,b \in V(H)$ such that $w \in I_G[a,b]$. If $a = v$ or $b = v$, then $d_H(a,b) = d_G(a,b) = 1$, a contradiction. Thus $a,b \in S$. Since $av, bv \in E(G)$, $d_G(a,b) = 2$ and $aw, bw \in E(G)$. Hence $w \in I_G[S]$ and $w \in N_G[S]$. Thus, S is a geodetic dominating set in G. This is a contradiction since $V(H)$ is a minimal geodetic dominating set in G and S is a proper subset of $V(H)$.

By Theorem 2.2, if H is complete, then $V(H)$ is the γ_g^+-set in $G = H$. Suppose that H is noncomplete having no vertex u with $e_H(u) = 1$. For each $u \in V(H)$, choose $v \in V(H)$ such that $d_H(u,v) = 2$. Corresponding to each pair u and v, add to H the vertex $x_{u,v}$ and the edges $ux_{u,v}$ and $vx_{u,v}$. Let G be the resulting graph of minimum order obtained in this way. Then $|V(G) \setminus V(H)| \leq |V(H)|$. We claim that $V(H)$ is a γ_g^+-set in G. Let $x \in V(G) \setminus V(H)$. Then $x = x_{u,v}$ for some $u, v \in V(H)$ with $d_H(u,v) = d_G(u,v) = 2$. More precisely, $xu, xv \in E(G)$. Thus, $x \in I_G[u,v]$ and $x \in N_G[u]$. In other words, $V(H)$ is a geodetic dominating set in G.

Let $u \in V(H)$, and let $v \in V(H)$ such that $d_H(u,v) = 2$. Corresponding to u and v is a $x_{u,v} \in V(G) \setminus V(H)$. By its construction, $x_{u,v} \notin I_G[V(H) \setminus \{u\}]$. Since u is arbitrary, $V(H)$ is a minimal geodetic dominating set in G. Finally, let $S \subseteq V(G)$ be a minimal geodetic dominating set in G. For the triple u, v, $x_{u,v}$, if $u, v \in S$, then $x_{u,v} \notin S$, or, equivalently, if $x_{u,v} \in S$, then $u \notin S$ or $v \notin S$. Thus

$$|S| = |S \setminus V(H)| + |V(H) \cap S| \leq |V(H) \setminus S| + |V(H) \cap S| = |V(H)|.$$

Since S is arbitrary, $V(H)$ is a γ_g^+-set in G. ■

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