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A NOTE ON LONGEST PATHS IN CIRCULAR ARC GRAPHS

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Abstract

As observed by Rautenbach and Sereni [SIAM J. Discrete Math. **28** (2014) 335–341] there is a gap in the proof of the theorem of Balister *et al.* [Combin. Probab. Comput. **13** (2004) 311–317], which states that the intersection of all longest paths in a connected circular arc graph is nonempty. In this paper we close this gap.

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1. Introduction

It is easy to prove that every two longest paths in a connected graph have a nonempty intersection. Gallai [2] asked if the intersection of all longest paths is nonempty. This is not true in general but holds for some graph classes. See [5] for a survey. In [1] Balister *et al.* proved that it is true for interval graphs and circular arc graphs. However, as pointed out by Rautenbach and Sereni [4], there is a gap in the proof for the class of circular arc graphs. The gab stems from being able to reorder a longest path such that certain symmetric properties hold at the beginning and the end of the path. While the properties are symmetric, Balister *et al.* did not prove that they can hold for the same path reordering. Rautenbach and Sereni proved the weaker result that in a connected circular arc graph, there is a set of at most three vertices such that every longest path intersects this set. In Lemma 3 we close their gap by extending Lemma 3.2 from [1].

We follow the notation in [1]. A graph G is a *circular arc graph* if there exists a function ϕ of its vertex set $V(G)$ into a collection of open arcs of a circle such that, for every two distinct vertices u and w of G, uw is an edge of G if and only if $\phi(u) \cap \phi(w) \neq \emptyset$, that is, the class of circular arc graphs are the intersection graphs of arcs in a circle. Let *interval graphs* be the intersection graphs of open intervals of the real line. Note that one can assume that all endpoints of the arcs and intervals are distinct.

2. Result

We review the approach of Balister *et al.* Let G be a connected circular arc graph. Let C be a circle and $\mathcal{F}$ be a finite collection of open arcs of C that correspond to the vertices of G. If the union of arcs in $\mathcal{F}$ does not cover C, then G is an interval graph and hence the statement follows by a result of [1]. Therefore, we may assume that the union of arcs in $\mathcal{F}$ covers C. We choose a set $\mathcal{K} \subseteq \mathcal{F}$ such that $\mathcal{K} = \{K_0, \ldots, K_{n-1}\}$,

- $C = K_0 \cup \cdots \cup K_{n-1}$,
- n is minimal, and
- no K_i is contained in another arc, i.e. $K_i \subseteq A \in \mathcal{F} \Rightarrow K_i = A$.

We cyclically order the elements of $\mathcal{K}$ clockwise and consider all indices of elements of $\mathcal{K}$ modulo n. A *chain* $\mathcal{P}$ of length t is a t-tuple $(J_1, \ldots, J_t)$ of distinct arcs (in $\mathcal{F}$) such that $J_i \cap J_{i+1} \neq \emptyset$ for every $1 \leq i \leq t-1$. This corresponds to a path in G on t vertices. The chain $\mathcal{P}$ is a *longest chain*, if there is no chain of larger length than $\mathcal{P}$. For a chain $\mathcal{P} = (J_1, \ldots, J_t)$, let the *support* Supp $\mathcal{P}$ of $\mathcal{P}$ be the subset of C defined by

$$J_1 \cup (J_2 \cap J_3) \cup \cdots \cup (J_{t-2} \cap J_{t-1}) \cup J_t.$$

Note that if there is an arc A in $\mathcal{F}$ that is not contained in the chain $\mathcal{P}$ of length t and intersects Supp $\mathcal{P}$, then there is a chain of length $t+1$ consisting of the arc A and all arcs of $\mathcal{P}$. This implies that for a longest chain $\mathcal{P}$ in $\mathcal{F}$, an arc A is contained in $\mathcal{P}$ if and only if it intersects Supp $\mathcal{P}$.

For two points x, y on the circle C, let $[x, y]$ be the arc from x to y in clockwise direction. For an arc $A \in \mathcal{F}$, let $\ell(A)$ and $r(A)$ be the two endpoints of A such that $\ell(A), A, r(A)$ are consecutive on C in clockwise direction.

Now, we mention two results, which we use later.

