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THE MEDIAN PROBLEM ON k-PARTITE GRAPHS

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Abstract

In a connected graph G, the status of a vertex is the sum of the distances of that vertex to each of the other vertices in G. The subgraph induced by the vertices of minimum (maximum) status in G is called the median (anti-median) of G. The median problem of graphs is closely related to the optimization problems involving the placement of network servers, the core of the entire networks. Bipartite graphs play a significant role in designing very large interconnection networks. In this paper, we answer a problem on the structure of medians of bipartite graphs by showing that any bipartite graph is the median (or anti-median) of another bipartite graph. Also, with a different construction, we show that the similar results hold for k-partite graphs, $k \geq 3$. In addition, we provide constructions to embed another graph as center in both bipartite and k-partite cases. Since any graph is a k-partite graph, for some k, these constructions can be applied in general.

Keywords: networks, distance, median, bipartite, k-partite.

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1. Introduction

Let $G = (V, E)$ be a graph on n vertices with vertex set V and edge set E. A graph is bipartite if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y, and a graph is k-partite if its vertex set can be partitioned into k nonempty subsets such that no edge in G has its both ends in the same subset. Degree of a vertex v, $d(v)$, is the number vertices adjacent to v and by $N(v)$ we denote the neighbor set of v. The smallest and largest degrees of vertices in G are respectively denoted by $\delta(G)$ and $\Delta(G)$.

The distance between two vertices u and v is the number of edges on a shortest path between u and v, and it is denoted by $d(u, v)$. The eccentricity of u is $e(u) = \max_v d(u, v)$. The center $C(G)$ of a graph G is the subgraph of G induced by the vertices of minimum eccentricity. The status of a vertex $v \in V(G)$, denoted by $S_G(v)$, is the sum of the distances from v to all other vertices in G. The subgraph induced by the vertices of minimum (maximum) status in G is known as the median (anti-median) of G, denoted by $M(G)$ $(AM(G))$. The status difference [5] in a graph G is $\mathrm{SD}(G) = \max_{u,v \in V(G)} (S_G(u) - S_G(v))$.

Given a graph G the problem of finding a graph H such that $M(H) \cong G$ is referred to as the median problem. In [6], it is shown that any graph $G = (V, E)$ is the median of some connected graph. In [3] the notion of anti-median of a graph was introduced and proved that every graph is the anti-median graph of some graph. The problem of simultaneous embedding of median and anti-median is discussed in [1]. Another construction, which generalises all the previously mentioned constructions, can be found in [5].

The median vertices have the minimum average distance in a graph and thus the median problem is significant among the optimization problems involving the placement of network servers. However, the median constructions for general graphs cannot be directly applied to many networks as their underlying graph belong to different classes of graphs. It can be seen that the underlying graphs of many networks are bipartite. For example, most of the analysis in network communities are done using preference networks [4] and they are modelled using bipartite graphs.

It is well known that the median of a tree is a vertex or an edge. This operator was also studied for some classes of graphs in [7] and [8]. In this paper we show that any bipartite graph is the median of another bipartite graph. With a different construction, we show that the similar result also hold for k-partite graphs. The analogous results for anti-median problem on these graph classes are also obtained. Since any graph is a k-partite graph, for some k, these constructions can be applied in general. For all other basic concepts and notations not mentioned in this paper we refer to [2].

2. Bipartite Graphs with Prescribed Median and Anti-median

Lemma 1. *Given a bipartite graph G of n vertices, there exists a connected bipartite graph H' such that G is an induced subgraph of H' and all the vertices of G in H' have equal status in H'.*

Proof. Let X, Y be a bipartition of $V(G)$ and X', Y' be the copy of X, Y such that v' denote the copy of a vertex $v \in V(G)$. Consider two new vertices v_x and v_y. Make v_y adjacent to all vertices of $X \cup X'$ and v_x adjacent to all vertices of $Y \cup Y'$. Also, for each $v \in X$ ($v \in Y$) make v' adjacent to $Y \backslash N(v)$ ($X \backslash N(v)$). It follows that H' is bipartite and $S_{H'}(v) = 4n + 1$, for all $v \in V(G)$. ■

The graph H' is called the bipartite gadget graph of G. Let $|X| = n_1$ and $|Y| = n_2$. Then we have, in H', $S_{H'}(v_x) = 4n + 1 - (2n_1 - 2)$, $S_{H'}(v_y) = 4n + 1 - (2n_2 - 2)$ and $4n + 1 \leq S_{H'}(v') \leq 4n + 1 + 2\Delta(G) + 2 + 2\max(n_1, n_2)$, for each $v \in V(G)$.

