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DOMINATION, ETERNAL DOMINATION AND CLIQUE COVERING

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Abstract

Eternal and m-eternal domination are concerned with using mobile guards to protect a graph against infinite sequences of attacks at vertices. Eternal domination allows one guard to move per attack, whereas more than one guard may move per attack in the m-eternal domination model. Inequality chains consisting of the domination, eternal domination, m-eternal domination, independence, and clique covering numbers of graph are explored in this paper.

Among other results, we characterize bipartite and triangle-free graphs with domination and eternal domination numbers equal to two, trees with equal m-eternal domination and clique covering numbers, and two classes of graphs with equal domination, eternal domination and clique covering numbers.

Keywords: dominating set, eternal dominating set, independent set, clique cover.

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1. Introduction

A *dominating set* of a finite, undirected graph $G = (V, E)$ is a set $D \subseteq V$ such that each vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality amongst all dominating sets of G is the *domination number*, $\gamma(G)$. By imposing conditions on the subgraph $G[D]$ of G induced by D, one can obtain several varieties of dominating sets and their associated parameters. For example, if $G[D]$ is connected, then D is a *connected dominating set* and the corresponding parameter is the *connected domination number* $\gamma_c(G)$.

Domination theory can be considered the precursor to the study of graph protection: one may view a dominating set as an immobile set of guards protecting a graph. A thorough survey of domination theory can be found in [8]. In this paper, we consider two forms of dynamic domination which aim to protect a graph against an infinite sequence of attacks occurring at the vertices of the graph.

Let $\{D_i\}$, $D_i \subseteq V$, $i \geq 1$, be a collection of sets of vertices of the same cardinality, with one guard located on each vertex of D_i. The two problems considered in this paper can each be modeled as a two-player game between a *defender* and an *attacker*: the defender chooses D_1 as well as each D_i, $i > 1$, while the attacker chooses the infinite sequence of vertices corresponding to the locations of the attacks $r_1, r_2, \ldots$. Players alternate turns, with the defender first choosing the initial location of guards. The attacker goes next and chooses a vertex to attack. Each attack is dealt with by the defender by choosing the next D_i subject to some constraints that depend on the particular game (see below). The defender wins the game if they can successfully defend any sequence of attacks, subject to the constraints of the game described below; the attacker wins otherwise.

We say that a vertex is *protected* if there is a guard on the vertex or on an adjacent vertex. A vertex v is *occupied* if there is a guard on v, otherwise v is *unoccupied*. An attack at an unoccupied vertex x is *defended* if a guard moves to the attacked vertex. If the guard moves to x from v, we also say v *defends* x.

For the **eternal domination problem**, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving one guard to r_i from an adjacent vertex $v \in D_i$. If the defender can win the game with the sets $\{D_i\}$, then each D_i is an *eternal dominating set* (*EDS*). The size of a smallest EDS of G is the *eternal domination number* $\gamma^\infty(G)$. This problem was first studied by Burger *et al.* in [4] and will sometimes be referred to as the *one-guard moves* model. It has been subsequently studied in [1, 6, 10] and other papers.

For the **m-eternal dominating set problem**, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and

D_{i+1} is obtained from D_i by moving guards to neighboring vertices. That is, each guard in D_i may move to an adjacent vertex, as long as one guard moves to r_i. Thus it is required that $r_i \in D_{i+1}$. The size of a smallest m-*eternal dominating set* (m-*EDS*) (defined similarly to an EDS) of G is the m-*eternal domination number* $\gamma_{\rm m}^{\infty}(G)$. This "multiple guards move" version of the problem was introduced by Goddard, Hedetniemi and Hedetniemi [5]. We refer to this as the "all-guards move" model of eternal domination. This problem has been subsequently studied in [7, 11] and other papers.