Lemma 1 (Balister *et al.* [1])**.** *If $\mathcal{P}$ is a longest chain in $\mathcal{F}$, then $\mathcal{P} \cap \mathcal{K} = \{K_i : i \in I\}$ is nonempty and I is a contiguous set of elements of $\mathbb{Z}_n$.*

The next lemma is due to Keil [3] and explicitly formulated as Lemma 2.3 in [1].

Lemma 2 (Keil [3]). *Let* $X = \{x_1, \ldots, x_{t+1}\}$ *be a set of real numbers, and let* $J_1, \ldots, J_t$ *be a sequence of open intervals with* $x_k, x_{k+1} \in J_k$ *for every* $1 \le k \le t$. *If* $x_{i_1} < \cdots < x_{i_{t+1}}$ *are the elements of* X *in increasing order, then the intervals have a permutation* $J_{j_1}, \ldots, J_{j_t}$ *such that* $x_{i_k}, x_{i_{k+1}} \in J_{j_k}$, *for every* $1 \le k \le t$.

Let $\mathcal{P} = (J_1, \ldots, J_t)$ be a chain such that $\mathcal{K} \not\subseteq \mathcal{P}$ and let $\{x_1, \ldots, x_{t+1}\} \subset \operatorname{Supp} \mathcal{P}$ be a set of distinct points such that $x_k, x_{k+1} \in J_k$, for every $1 \le k \le t$. Without loss of generality, we may assume, by Lemma 2, that $x_1, x_2, \ldots, x_{t+1}$ are consecutive points on C in clockwise direction. One might have to replace $\mathcal{P}$ by another chain having exactly the same arcs. Let $p, q \in \{1, \ldots, t\}$ such that $p < q$. If $[x_p, x_{p+1}], [x_q, x_{q+1}] \subseteq J_p \cap J_q$, then the reordering

$$(J_1, \ldots, J_{p-1}, J_q, J_{p+1}, \ldots, J_{q-1}, J_p, J_{q+1}, \ldots, J_t)$$

of $\mathcal{P}$ is a chain of the same length as $\mathcal{P}$. See Figure 1 for illustration. In this situation it is possible to *swap* J_p and J_q in $\mathcal{P}$.

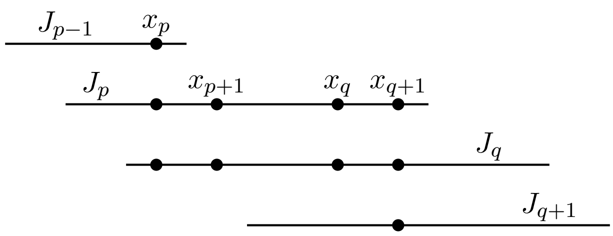

Figure 1. J_p and J_q can be swapped.

For $i \in \{0, \ldots, n-1\}$, let ΔK_i be the set of all points x such that $\ell(K_{i+1})$, x, $r(K_i)$ are consecutive points in clockwise direction on C. Note that for $n \ge 3$, we have $\Delta K_i = K_i \cap K_{i+1}$. We use this notation because Balister *et al.* omitted the case $n = 2$. For an arc A, note that $A \subset K_i \cup K_{i+1}$ implies the connectedness of $A \setminus \Delta K_{i+1}$ if n is at least 3.

Lemma 3 is our main contribution. Balister *et al.* only proved Lemma 3 with the properties (a)–(c). We extend this result.

Lemma 3. *If* $\mathcal{P}$ *is a longest chain in* $\mathcal{F}$ *and* $\mathcal{P} \cap \mathcal{K} = \{K_{a+1}, \ldots, K_{b-1}\} \ne \mathcal{K}$, *then the arcs in* $\mathcal{P}$ *have a reordering into a chain* $\mathcal{P}^*$ *such that in this reordering*

(a) K_{a+1} *precedes* K_{b-1} *in* $\mathcal{P}^*$ *provided they are distinct.*

(b) *If* A *precedes* K_{b-1} *in* $\mathcal{P}^*$, *then* $\Delta K_{b-1} \not\subseteq A$.

(c) *If* A *precedes* K_{a+1} *in* $\mathcal{P}^*$, *then* $A \subseteq K_a \cup K_{a+1}$ *and* $A \setminus \Delta K_{a+1}$ *is connected.*

(d) *If* K_{b-1} *precedes* A *in* $\mathcal{P}^*$, *then* $A \subseteq K_{b-1} \cup K_b$ *and* $A \setminus \Delta K_b$ *is connected.*

(e) *If* K_{a+1} *precedes* A *in* $\mathcal{P}^*$, *then* $\Delta K_a \not\subseteq A$.