Theorem 2. *Given a bipartite graph G there exists a bipartite graph H such that $M(H) \cong G$.*

Proof. The proof is by construction. Let H' be the bipartite gadget graph of the graph G. Choose a positive integer $s > \max(n_1, n_2) - 1$. Introduce s copies of K_2 and make one end of each K_2 adjacent to all the vertices of X and the other end to all the vertices of Y. Denote this graph by H. Then for each vertex $v \in V(G)$, $S_H(v) = 4n + 1 + 3s$. Also, for each $v \in V(H' \backslash G)$ the status is increased by $5s$. Let x be an arbitrary vertex from the newly added s copies of K_2. It easy to verify that $S_H(x) = 4n + 1 + 5s$. Hence $S_H(v) < S_H(u)$, for all $v \in V(G)$, for all $u \in V(H \backslash G)$, thus $M(H) \cong G$. ■

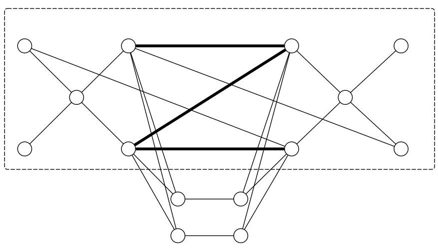

Figure 1. A graph with P_4 as the median. Here, the subgraph in the dotted box is the bipartite gadget graph of P_4.

Theorem 3. *Given a bipartite graph G there exists a bipartite graph H such that $AM(H) \cong G$*

Proof. The proof is by construction. Let H' be the bipartite gadget graph of the graph G. Consider the complete bipartite graph $K_{s,s}$, where $s > \max(n_1, n_2) + \Delta(G)+1$. Make the s vertices in one partition of $K_{s,s}$ adjacent to $v_y \cup Y'$ and the other s vertices to $v_x \cup X'$. Denote this graph by H. Then $S_H(v) = 4n + 1 + 5s$ for all the vertices in the subgraph G of H and for each other vertex in the subgraph H' of H, the status is increased by $3s$. For any vertex x in $K_{s,s}$, $S_H(x) = 4n+1+3s$. Thus $S_H(v) > S_H(u)$, for all $v \in V(G)$, for all $u \in V(H\backslash G)$, hence $AM(H) \cong G$. ■

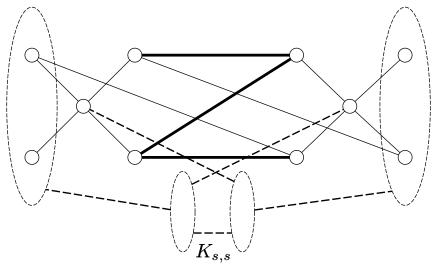

Figure 2. A graph with P_4 as the anti-median. Here the dotted circles represent a set of vertices and the dotted lines represent all possible edges between its two ends.

Remark 4. The number of vertices used in both constructions in Theorems 2 and 3 is $2(n + s + 1)$, where the value of s depends on the corresponding construction rules.

3. k-partite Graphs with Prescribed Median and Anti-median

In the following section we assume that $k \geq 3$.

Theorem 5. *Given a k-partite graph G there exists a k-partite graph H such that $M(H) \cong G$.*

Proof. The proof is by construction. Consider two functions f and g defined on an index set $I = \{1, 2, \ldots, k\}$ as

$$f(i) = \begin{cases} 1, & \text{if } i = k, \\ i+1, & \text{if } i \neq k, \end{cases} \quad \text{and } g(i) = \begin{cases} k, & \text{if } i = 1, \\ i-1, & \text{if } i \neq 1. \end{cases}$$

Let $\{X_i\}_{i \in I}$ be a partition of $V(G)$ with $|X_i| = n_i$. For each vertex $v \in X_i$, introduce three vertices $v_1 \in X_{g(i)}$, $v_2 \in X_{f(i)}$ and $v_3 \in X_i$ such that v_1 and v_2 are adjacent to both v and v_3. We refer v_1 and v_2 as the ortho vertices of v, and v_3 as the para vertex of v. Denote this graph as the k-partite gadget graph of G.

Make v_1 adjacent to $X_i \cup X_{f(i)} \setminus N_{X_{f(i)}}(v)$, v_2 adjacent to $X_i \cup \bigcup_{j=f(i)+1}^{g(i)} (X_j \setminus N_{X_j}(v))$ and v_3 adjacent to $\bigcup_{j\neq i} X_j$. Denote this graph by H.