It is clear from the definitions that $\gamma^{\infty}(G) \geq \gamma_{\rm m}^{\infty}(G) \geq \gamma(G)$ for all graphs G. A survey on several variations of eternal dominating sets, including the two just defined, can be found in [13]. Our focus in this paper is comparing these graph protection parameters to other parameters which will be defined and reviewed in the next section. We pay special attention to the study of graph classes that satisfy equality in bounds on γ^{∞} and $\gamma_{\rm m}^{\infty}$. After providing definitions, background and known results in Section 2, we consider m-eternal domination in graphs with $\alpha = 3$ in Section 3 as initiation of the study of graphs G for which $\gamma_{\rm m}^{\infty}(G) = \alpha(G)$. In Section 4 we characterize bipartite graphs with $\gamma = \gamma^{\infty}$, and bipartite and triangle-free graphs with $\gamma = \gamma_{\rm m}^{\infty} = 2$. As the main result of this paper, trees with equal m-eternal domination and clique covering numbers are characterized in Section 5, and in Section 6 we consider the problem of whether $\gamma(G) = \gamma^{\infty}(G)$ implies that $\gamma(G) = \theta(G)$. We end with a number of open problems and questions in Section 7.

2. Definitions and Background

The *open* and *closed neighborhoods* of $X \subseteq V$ are $N(X) = \{v \in V : v \text{ is adjacent to a vertex in } X\}$ and $N[X] = N(X) \cup X$, respectively, and $N(\{v\})$ and $N[\{v\}]$ are abbreviated, as usual, to $N(v)$ and $N[v]$. The set $\overline{N[v]}$ is the set of all vertices not dominated by v. For any $v \in X$, the *private neighborhood* $\mathrm{pn}(v, X)$ *of* v *with respect to* X is the set of all vertices in $N[v]$ that are not contained in the closed neighborhood of any other vertex in X, i.e., $\mathrm{pn}(v, X) = N[v] - N[X - \{v\}]$. The elements of $\mathrm{pn}(v, X)$ are the *private neighbors of* v *relative to* X. The *external private neighborhood*, $\mathrm{epn}(v, X)$, is defined similarly, except that $N(v)$ replaces $N[v]$ in the definition.

In a tree T, a *leaf* is a degree one vertex, a *stem* is a vertex adjacent to a leaf, and a branch vertex is a vertex of degree at least three. For any $v \in V(T)$, a v-*endpath* is a path from v to a leaf, all of whose internal vertices have degree two in T. An *end-branch-vertex* is a branch vertex v such that exactly one edge incident with v does not lie on a v-endpath. Every tree with at least two branch vertices has at least two end-branch vertices. A (non-trivial) *star* is a tree $K_{1,r}$,

$r \geq 1$.

We denote the minimum and maximum degree of a graph G by $\delta(G)$ and $\Delta(G)$ respectively, and its independence number by $\alpha(G)$. The *clique covering number* $\theta(G)$ is the minimum number k of sets in a partition $V = V_1 \cup \cdots \cup V_k$ of V such that each $G[V_i]$ is complete. Hence $\theta(G)$ equals the chromatic number $\chi(\overline{G})$ of the complement $\overline{G}$ of G. Since $\chi(G) = \omega(G)$ (the size of a maximum clique) if G is perfect, and G is perfect if and only if $\overline{G}$ is perfect, $\alpha(G) = \theta(G)$ for all perfect graphs.

As first observed by Burger *et al.* [4], γ^∞ lies between the independence and clique covering numbers, giving the inequality chain below.

Fact 1. *For any graph G, $\gamma(G) \leq \alpha(G) \leq \gamma^\infty(G) \leq \theta(G)$.*

Since $\alpha(G) = \theta(G)$ for perfect graphs, the rightmost two bounds in Fact 1 are tight for perfect graphs. A topic that has received much attention is finding classes of non-perfect graphs that satisfy equality in one or more of the bounds in Fact 1. A number of graphs classes have been shown to satisfy $\gamma^\infty(G) = \theta(G)$, such as circular-arc graphs [15] and series-parallel graphs [1]. It is, as of yet, not known whether $\gamma^\infty(G) = \theta(G)$ for all planar graphs G.

The following upper bound is due to Klostermeyer and MacGillivray [10]; Goldwasser and Klostermeyer [6] show that the bound is sharp.

Theorem 2 [10]**.** *For any graph G,*

$$\gamma^\infty(G) \leq \binom{\alpha(G)+1}{2}.$$

Goddard *et al.* [5] determine $\gamma_{\mathrm{m}}^\infty(G)$ exactly for complete graphs, paths, cycles, and complete bipartite graphs. Further, they show that $\gamma_{\mathrm{m}}^\infty(G) = \gamma(G)$ for all Cayley graphs G obtainable from Abelian groups. Their assertion that this equality holds for all Cayley graphs is shown to be false in [3].