Here is the gap of Balister *et al.* Indeed (b) and (c) is symmetric to (d) and (e) (they proved that (b) and (c) holds), however, forcing both at the same time is a stronger assertion.

Proof of Lemma 3. Let $\mathcal{P} = (J_1, \dots, J_t)$ and let $\{x_1, \dots, x_{t+1}\} \subset \operatorname{Supp} \mathcal{P}$ be a set of distinct points such that $x_k, x_{k+1} \in J_k$ for every $1 \le k \le t$. Without loss of generality, we may assume, by Lemma 2, that $x_1, x_2, \dots, x_{t+1}$ are consecutive points on C in clockwise direction. It is important to keep in mind that every x_i belongs to $(K_{a+1} \cup \dots \cup K_{b-1}) \setminus (K_a \cup K_b)$, because K_a and K_b do not belong to $\mathcal{P}$.

First, we prove (c) and (e). Let $\mathcal{P}' = (J_{j_1}, \dots, J_{j_s})$ be a subsequence of $\mathcal{P}$ such that $A \in \mathcal{P}'$ if and only if

(i) K_{a+1} precedes A in $\mathcal{P}$ and $\Delta K_a \subseteq A$, or

(ii) A precedes K_{a+1} in $\mathcal{P}$ and $A \not\subseteq K_a \cup K_{a+1}$ if $n \ge 3$, and $A \setminus \Delta K_{a+1}$ is disconnected if $n = 2$.

If $n \ge 3$, then we observe the following. If $A \in \mathcal{P}'$ satisfies requirement (i), then, by the choice of $\mathcal{K}$, we conclude that $\ell(K_a)$, $\ell(A)$, $\ell(K_{a+1})$, $r(K_a)$, $r(A)$, $r(K_{a+1})$ are consecutive points in clockwise direction on C. If $A \in \mathcal{P}'$ satisfies requirement (ii), then $\ell(K_a)$, $\ell(K_{a+1})$, $\ell(A)$, $r(K_a)$, $r(K_{a+1})$, $r(A)$ or $\ell(K_a)$, $\ell(K_{a+1})$, $r(K_a)$, $\ell(A)$, $r(K_{a+1})$, $r(A)$ are consecutive points in clockwise direction on C, because $A \cap (K_{a+1} \setminus K_a) \neq \emptyset$ and because of the choice of $\mathcal{K}$.

Suppose $n = 2$. If $A \in \mathcal{P}'$ satisfies requirement (i), then $\ell(A)$, $\ell(K_{a+1})$, $r(K_a)$, $r(A)$ are consecutive points in clockwise direction on C and if $A \in \mathcal{P}'$ satisfies requirement (ii), then $\ell(A)$, $\ell(K_a)$, $r(K_{a+1})$, $r(A)$ are consecutive points in clockwise direction on C.

Let $L = \{i \in [t] : J_i \in \mathcal{P}$ and J_i satisfies requirement (i)$\}$ and $R = \{i \in [t] : J_i \in \mathcal{P}$ and J_i satisfies requirement (ii)$\}$.

Let $L_{\mathcal{P}} = \{J_i \in \mathcal{P} : i \in L\}$ and $R_{\mathcal{P}} = \{J_i \in \mathcal{P} : i \in R\}$; that is, $L_{\mathcal{P}}$ and $R_{\mathcal{P}}$ partition $\mathcal{P}'$. Furthermore, all arcs in $R_{\mathcal{P}}$ precede the arcs in $L_{\mathcal{P}}$. Note that all arcs in $\mathcal{P} \setminus \mathcal{P}'$ satisfy the requirements (c) and (e).

