

## THE MEDIAN PROBLEM ON $k$ -PARTITE GRAPHS

KARUVACHERY PRAVAS

*Government Polytechnic College*  
*Koratty-680 308*  
*India*

**e-mail:** pravask@gmail.com

AND

AMBAT VIJAYAKUMAR

*Cochin University of Science and Technology*  
*Cochin-682022*  
*India*

**e-mail:** vambat@gmail.com

### Abstract

In a connected graph  $G$ , the status of a vertex is the sum of the distances of that vertex to each of the other vertices in  $G$ . The subgraph induced by the vertices of minimum (maximum) status in  $G$  is called the median (anti-median) of  $G$ . The median problem of graphs is closely related to the optimization problems involving the placement of network servers, the core of the entire networks. Bipartite graphs play a significant role in designing very large interconnection networks. In this paper, we answer a problem on the structure of medians of bipartite graphs by showing that any bipartite graph is the median (or anti-median) of another bipartite graph. Also, with a different construction, we show that the similar results hold for  $k$ -partite graphs,  $k \geq 3$ . In addition, we provide constructions to embed another graph as center in both bipartite and  $k$ -partite cases. Since any graph is a  $k$ -partite graph, for some  $k$ , these constructions can be applied in general.

**Keywords:** networks, distance, median, bipartite,  $k$ -partite.

**2010 Mathematics Subject Classification:** 05C12.

## 1. INTRODUCTION

Let  $G = (V, E)$  be a graph on  $n$  vertices with vertex set  $V$  and edge set  $E$ . A graph is bipartite if its vertex set can be partitioned into two nonempty subsets  $X$  and  $Y$  such that each edge of  $G$  has one end in  $X$  and the other in  $Y$ , and a graph is  $k$ -partite if its vertex set can be partitioned into  $k$  nonempty subsets such that no edge in  $G$  has its both ends in the same subset. Degree of a vertex  $v$ ,  $d(v)$ , is the number vertices adjacent to  $v$  and by  $N(v)$  we denote the neighbor set of  $v$ . The smallest and largest degrees of vertices in  $G$  are respectively denoted by  $\delta(G)$  and  $\Delta(G)$ .

The distance between two vertices  $u$  and  $v$  is the number of edges on a shortest path between  $u$  and  $v$ , and it is denoted by  $d(u, v)$ . The eccentricity of  $u$  is  $e(u) = \max_v d(u, v)$ . The center  $C(G)$  of a graph  $G$  is the subgraph of  $G$  induced by the vertices of minimum eccentricity. The status of a vertex  $v \in V(G)$ , denoted by  $S_G(v)$ , is the sum of the distances from  $v$  to all other vertices in  $G$ . The subgraph induced by the vertices of minimum (maximum) status in  $G$  is known as the median (anti-median) of  $G$ , denoted by  $M(G)$  ( $AM(G)$ ). The status difference [5] in a graph  $G$  is  $SD(G) = \max_{u,v \in V(G)} (S_G(u) - S_G(v))$ .

Given a graph  $G$  the problem of finding a graph  $H$  such that  $M(H) \cong G$  is referred to as the median problem. In [6], it is shown that any graph  $G = (V, E)$  is the median of some connected graph. In [3] the notion of anti-median of a graph was introduced and proved that every graph is the anti-median graph of some graph. The problem of simultaneous embedding of median and anti-median is discussed in [1]. Another construction, which generalises all the previously mentioned constructions, can be found in [5].

The median vertices have the minimum average distance in a graph and thus the median problem is significant among the optimization problems involving the placement of network servers. However, the median constructions for general graphs cannot be directly applied to many networks as their underlying graph belong to different classes of graphs. It can be seen that the underlying graphs of many networks are bipartite. For example, most of the analysis in network communities are done using preference networks [4] and they are modelled using bipartite graphs.

It is well known that the median of a tree is a vertex or an edge. This operator was also studied for some classes of graphs in [7] and [8]. In this paper we show that any bipartite graph is the median of another bipartite graph. With a different construction, we show that the similar result also hold for  $k$ -partite graphs. The analogous results for anti-median problem on these graph classes are also obtained. Since any graph is a  $k$ -partite graph, for some  $k$ , these constructions can be applied in general. For all other basic concepts and notations not mentioned in this paper we refer to [2].