The inherent symmetry of Cayley graphs provides a sort of foothold for m-eternal domination; an open problem is to determine other classes of graphs where $\gamma_{\mathrm{m}}^\infty(G) = \gamma(G)$. Goddard *et al.* also prove the following fundamental bound.

Theorem 3 [5]**.** *For all graphs G, $\gamma(G) \leq \gamma_{\mathrm{m}}^\infty(G) \leq \alpha(G)$.*

In order to get a better upper bound on $\gamma_{\mathrm{m}}^\infty$, Goddard *et al.* define a *neo-colonization* to be a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_t\}$ of G such that each V_i induces a connected graph [5]. A part V_i is assigned *weight* $w(V_i) = 1$ if V_i induces a clique, and $w(V_i) = 1+\gamma_c(G[V_i])$ otherwise, where $\gamma_c(G[V_i])$ is the connected domination number of the subgraph induced by V_i. The *weight* $w(\mathcal{P})$ of a neo-colonization $\mathcal{P}$ is the sum of the weights of its parts. Define $\theta_c(G)$ to be the minimum weight of any neo-colonization of G. Goddard *et al.* [5] prove that $\gamma_{\mathrm{m}}^\infty(G) \leq \theta_c(G) \leq$

$\gamma_c(G) + 1$. In general, however, $\alpha(G)$ and $\theta_c(G)$ are not comparable: consider $\theta_c(K_{1,5}) < \alpha(K_{1,5})$, $\theta_c(K_n) = \alpha(K_n)$, and $\theta_c(C_5) = 3 > \alpha(C_5) = 2$. On the other hand, $\theta_c(G) \le \alpha(G)$ for all perfect graphs G because $\theta_c(G) \le \theta(G)$ for all graphs and $\theta(G) = \alpha(G)$ if G is perfect.

Let $\tau(G)$ denote the size of a smallest vertex cover of G. For a bipartite graph $G = (V, E)$, let C be a minimum vertex cover of G and M a maximum matching of G that is formed from C and a neighbor of each vertex in C. If the end-vertices of M, M_c, yield the set V, then $\theta_c(G) = \alpha(G) = |M| = \tau(G)$ and we are done. Otherwise, $|M_c| < |V|$. Let M_u be $V - M_c$.

Proposition 4. *Let G be a bipartite graph. Then $\theta_c(G) \le \tau(G) + |M_u| = \alpha(G)$.*

Proof. Observe that $\alpha(G) = \tau(G) + |M_u|$. Partition V into sets such that each set contains the two end-vertices from one edge in M; each vertex in M_u is placed in a set with a neighbor (which is a vertex in M_c). Note that each such set induces a star. From this partitioning, we see that a neo-colonization exists consisting only of stars—and a star that is a K_2 has weight one and a star that is a $K_{1,m}, m > 1$ has weight two. Therefore $\theta_c(G) \le \tau(G) + |M_u|$. ■

As shown in [11], $\gamma_{\mathrm{m}}^{\infty}(T) = \theta_c(T)$ for all trees T. There exist graphs with $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G) < \alpha(G)$, such as C_4 with a pendant vertex attached to one of its vertices. Additional results comparing the vertex cover and eternal domination numbers can be found in [12].

The following fact and its converse for $k = 2$ (Proposition 6) can be useful.

Fact 5. *A necessary condition for $\gamma^{\infty}(G) = k$, or $\gamma_{\mathrm{m}}^{\infty}(G) = k$, is that every vertex of G be contained in a dominating set of size k.*

If $k = 1$, then this condition is also sufficient, and if $k \ge 3$, then it is not sufficient: let T be the tree obtained by joining a new leaf to each stem of P_{3k-4}. Then every vertex of T is contained in some dominating set of size k, but $\gamma^{\infty}(T) = \gamma_{\mathrm{m}}^{\infty}(T) > k$ (first attack one leaf, then attack another leaf at distance $3k - 5$ from the first leaf). For $k = 2$, the condition is not sufficient for γ^{∞} (if $G = K_{m,n}$, $n \ge m \ge 3$, then any pair of vertices from different partite sets form a dominating set, but $\gamma^{\infty}(G) = n$). We show that it is sufficient for $\gamma_{\mathrm{m}}^{\infty}$.