Claim 4. *Let L and R be nonempty, and consider $p \in R$ and $q \in L$. It is possible to swap J_p and J_q in $\mathcal{P}$, the reordering of $\mathcal{P}$ is still a chain and the sets L and R lose exactly q and p, respectively.*

Proof. By our observations above and since J_p precedes J_q, we conclude that $\ell(J_q)$, $\ell(J_p)$, $r(J_q)$ and $r(J_p)$ are consecutive points in clockwise direction on C. Since J_p precedes J_q, we obtain $[x_p, x_{p+1}], [x_q, x_{q+1}] \subseteq J_p \cap J_q$. Thus it is possible to swap J_p and J_q in $\mathcal{P}$. After this swap both arcs do not satisfy the requirements (i) and (ii) any more and in addition the relative positions of all other arcs concerning K_{a+1} do not change. This completes the proof of the claim. □

Claim 5. *Each element $J_p \in R_{\mathcal{P}}$ can be swapped with K_{a+1} in $\mathcal{P}$ and the reordering of $\mathcal{P}$ is still a chain.*

Proof. Let q be such that $J_q = K_{a+1}$, that is $p < q$ by the definition of R. By our observations above, we know that $\ell(K_{a+1})$, $\ell(J_p)$, $r(K_{a+1})$ and $r(J_p)$ are consecutive points in clockwise direction on C. Since J_p precedes K_{a+1}, we obtain $[x_p, x_{p+1}], [x_q, x_{q+1}] \subseteq J_p \cap K_{a+1}$. Thus it is possible to swap J_p and K_{a+1} in $\mathcal{P}$ and the reordering of $\mathcal{P}$ is still a chain. □

Claim 6. *Each element $J_q \in L_{\mathcal{P}}$ can be swapped with K_{a+1} in $\mathcal{P}$ and the reordering of $\mathcal{P}$ is still a chain.*

Proof. Let p be such that $J_p = K_{a+1}$, that is $p < q$ by the definition of L. By our observations above, we know that $\ell(J_q)$, $\ell(K_{a+1})$, $r(J_q)$ and $r(K_{a+1})$ are consecutive points in clockwise direction on C. Since K_{a+1} precedes J_q, we obtain $[x_p, x_{p+1}], [x_q, x_{q+1}] \subseteq K_{a+1} \cap J_q$. Thus it is possible to swap K_{a+1} and J_q and the reordering of $\mathcal{P}$ is still a chain. □

Let $\gamma \in \mathbb{N}$ be such that $K_{a+1} = J_\gamma$ and $f(\mathcal{P}')$ be defined by

$$\max\{\{\gamma\} \cup L \cup R\} - \min\{\{\gamma\} \cup L \cup R\}.$$

Let $\alpha = \min\{\{\gamma\} \cup L \cup R\}$ and $\beta = \max\{\{\gamma\} \cup L \cup R\}$. Note that α does not decrease and β does not increase if we reorder $\mathcal{P}$ as described in Claims 4–6. In particular, $f(\mathcal{P}')$ does not increase. After swapping two elements in $\mathcal{P}'$, by Claim 4 the subsequence loses two elements. Using Claim 4 iteratively, we can assume that $L = \emptyset$ or $R = \emptyset$. If $\mathcal{P}' = \emptyset$, then this completes the proof of (c) and (e). Therefore, we assume that $\mathcal{P}' \neq \emptyset$ and $\mathcal{P}' = L_{\mathcal{P}}$ or $\mathcal{P}' = R_{\mathcal{P}}$. We distinguish the two possible cases.

(I) If $\mathcal{P}' = L_{\mathcal{P}}$, then we have $K_{a+1} = J_\alpha$ and $\beta = \max\{L\}$, and

(II) if $\mathcal{P}' = R_{\mathcal{P}}$, then we have $\alpha = \min\{R\}$ and $K_{a+1} = J_\beta$.

Note that $f(\mathcal{P}') = 0$ if and only if $\mathcal{P}' = \emptyset$. By Claims 5 and 6, it is possible to swap K_{a+1} with each element of $\mathcal{P}'$. In the first case swap K_{a+1} with J_β and in the second case with J_α. Denote this reordering of $\mathcal{P}$ by $\mathcal{P}$ again and define $\mathcal{P}'$, L and R as before. Consider first case (I). Note that $L = \emptyset$ and $R \subseteq \{\alpha+1, \ldots, \beta-1\}$. In case (II), we have $L \subseteq \{\alpha+1, \ldots, \beta-1\}$ and $R = \emptyset$. In both cases $f(\mathcal{P}')$ decreases by at least 1. After iterating this procedure at most $\beta - \alpha$ times, we have $f(\mathcal{P}') = 0$. Hence there is a reordering of $\mathcal{P}$ such that the requirements (c) and (e) are fulfilled. From now on, we assume that $\mathcal{P}$ fulfills requirements (c) and (e).