Consider a vertex v in X_1. Then, $S_H(v) = 6\sum_{i=2}^{k} n_i + 4n_1 + 2(n_1-1) = 6n-2$. Hence $S_H(v) = 6n-2$, for all $v \in V(G)$.

For each vertex $v \in V(G)$ we get $7n + d_{X_2}(v) + 2\sum_3^k n_i \leq S_H(v_1) \leq 7n + 3d_{X_2}(v) + 3\sum_3^k n_i$, $7n-3+n_2+d(v)-d_{X_2}(v) \leq S_H(v_2) \leq 7n-3+3n_2+3d(v)-3d_{X_2}(v)$ and $7n-2-\max_i(n_i) \leq S_H(v_3) \leq 8n-4+\min_i(n_i)$. Hence $M(H) \cong G$. ■

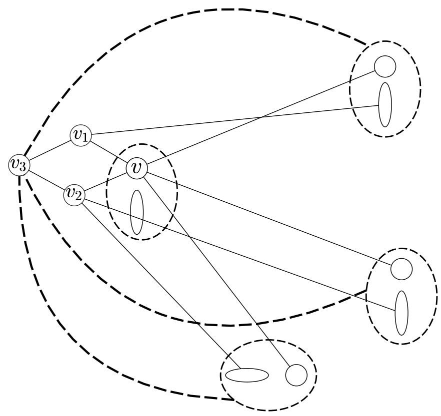

Figure 3. Construction in Theorem 5. Here the dotted circles represent a set of vertices and the dotted lines represent all possible edges between its two ends.

Theorem 6. *Given a k-partite graph G there exists a k-partite graph H' such that $AM(H') \cong G$*

Proof. The proof is by construction. Let H be the graph obtained using the construction in Theorem 5. Consider a complete k-partite graph $K_{r,r,\ldots,r}$, where $r > \frac{2n+1}{k}$ and let $\{Y_i\}_{i\in I}$ be its k-partition. For each vertex $v \in X_i$ make v_3 adjacent to $\bigcup_{j\neq i} Y_j$, v_1 adjacent to $\bigcup_{j\neq f(i)} Y_j$ and v_2 adjacent to $\bigcup_{j\neq g(i)} Y_j$. In the new graph H', $S_{H'}(v) = S_H(v) + 2kr$, for all $v \in V(G)$ and $S_{H'}(v_s) = S_H(v_s) + (k+1)r$, for $s = 1,2,3$ and hence $AM(H') \cong G$. ■

4. EMBEDDING CENTER WITH MEDIAN CONSTRUCTIONS

The constructions of a graph with prescribed median naturally faces the following problem. The addition of a vertex in any part of the graph changes the status of

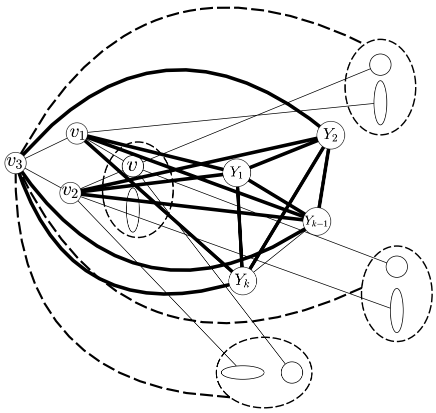

Figure 4. Construction in Theorem 6. Here the shaded graph in the background is the graph in Figure 3.

each vertex in that graph, thus changing the median preferences in that graph. In this section we embed another k-partite graph as the center of the newly constructed graph keeping the median same in the graphs, which are obtained using previous theorems.

Theorem 7. *Given two bipartite graphs G and J there exists a bipartite graph H with $M(H) \cong G$ and $C(H) \cong J$.*

Proof. The proof is by construction. Let H' be the bipartite gadget graph of G. For $k \geq 3$ introduce two paths $x_1, x_2, \ldots, x_{k-1}$ and $y_1, y_2, \ldots, y_{k-1}$ of length $k-2$. Let $u_1, u_2, \ldots, u_{k+1}$ and $v_1, v_2, \ldots, v_{k+1}$ be two paths of length k. Let R and S be the bipartition of J such that $|R| \leq |S|$. Make x_1 adjacent to $X \cup \{v_x, y_1\}$, y_1 to $Y \cup \{v_y\}$, x_{k-1} to $R \cup \{y_{k-1}\}$, y_{k-1} to S, u_1 to $R \cup \{v_1\}$, v_1 to S and u_{k+1} to v_{k+1}. Attach $|S| - |R| + 1$ vertices to x_1 and a vertex w to y_1. Denote this graph by H_0. Introduce s copies of K_2, where $s > \mathrm{SD}(H_0)/2$, and make them adjacent to X and Y of G, as in Theorem 2. Denote this new graph by H. Clearly $C(H) \cong J$ with $e(v) = k + 2$, for all $v \in V(J)$.