Proposition 6. *If every vertex of the graph $G \ne K_n$ is contained in a dominating set of size two, then $\gamma_{\mathrm{m}}^{\infty}(G) = 2$.*

Proof. Suppose every vertex of G is in a dominating set of size two. Let $D = \{u, v\}$ be any dominating set and consider any $x \in V - \{u, v\}$. We need to show that guards occupying u and v can move to x and to a vertex y such that $\{x, y\}$ is a dominating set; that is, G has a dominating set $\{x, y\}$ such that $ux \in E(G)$ and $v \in N[y]$, or $vx \in E(G)$ and $u \in N[y]$. Since D dominates x, assume without

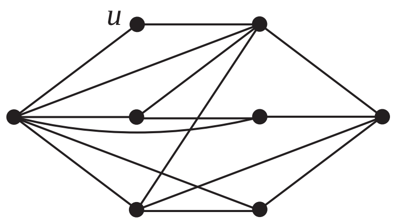

Figure 1. $\gamma_{\rm m}^{\infty}(G) = \alpha(G) = 3$.

loss of generality that $vx \in E(G)$. By the hypothesis there exists a vertex y such that $D' = \{x, y\}$ is a dominating set. If $y \in N[u]$, we are done. If $y \notin N[u]$, then $y \in N[v]$ because D dominates y, and $u \in N[x]$ because D' dominates u. But then $ux \in E(G)$ and $v \in N[y]$, as required. ■

3. m-Eternal Domination and Independence

Clearly, if $\alpha(G) = 1$ or 2, then $\gamma_{\rm m}^{\infty}(G) = \alpha(G)$. We next examine graphs with independence number three, in which case $\gamma_{\rm m}^{\infty}(G) \in \{2, 3\}$ (Theorem 3). Classifying the graphs with $\alpha(G) = 3$ and $\gamma_{\rm m}^{\infty}(G) = 2$, or equivalently $\alpha(G) = 3$ and $\gamma_{\rm m}^{\infty}(G) = 3$, will make a valuable contribution to the study of graphs with $\gamma_{\rm m}^{\infty}(G) = \alpha(G)$, but even this apparently "small" case may be difficult as there is no known characterization of graphs with $\gamma = 2$ and $\alpha = 3$.

The statement "$\alpha(G) = 3$ and any three independent vertices of G have a common neighbor" does not imply that $\gamma_{\rm m}^{\infty}(G) = 2$: for the graph G in Figure 1, $\alpha(G) = 3$ and any three independent vertices of G have a common neighbor. However, the vertex u is not in any dominating set of size two. By Fact 5, $\gamma_{\rm m}^{\infty}(G) > 2$, hence by Theorem 3, $\gamma_{\rm m}^{\infty}(G) = 3$.

We need to impose a stronger condition for the next result.

Proposition 7. *Let $G = (V, E)$ be a graph with $\alpha(G) = 3$. If G has a vertex v that dominates all three vertices in all maximum independent sets, then $\gamma_{\rm m}^{\infty}(G) = 2$.*

Proof. Since $\alpha(G) = 3$, $\gamma_{\rm m}^{\infty}(G) \geq 2$. If $N[v] = V$ and $u \in V - \{v\}$ is arbitrary, then $\{u, v\}$ is a domination set and the result follows from Proposition 6. Hence assume $X = \overline{N[v]} \neq \emptyset$. For any distinct $x, x' \in X$, $xx' \in E$, otherwise $\{v, x, x'\}$ is an independent set not dominated by v. Thus X is a clique. For any $x \in X$ and any two distinct vertices $u, w \in \overline{N[x]}$, $uw \in E(G)$, otherwise $\{x, u, w\}$ is an independent set not dominated by v; that is, $\overline{N[x]}$ is a clique. Since X is a clique, $\{v, x\}$ dominates G for any $x \in X$. For any $u \in N(v)$, if u is adjacent to all vertices in X, then $\{u, v\}$ dominates G, and if u is nonadjacent to some $x \in X$, then the fact that $\overline{N[x]}$ is a clique implies that $\{u, x\}$ dominates G. Hence each vertex of G is contained in a dominating set of size two, and by Proposition 6, $\gamma_{\mathrm{m}}^{\infty}(G) = 2$. ■