## 2. BIPARTITE GRAPHS WITH PRESCRIBED MEDIAN AND ANTI-MEDIAN

**Lemma 1.** *Given a bipartite graph  $G$  of  $n$  vertices, there exists a connected bipartite graph  $H'$  such that  $G$  is an induced subgraph of  $H'$  and all the vertices of  $G$  in  $H'$  have equal status in  $H'$ .*

**Proof.** Let  $X, Y$  be a bipartition of  $V(G)$  and  $X', Y'$  be the copy of  $X, Y$  such that  $v'$  denote the copy of a vertex  $v \in V(G)$ . Consider two new vertices  $v_x$  and  $v_y$ . Make  $v_y$  adjacent to all vertices of  $X \cup X'$  and  $v_x$  adjacent to all vertices of  $Y \cup Y'$ . Also, for each  $v \in X$  ( $v \in Y$ ) make  $v'$  adjacent to  $Y \setminus N(v)$  ( $X \setminus N(v)$ ). It follows that  $H'$  is bipartite and  $S_{H'}(v) = 4n + 1$ , for all  $v \in V(G)$ . ■

The graph  $H'$  is called the bipartite gadget graph of  $G$ . Let  $|X| = n_1$  and  $|Y| = n_2$ . Then we have, in  $H'$ ,  $S_{H'}(v_x) = 4n + 1 - (2n_1 - 2)$ ,  $S_{H'}(v_y) = 4n + 1 - (2n_2 - 2)$  and  $4n + 1 \leq S_{H'}(v') \leq 4n + 1 + 2\Delta(G) + 2 + 2\max(n_1, n_2)$ , for each  $v \in V(G)$ .

**Theorem 2.** *Given a bipartite graph  $G$  there exists a bipartite graph  $H$  such that  $M(H) \cong G$ .*

**Proof.** The proof is by construction. Let  $H'$  be the bipartite gadget graph of the graph  $G$ . Choose a positive integer  $s > \max(n_1, n_2) - 1$ . Introduce  $s$  copies of  $K_2$  and make one end of each  $K_2$  adjacent to all the vertices of  $X$  and the other end to all the vertices of  $Y$ . Denote this graph by  $H$ . Then for each vertex  $v \in V(G)$ ,  $S_H(v) = 4n + 1 + 3s$ . Also, for each  $v \in V(H' \setminus G)$  the status is increased by  $5s$ . Let  $x$  be an arbitrary vertex from the newly added  $s$  copies of  $K_2$ . It easy to verify that  $S_H(x) = 4n + 1 + 5s$ . Hence  $S_H(v) < S_H(u)$ , for all  $v \in V(G)$ , for all  $u \in V(H \setminus G)$ , thus  $M(H) \cong G$ . ■

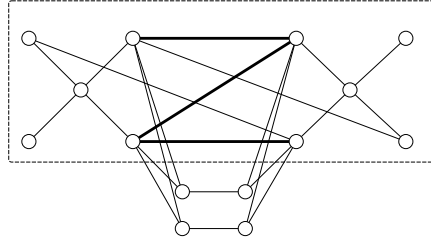


Figure 1. A graph with  $P_4$  as the median. Here, the subgraph in the dotted box is the bipartite gadget graph of  $P_4$ .

**Theorem 3.** *Given a bipartite graph  $G$  there exists a bipartite graph  $H$  such that  $AM(H) \cong G$*

**Proof.** The proof is by construction. Let  $H'$  be the bipartite gadget graph of the graph  $G$ . Consider the complete bipartite graph  $K_{s,s}$ , where  $s > \max(n_1, n_2) + \Delta(G) + 1$ . Make the  $s$  vertices in one partition of  $K_{s,s}$  adjacent to  $v_y \cup Y'$  and the other  $s$  vertices to  $v_x \cup X'$ . Denote this graph by  $H$ . Then  $S_H(v) = 4n + 1 + 5s$  for all the vertices in the subgraph  $G$  of  $H$  and for each other vertex in the subgraph  $H'$  of  $H$ , the status is increased by  $3s$ . For any vertex  $x$  in  $K_{s,s}$ ,  $S_H(x) = 4n + 1 + 3s$ . Thus  $S_H(v) > S_H(u)$ , for all  $v \in V(G)$ , for all  $u \in V(H \setminus G)$ , hence  $AM(H) \cong G$ . ■

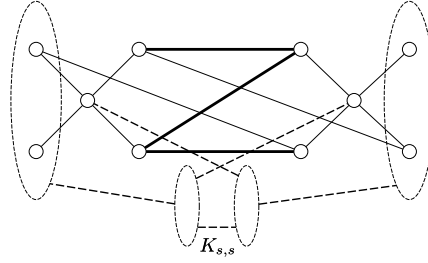


Figure 2. A graph with  $P_4$  as the anti-median. Here the dotted circles represent a set of vertices and the dotted lines represent all possible edges between its two ends.