$S_H(x) = S_H(y) = 4(n + k^2) + k(|R| + |S| + 6) + 3|S| - 2|R| + 3s + 8$, for all $x \in X$, $y \in Y$. For a vertex $v \in V(H)$, let $S^*(u) = d(u, v') + d(u, v'')$, where v' and v'' are the end vertices of a K_2 among the s copies of K_2 in H. $S^*(u) = 3$, when $u \in V(G)$ and $S^*(u) \geq 5$, when $u \in V(H \backslash G) \backslash \{v', v''\}$. Hence $M(H) = G$, when $s > \mathrm{SD}(H_0)/2$. ∎

Theorem 8. *Given two k-partite graphs G and J there exists a k-partite graph W such that $M(W) \cong G$ and $C(W) \cong J$.*

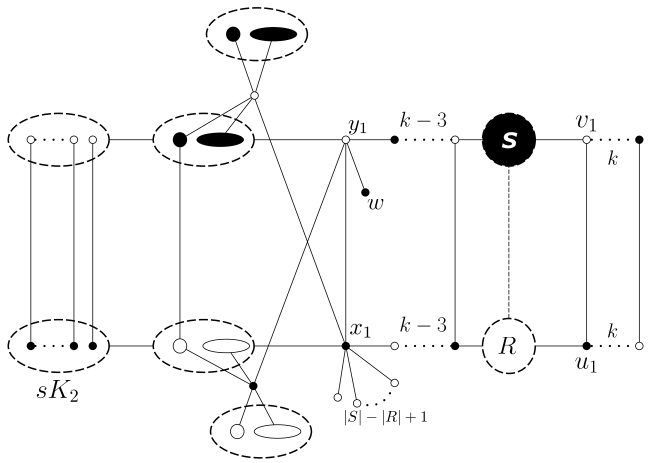

Figure 5. Construction in Theorem 7. Here the white-black coloring illustrates the bipartition of the graph. The dotted circles represent a set of vertices and a line between them represent all possible edges between its two ends.

Proof. The proof is by construction. Let H be the graph obtained from graph G as in Theorem 5. Introduce k paths P_{x_i,y_i} of length $r-2$ with end vertices x_i and y_i, where $i \in I$. A vertex in P_{x_i,y_i}, at distance t from x_i, is denoted by $P_{x_i,y_i}[t]$. For each $t = 0, \ldots, r-3$, make the vertices $P_{x_i,y_i}[t]$, for all i, adjacent so that they induce a complete graph. Similarly introduce k paths $R_{x'_i,y'_i}$ of length r with end vertices x'_i and y'_i and make adjacencies $R_{x'_i,y'_i}[t]$ for each t and every i.

Let $\{Y_i\}_{i\in I}$ be the k-partition of the graph J and let J' be the k-partite gadget graph of J. Let $P(Y_i)$ and $O(Y_i)$ be respectively the sets of para vertices and ortho vertices of Y_i. For each $i, j \in I$ make x_i adjacent to X_i, y_i adjacent to $Y_i \cup P(Y_i) \bigcup_{j\neq i} O(Y_j)$ and x'_i adjacent to Y_i. Denote this graph by W_0. Introduce s copies of K_k, where $s >$SD$(W_0)/2$, and let $\{Y'_i\}_{i\in I}$ denote their k-partition. For each $i \in I$, make all the vertices of Y'_i adjacent to X_i. Denote this graph by W. It can be verified that $C(W) \cong J$, with $e(v) = r+1$, for all $v \in V(J)$ and $S_H(v) = 6n + |J|(4r-2) + k(2r^2 - r - 1) + s(2k-1)$, for all $v \in V(G)$. Let $S^*(u) = \sum_{v\in K} d(u,v)$, where K is one of the s copies of K_k. We can see that $S^*(u) = 2k-1$ when $u \in V(G)$ and $S^*(u) \geq 2k+1$ when $u \in V(W\backslash G\backslash K)$. Hence $\mathrm{M}(W) \cong G$. ∎

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