If $a+1 = b-1$, then $\mathcal{P}$ fulfills the requirements (a), (b) and (d). Note that this is also true if $|\mathcal{K}| = 2$. Thus we assume that K_{a+1} and K_{b-1} are distinct. This implies $n \geq 3$. Note that K_{a+1} precedes K_{b-1} by requirement (c). Let $\tilde{\mathcal{P}} = (J_{k_1}, \ldots, J_{k_{s'}})$ be the subsequence of $\mathcal{P}$ such that $A \in \tilde{\mathcal{P}}$ if and only if (i') K_{b-1} precedes A and $A \not\subseteq \Delta K_{b-1}$, or

(ii') A precedes K_{b-1} and $K_{b-1} \cap K_b \subseteq A$.

Note that $K_{a+1} \notin \tilde{\mathcal{P}}$. Let $\tilde{L} = \{i \in [t] : J_i \in \mathcal{P}$ and J_i satisfies requirement (i')$\}$ and $\tilde{R} = \{i \in [t] : J_i \in \mathcal{P}$ and J_i satisfies requirement (ii')$\}$. Let $\tilde{\gamma} \in \mathbb{N}$ be such that $K_{b-1} = J_{\tilde{\gamma}}$ and $\tilde{\alpha} = \min\{\{\tilde{\gamma}\} \cup \tilde{L} \cup \tilde{R}\}$. Note that $\gamma < \tilde{\alpha}$. This implies that K_{a+1} precedes all arcs in $\tilde{\mathcal{P}}$ and hence arguing as above for K_{b-1}, the relative order in the ordering of $\mathcal{P}$ of all arcs of $\mathcal{P}$ concerning K_{a+1} does not change. This shows that there is a reordering $\mathcal{P}^*$ of $\mathcal{P}$ such that $\mathcal{P}^*$ fulfills the requirements of Lemma 3. ■

Theorem 7. *If G is a connected circular arc graph, then the intersection of all longest paths is nonempty.*

Proof. We can assume that G is not an interval graph, otherwise the statement follows by a result of [1]. As above, let $\mathcal{F}$ be the finite collection of arcs of a circle C that correspond to the vertices of G. We choose $\mathcal{K}$ as above. If $n = 1$, then every longest chain contains K_0 and we are done. Let $\mathcal{P}$ a longest chain such that $|\mathcal{P} \cap \mathcal{K}|$ is as small as possible. If $|\mathcal{P} \cap \mathcal{K}| = n$, then every longest chain contains all arcs of $\mathcal{K}$ and we are done, too. Therefore, we assume that $n \geq 2$ and $|\mathcal{P} \cap \mathcal{K}| < n$. That is, by Lemma 1, $\mathcal{P} \cap \mathcal{K} = \{K_{a+1}, \ldots, K_{b-1}\}$. We prove Theorem 7 by showing that every longest chain contains K_{b-1}. We assume, for contradiction, that there is a longest chain $\mathcal{Q}$ such that $K_{b-1} \notin \mathcal{Q}$. Let $\mathcal{Q} \cap \mathcal{K} = \{K_{\ell+1}, \ldots, K_{m-1}\}$. Our assumption and choice of $\mathcal{P}$ imply that $K_{b-1} \in \mathcal{P} \setminus \mathcal{Q}$, $K_{\ell+1} \in \mathcal{Q} \setminus \mathcal{P}$ and $K_b, \ldots, K_\ell \notin \mathcal{P} \cup \mathcal{Q}$. Let $\mathcal{R}$ be the chain $(K_b, \ldots, K_\ell)$. Note that $\mathcal{R} = \emptyset$ if $b = \ell + 1$.

For a k-tuple $\mathcal{A} = (A_1, \ldots, A_k)$, let the reversed k-tuple $\mathcal{A}^r$ be defined by $(A_k, \ldots, A_1)$. If $\mathcal{B} = (B_1, \ldots, B_{k'})$, then let $\mathcal{AB} = (A_1, \ldots, A_k, B_1, \ldots, B_{k'})$ and $\mathcal{A}B_1 = (A_1, \ldots, A_k, B_1)$. We reorder $\mathcal{P}$ and $\mathcal{Q}$ such that the reorderings $\mathcal{P}^*$ and $\mathcal{Q}^*$ satisfy the requirements of Lemma 3. Let $\mathcal{P}^* = \mathcal{P}_1 K_{b-1} \mathcal{P}_2$ and $\mathcal{Q}^* = \mathcal{Q}_1 K_{\ell+1} \mathcal{Q}_2$. Note that