**Remark 4.** The number of vertices used in both constructions in Theorems 2 and 3 is  $2(n + s + 1)$ , where the value of  $s$  depends on the corresponding construction rules.

### 3. $k$ -PARTITE GRAPHS WITH PRESCRIBED MEDIAN AND ANTI-MEDIAN

In the following section we assume that  $k \geq 3$ .

**Theorem 5.** Given a  $k$ -partite graph  $G$  there exists a  $k$ -partite graph  $H$  such that  $M(H) \cong G$ .

**Proof.** The proof is by construction. Consider two functions  $f$  and  $g$  defined on an index set  $I = \{1, 2, \dots, k\}$  as

$$f(i) = \begin{cases} 1, & \text{if } i = k, \\ i + 1, & \text{if } i \neq k, \end{cases} \quad \text{and } g(i) = \begin{cases} k, & \text{if } i = 1, \\ i - 1, & \text{if } i \neq 1. \end{cases}$$

Let  $\{X_i\}_{i \in I}$  be a partition of  $V(G)$  with  $|X_i| = n_i$ . For each vertex  $v \in X_i$ , introduce three vertices  $v_1 \in X_{g(i)}$ ,  $v_2 \in X_{f(i)}$  and  $v_3 \in X_i$  such that  $v_1$  and  $v_2$  are adjacent to both  $v$  and  $v_3$ . We refer  $v_1$  and  $v_2$  as the ortho vertices of  $v$ , and  $v_3$  as the para vertex of  $v$ . Denote this graph as the  $k$ -partite gadget graph of  $G$ .

Make  $v_1$  adjacent to  $X_i \cup X_{f(i)} \setminus N_{X_{f(i)}}(v)$ ,  $v_2$  adjacent to  $X_i \cup \bigcup_{j=f(i)+1}^{g(i)} (X_j \setminus N_{X_j}(v))$  and  $v_3$  adjacent to  $\bigcup_{j \neq i} X_j$ . Denote this graph by  $H$ .

Consider a vertex  $v$  in  $X_1$ . Then,  $S_H(v) = 6 \sum_{i=2}^k n_i + 4n_1 + 2(n_1 - 1) = 6n - 2$ . Hence  $S_H(v) = 6n - 2$ , for all  $v \in V(G)$ .

For each vertex  $v \in V(G)$  we get  $7n + d_{X_2}(v) + 2 \sum_3^k n_i \leq S_H(v_1) \leq 7n + 3d_{X_2}(v) + 3 \sum_3^k n_i$ ,  $7n - 3 + n_2 + d(v) - d_{X_2}(v) \leq S_H(v_2) \leq 7n - 3 + 3n_2 + 3d(v) - 3d_{X_2}(v)$  and  $7n - 2 - \max_i(n_i) \leq S_H(v_3) \leq 8n - 4 + \min_i(n_i)$ . Hence  $M(H) \cong G$ . ■

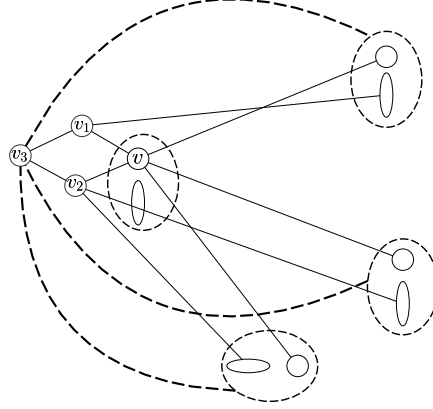


Figure 3. Construction in Theorem 5. Here the dotted circles represent a set of vertices and the dotted lines represent all possible edges between its two ends.

**Theorem 6.** *Given a  $k$ -partite graph  $G$  there exists a  $k$ -partite graph  $H'$  such that  $AM(H') \cong G$*

**Proof.** The proof is by construction. Let  $H$  be the graph obtained using the construction in Theorem 5. Consider a complete  $k$ -partite graph  $K_{r,r,\dots,r}$ , where  $r > \frac{2n+1}{k}$  and let  $\{Y_i\}_{i \in I}$  be its  $k$ -partition. For each vertex  $v \in X_i$  make  $v_3$  adjacent to  $\bigcup_{j \neq i} Y_j$ ,  $v_1$  adjacent to  $\bigcup_{j \neq f(i)} Y_j$  and  $v_2$  adjacent to  $\bigcup_{j \neq g(i)} Y_j$ . In the new graph  $H'$ ,  $S_{H'}(v) = S_H(v) + 2kr$ , for all  $v \in V(G)$  and  $S_{H'}(v_s) = S_H(v_s) + (k+1)r$ , for  $s = 1, 2, 3$  and hence  $AM(H') \cong G$ . ■