Note that $\gamma_{\mathrm{m}}^{\infty}(C_6) = 2$, $\alpha(C_6) = 3$, and no maximum independent set is dominated by a single vertex. This example can be generalized as follows to obtain a class of graphs G such that $\gamma_{\mathrm{m}}^{\infty}(G) = 2$ and $\alpha(G) = 3$. In $C_6 = v_0, v_1, \ldots, v_5, v_0$, replace each v_i by a complete graph H_i of any order, and join each vertex of H_i, $i = 0, \ldots, 5$, to each vertex of $H_{i+1 \pmod 6}$ and to each vertex of $H_{i-1 \pmod 6}$ to form the graph H. Note that $\alpha(H) = 3$ and, by Proposition 6, $\gamma_{\mathrm{m}}^{\infty}(H) = 2$—for any $u \in H_i$ and any $v \in H_{i+3 \pmod 3}$, $\{u, v\}$ dominates H, $i = 0, \ldots, 5$. Any graph G with $\alpha(G) = 3$ that has H as spanning subgraph also has $\gamma_{\mathrm{m}}^{\infty}(G) = 2$.

4. Bipartite Graphs with $\gamma = \gamma^{\infty}$ or $\gamma = \gamma_{\mathrm{m}}^{\infty}$

In this section we consider bipartite graphs G such that $\gamma(G) = \gamma^{\infty}(G)$ or $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G)$. The former condition is more restrictive and this class of graphs is easy to characterize. The second class is larger and more difficult to characterize, and as a first step in this investigation we impose the further condition that $\gamma(G) = 2$. Recall that a graph is *well-covered* if every maximal independent set is maximum independent. For a matching M in G, let $M(x)$ denote the vertex matched with x.

Theorem 8 [14]**.** *A bipartite graph G without isolated vertices is well-covered if and only if G has a perfect matching M such that, for every pair $(x, M(x))$, the subgraph induced by $N(x) \cup N(M(x))$ is complete bipartite.*

Proposition 9. *Let G be a bipartite graph without isolated vertices. Then $\gamma(G) = \gamma^{\infty}(G)$ if and only if $\gamma(G) = n/2$.*

Proof. If $\gamma(G) = n/2$, then G is well-covered. By Theorem 8, G has a perfect matching. Since G is bipartite, $\theta(G) = n/2$, which implies $\gamma^{\infty}(G) = n/2$. On the other hand, if $\gamma(G) = \gamma^{\infty}(G)$, then, by Fact 1, $\gamma^{\infty}(G) = \alpha(G)$. Since $\alpha(G) \geq n/2$ for any bipartite graph, the result follows. ■

Note that $\gamma(G) = n/2$ if and only if each component of G is a 4-cycle or the corona of a connected graph H with K_1, c.f. [8]. We strengthen Proposition 9 to triangle-free graphs in Corollary 18.

If $\gamma(G) = 1$, then $\gamma_{\mathrm{m}}^{\infty}(G) = 1$ if G is complete, and $\gamma_{\mathrm{m}}^{\infty}(G) = 2$ otherwise. Now we turn to describing the bipartite graphs with $\gamma_{\mathrm{m}}^{\infty} = \gamma = 2$. Let $\mathfrak{C}$ be the class of all graphs obtained from $K_{m,m}$, $m \geq 2$, by deleting a matching M of size k, where $0 \leq k \leq m$, or from $K_{m,n}$, $n > m \geq 2$, by deleting a matching M of size

ℓ, $0 \le \ell \le m-1$. For example, $\mathfrak{C}$ contains the graphs $2K_2$, P_4, C_6, $K_{m,n}$, $K_{2,3}-e$. If $G \in \mathfrak{C}$ and v is a vertex of G incident with an edge of the removed matching M, then v is a *depleted vertex*, otherwise v is a *full vertex*. Note that each $G \in \mathfrak{C}$ that has a full vertex, has a full vertex in each of its partite sets.

Theorem 10. *If G is bipartite, then $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G) = 2$ if and only if $G \in \mathfrak{C}$.*

Proof. Let G have partite sets A and B. Suppose $G \in \mathfrak{C}$. Then $\gamma_{\mathrm{m}}^{\infty}(G) \ge \gamma(G) \ge 2$. If $x \in A$ is full, then there exists $y \in B$ that is full, and $\{x, y\}$ dominates G. If $x \in A$ is depleted, let $y \in B$ be the vertex such that xy belongs to the deleted matching. Then $\{x, y\}$ dominates G. Hence each vertex of A, and similarly each vertex of B belongs to a dominating set of size two. By Proposition 6, $\gamma_{\mathrm{m}}^{\infty}(G) = \gamma(G) = 2$.