(i) if $A \in \mathcal{P}_1$, then $\Delta K_{b-1} \not\subseteq A$,

(ii) if $A \in \mathcal{P}_2$, then $A \subseteq K_{b-1} \cup K_b$ and $A \setminus \Delta K_b$ is connected,

(iii) if $A \in \mathcal{Q}_1$, then $A \subseteq K_\ell \cup K_{\ell+1}$ and $A \setminus \Delta K_{\ell+1}$ is connected, and

(iv) if $A \in \mathcal{Q}_2$, then $\Delta K_\ell \not\subseteq A$.

Let $\mathcal{C}_1 = \mathcal{P}_1 K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_1^r$ and $\mathcal{C}_2 = \mathcal{P}_2^r K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_2$.

Claim 8. *$\mathcal{C}_1$ is a chain.*

Proof. It suffices to show that $\mathcal{P}_1 \cap \mathcal{Q}_1 = \emptyset$. We assume, for contradiction, that there is an arc $A \in \mathcal{P}_1 \cap \mathcal{Q}_1$. Suppose $n = 2$. Thus $\mathcal{K} = \{K_{b-1}, K_{\ell+1}\}$. By (iii), $A \setminus \Delta K_{\ell+1}$ is connected and by (i) $\Delta K_{b-1} \not\subseteq A$. This implies that $A \subseteq K_{\ell+1}$ or

$A \subseteq K_{b-1}$. Since $A \in \mathcal{P} \cap \mathcal{Q}$, this implies $K_{\ell+1} \in \mathcal{P} \cap \mathcal{Q}$ or $K_{b-1} \in \mathcal{P} \cap \mathcal{Q}$, which is a contradiction.

Now we assume $n \geq 3$. By (iii), $A \subseteq K_\ell \cup K_{\ell+1}$. Since $A \in \mathcal{P} \cap \mathcal{Q}$ and hence A meets $K_{\ell+1} \setminus K_\ell$, we observe that $r(K_\ell)$, $r(A)$ and $r(K_{\ell+1})$ are consecutive points on C. If $A \subseteq K_{\ell+1}$, then $\operatorname{Supp} \mathcal{P} \cap K_{\ell+1} \neq \emptyset$ and hence $K_{\ell+1} \in \mathcal{P}$, which is a contradiction. Thus $\ell(K_\ell)$, $\ell(A)$, $\ell(K_{\ell+1})$, $r(K_\ell)$, $r(A)$ and $r(K_{\ell+1})$ are consecutive points on C.

By (i), $K_{b-1} \cap K_b \not\subseteq A$. This implies that $b \neq \ell+1$ and hence $\mathcal{R}$ is not empty. Thus $K_\ell \notin \mathcal{P}$. Since $A \in \mathcal{P}$, it is $A \cap \operatorname{Supp} \mathcal{P} \neq \emptyset$ and hence $\operatorname{Supp} \mathcal{P} \cap (K_\ell \cup K_{\ell+1}) \neq \emptyset$. Thus $\mathcal{P}$ contains K_ℓ or $K_{\ell+1}$. This is a contradiction and completes the proof of the claim. $\square$

Claim 9. *$\mathcal{C}_2$ is a chain.*

Proof. Using (ii) and (iv) instead of (i) and (iii) this is the completely symmetric case to Claim 8. $\square$

Note that $|\mathcal{C}_1| + |\mathcal{C}_2| \geq |\mathcal{P}| + |\mathcal{Q}| + 2$. This implies that $|\mathcal{C}_1| > |\mathcal{P}|$ or $|\mathcal{C}_2| > |\mathcal{P}|$, which is a contradiction to the choice of $\mathcal{P}$. $\blacksquare$

Remark. As pointed out by a referee, for the proof of Theorem 7 it is enough to require in Lemma 3 only (a), (b) and (d), or equivalently, (a), (c) and (e). This is due to the fact that one can apply (b) and (d) to $\mathcal{Q}^r$ instead of (c) and (e) to $\mathcal{Q}$. Since proving (a)-(e) results only in a slightly longer proof, we prove (a)–(e).

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