#### 4. EMBEDDING CENTER WITH MEDIAN CONSTRUCTIONS

The constructions of a graph with prescribed median naturally faces the following problem. The addition of a vertex in any part of the graph changes the status of

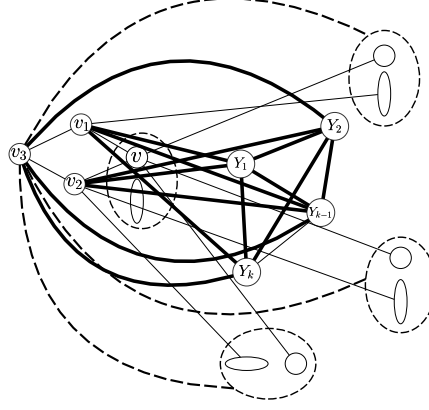


Figure 4. Construction in Theorem 6. Here the shaded graph in the background is the graph in Figure 3.

each vertex in that graph, thus changing the median preferences in that graph. In this section we embed another  $k$ -partite graph as the center of the newly constructed graph keeping the median same in the graphs, which are obtained using previous theorems.

**Theorem 7.** *Given two bipartite graphs  $G$  and  $J$  there exists a bipartite graph  $H$  with  $M(H) \cong G$  and  $C(H) \cong J$ .*

**Proof.** The proof is by construction. Let  $H'$  be the bipartite gadget graph of  $G$ . For  $k \geq 3$  introduce two paths  $x_1, x_2, \dots, x_{k-1}$  and  $y_1, y_2, \dots, y_{k-1}$  of length  $k-2$ . Let  $u_1, u_2, \dots, u_{k+1}$  and  $v_1, v_2, \dots, v_{k+1}$  be two paths of length  $k$ . Let  $R$  and  $S$  be the bipartition of  $J$  such that  $|R| \leq |S|$ . Make  $x_1$  adjacent to  $X \cup \{v_x, y_1\}$ ,  $y_1$  to  $Y \cup \{v_y\}$ ,  $x_{k-1}$  to  $R \cup \{y_{k-1}\}$ ,  $y_{k-1}$  to  $S$ ,  $u_1$  to  $R \cup \{v_1\}$ ,  $v_1$  to  $S$  and  $u_{k+1}$  to  $v_{k+1}$ . Attach  $|S| - |R| + 1$  vertices to  $x_1$  and a vertex  $w$  to  $y_1$ . Denote this graph by  $H_0$ . Introduce  $s$  copies of  $K_2$ , where  $s > \text{SD}(H_0)/2$ , and make them adjacent to  $X$  and  $Y$  of  $G$ , as in Theorem 2. Denote this new graph by  $H$ . Clearly  $C(H) \cong J$  with  $e(v) = k + 2$ , for all  $v \in V(J)$ .

$S_H(x) = S_H(y) = 4(n + k^2) + k(|R| + |S| + 6) + 3|S| - 2|R| + 3s + 8$ , for all  $x \in X, y \in Y$ . For a vertex  $v \in V(H)$ , let  $S^*(u) = d(u, v') + d(u, v'')$ , where  $v'$  and  $v''$  are the end vertices of a  $K_2$  among the  $s$  copies of  $K_2$  in  $H$ .  $S^*(u) = 3$ , when  $u \in V(G)$  and  $S^*(u) \geq 5$ , when  $u \in V(H \setminus G) \setminus \{v', v''\}$ . Hence  $M(H) = G$ , when  $s > \text{SD}(H_0)/2$ . ■

**Theorem 8.** *Given two  $k$ -partite graphs  $G$  and  $J$  there exists a  $k$ -partite graph  $W$  such that  $M(W) \cong G$  and  $C(W) \cong J$ .*

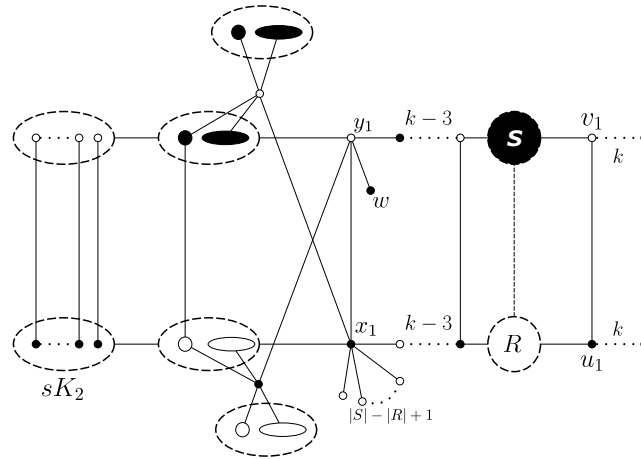


Figure 5. Construction in Theorem 7. Here the white-black coloring illustrates the bipartition of the graph. The dotted circles represent a set of vertices and a line between them represent all possible edges between its two ends.