Conversely, suppose $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G) = 2$. Then G does not have a universal vertex, so $|A|, |B| \ge 2$. Assume without loss of generality that $2 \le m = |A| \le n = |B|$.

Suppose $\deg v \le m-2$ for some $v \in B$; say v is nonadjacent to $u, u' \in A$. By Fact 5 there is a configuration of guards such that u is occupied. Since u' is protected, the other guard occupies u' or some vertex $w \in B - \{v\}$. But in either case v is unprotected, contradicting $\gamma_{\mathrm{m}}^{\infty}(G) = 2$. Hence $\deg v \ge m-1$ for each $v \in B$. Similarly, $\deg u \ge n-1$ for each $u \in A$. Therefore $G = K_{m,n}$ or G is obtained from $K_{m,n}$ by deleting edges of a matching.

Now suppose $m < n$ and $\deg u = n-1$ for each $u \in A$. Since $m < n$ there exists $v \in B$ such that $\deg v = m$. Let v be occupied. Since $|B - \{v\}| \ge 2$, the other guard occupies a vertex $u \in A$. Now v is adjacent to u, and $\deg u = n-1$; hence there exists $w \in B - \{v\}$ such that $uw \notin E(G)$. But then w is not protected, a contradiction as above. We deduce that $\deg u = n$ for at least one vertex $u \in A$. Therefore $G \in \mathfrak{C}$ as required. ■

It turns out that the class of triangle-free graphs with $\gamma_{\mathrm{m}}^{\infty} = \gamma = 2$ is almost the same as the class of bipartite graphs with this property.

Corollary 11. *A triangle-free graph G satisfies $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G) = 2$ if and only if $G = C_5$ or $G \in \mathfrak{C}$.*

Proof. Suppose $G \ncong C_5$ is a non-bipartite triangle-free graph such that $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G) = 2$. Then G has a shortest odd cycle $H \cong C_{2n+1}$, where $n \ge 2$. Since the component of G containing H is not complete and $\gamma(G) = 2$, G is connected. We obtain a contradiction by proving by induction on n that $H \ncong C_{2n+1}$ for all $n \ge 2$.

Suppose first that $H \cong C_5$; say H is the cycle $v_0, v_1, \ldots, v_4, v_0$. Since H is triangle-free, H is a chordless 5-cycle. Since $G \ncong C_5$, there exists a vertex

$x \in V(G) - V(H)$ that is adjacent to a vertex of H; say $xv_0 \in E(G)$. By Fact 5 there exists a vertex y such that $\{x, y\}$ is a dominating set of G.

Suppose x is not adjacent to any other vertex of H. Then y dominates $\{v_1, \ldots, v_4\}$. Since $G[\{v_1, \ldots, v_4\}] \cong P_4$ and no vertex of $G - H$ dominates more than two of $v_1, \ldots, v_4$, this is impossible. Hence x is adjacent to v_i for some $i = 1, \ldots, 4$. Since G is triangle-free, we may assume without loss of generality that $xv_2 \in E(G)$ and $xv_i \notin E(G)$ for $i = 1, 3, 4$. Then y dominates $\{v_1, v_3, v_4\}$. But G is triangle-free, so neither v_3 nor v_4 dominates v_1, and no other vertex of G dominates both v_3 and v_4. We deduce that $H \ncong C_5$.

Now suppose that for some $k \geq 3$, $H \ncong C_{2r+1}$ for all $r = 2, \ldots, k-1$ and suppose $H \cong C_{2k+1}$. Say H is the cycle $v_0, v_1, \ldots, v_{2k}, v_0$. Since $\gamma(C_{2k+1}) > 2$, $G \ncong H$. If H has a chord, then G has an odd cycle C_{2r+1} for $r < k$, which is not the case. Hence there is a vertex $x \in V(G) - V(H)$ such that x is adjacent to a vertex of H, say to v_0. As before, there is a vertex y such that $\{x, y\}$ is a dominating set of G. If x is not adjacent to any other vertex of H, we obtain a contradiction as in the case where $H = C_5$. On the other hand, if x is adjacent to some v_j, $j \in \{1, \ldots, 2k\} - \{2, 2k-1\}$, then G also has an odd cycle C_{2r+1} for $r < k$. Hence assume $xv_2 \in E(G)$. Then x is not adjacent to v_{2k-1}, hence y dominates all of $v_1, v_3, v_4, \ldots, v_{2k}$. As in the case where $H = C_5$, this is impossible.