**Proof.** The proof is by construction. Let  $H$  be the graph obtained from graph  $G$  as in Theorem 5. Introduce  $k$  paths  $P_{x_i, y_i}$  of length  $r - 2$  with end vertices  $x_i$  and  $y_i$ , where  $i \in I$ . A vertex in  $P_{x_i, y_i}$ , at distance  $t$  from  $x_i$ , is denoted by  $P_{x_i, y_i}[t]$ . For each  $t = 0, \dots, r - 3$ , make the vertices  $P_{x_i, y_i}[t]$ , for all  $i$ , adjacent so that they induce a complete graph. Similarly introduce  $k$  paths  $R_{x'_i, y'_i}$  of length  $r$  with end vertices  $x'_i$  and  $y'_i$  and make adjacencies  $R_{x'_i, y'_i}[t]$  for each  $t$  and every  $i$ .

Let  $\{Y_i\}_{i \in I}$  be the  $k$ -partition of the graph  $J$  and let  $J'$  be the  $k$ -partite gadget graph of  $J$ . Let  $P(Y_i)$  and  $O(Y_i)$  be respectively the sets of para vertices and ortho vertices of  $Y_i$ . For each  $i, j \in I$  make  $x_i$  adjacent to  $X_i$ ,  $y_i$  adjacent to  $Y_i \cup P(Y_i) \cup \bigcup_{j \neq i} O(Y_j)$  and  $x'_i$  adjacent to  $Y_i$ . Denote this graph by  $W_0$ . Introduce  $s$  copies of  $K_k$ , where  $s > \text{SD}(W_0)/2$ , and let  $\{Y'_i\}_{i \in I}$  denote their  $k$ -partition. For each  $i \in I$ , make all the vertices of  $Y'_i$  adjacent to  $X_i$ . Denote this graph by  $W$ . It can be verified that  $C(W) \cong J$ , with  $e(v) = r + 1$ , for all  $v \in V(J)$  and  $S_H(v) = 6n + |J|(4r - 2) + k(2r^2 - r - 1) + s(2k - 1)$ , for all  $v \in V(G)$ . Let  $S^*(u) = \sum_{v \in K} d(u, v)$ , where  $K$  is one of the  $s$  copies of  $K_k$ . We can see that  $S^*(u) = 2k - 1$  when  $u \in V(G)$  and  $S^*(u) \geq 2k + 1$  when  $u \in V(W \setminus G \setminus K)$ . Hence  $M(W) \cong G$ . ■

## REFERENCES

- [1] K. Balakrishnan, B. Brešar, M. Kovše, M. Changat, A.R. Subhamathi and S. Klavžar, *Simultaneous embeddings of graphs as median and antimedial*

- subgraphs*, Networks **56** (2010) 90–94.  
doi:10.002/net.20350
- [2] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Second Edition (Heidelberg, Springer, 2012).  
doi:10.1007/978-1-4614-4529-6
- [3] H. Bielak and M.M. Sysło, *Peripheral vertices in graphs*, Studia Sci. Math. Hungar. **18** (1983) 269–275.
- [4] H. Kautz, B. Selman and M. Shah, *Referral Web: combining social networks and collaborative filtering*, Communications of the ACM **40(3)** (1997) 63–65.  
doi:10.1145/245108.245123
- [5] K. Pravas and A. Vijayakumar, *Convex median and anti-median at pre-scribed distance*, communicated.
- [6] P.J. Slater, *Medians of arbitrary graphs*, J. Graph Theory **4** (1980) 389–392.  
doi:10.1002/jgt.3190040408
- [7] S.B. Rao and A. Vijayakumar, *On the median and the anti-median of a co-graph*, Int. J. Pure Appl. Math. **46** (2008) 703–710.
- [8] H.G. Yeh and G.J. Chang, *Centers and medians of distance-hereditary graphs*, Discrete Math. **265** (2003) 297–310.  
doi:10.1016/S0012-365X(02)00630-1

Received 24 February 2014

Revised 20 June 2014

Accepted 15 August 2014