By induction, $H \ncong C_{2n+1}$ for all $n \geq 2$. Therefore C_5 is the only non-bipartite triangle-free graph G such that $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G) = 2$. ■

5. Trees with $\gamma_{\mathrm{m}}^{\infty} = \theta$

In this section we prove our main result—a characterization of the class of trees T for which $\gamma_{\mathrm{m}}^{\infty}(T) = \theta(T)$. We begin by stating two reductions on trees from [11].

R1: Let x be a stem of T adjacent to $\ell \geq 2$ leaves and to exactly one vertex of degree at least two. Delete all leaves adjacent to x.

R2: Let x be a stem of degree two in T such that x is adjacent to exactly one leaf, y. Delete both x and y.

Lemma 12 [11]**.** *If T' is the result of applying reduction $R1$ or $R2$ to the tree T, then T' is a tree and $\gamma_{\mathrm{m}}^{\infty}(T) = 1 + \gamma_{\mathrm{m}}^{\infty}(T')$.*

It is shown in [11] that one can repeatedly apply these reductions, reducing T to a star $K_{1,r}$, $r \geq 1$, in such a way as to compute $\theta_c(T) = \gamma_{\mathrm{m}}^{\infty}(T)$. The characterization of trees with equal clique covering and m-eternal domination numbers follows.

Theorem 13. *Let T be a tree with at least two vertices. Then $\gamma_{\mathrm{m}}^{\infty}(T) = \theta(T)$ if and only if the reduction R2 can be applied repeatedly to T to obtain a star $K_{1,r}$, $r \in \{1, 2\}$.*

Proof. Suppose first that $T = K_{1,r}$, $r \geq 1$. Then either $T = K_2$ and $\gamma_{\mathrm{m}}^{\infty}(T) = \theta(T) = 1$, or $r \geq 2$, $\gamma_{\mathrm{m}}^{\infty}(T) = 2$ and $\theta(T) = r$, hence $\gamma_{\mathrm{m}}^{\infty}(T) = \theta(T) = 2$ if and only if $r = 2$ and thus $T = K_{1,2}$. Hence the theorem holds for stars. Assume the theorem holds for all trees of order less than n, where $n \geq 4$, and let T be a tree of order n. We may assume that T is not a star.

First assume that T can be reduced to K_2 or $K_{1,2}$ by repeatedly applying R2. Since T is not a star, T has a stem x of degree two that is adjacent to exactly one leaf, say y, such that $T' = T - \{x, y\}$ is either K_2, $K_{1,2}$ or can be reduced to one of these trees by repeatedly applying R2. By the induction hypothesis, $\gamma_{\mathrm{m}}^{\infty}(T') = \theta(T')$. By Lemma 12, $\gamma_{\mathrm{m}}^{\infty}(T) = 1 + \gamma_{\mathrm{m}}^{\infty}(T')$, and obviously $\theta(T) = \theta(T') + 1$, so that $\gamma_{\mathrm{m}}^{\infty}(T) = \theta(T)$.

Conversely, assume T cannot be reduced to K_2 or $K_{1,2}$ by repeatedly applying R2. Apply R2 to T repeatedly until a tree $T' \notin \{K_2, K_{1,2}\}$ is obtained to which R2 cannot be applied; say R2 is applied k times to obtain T'. By Lemma 12 applied k times, $\gamma_{\mathrm{m}}^{\infty}(T') = \gamma_{\mathrm{m}}^{\infty}(T) - k$. Similarly, each application of R2 reduces the clique partition number by 1, thus $\theta(T') = \theta(T) - k$. Therefore, if we can show that $\gamma_{\mathrm{m}}^{\infty}(T') < \theta(T')$, it will follow that $\gamma_{\mathrm{m}}^{\infty}(T) < \theta(T)$ and the proof will be complete. The remainder of the proof shows that $\theta_c(T') < \theta(T')$.

If T' is a star, then $T' = K_{1,r}$, $r \geq 3$, and $\theta_c(T') = \gamma_{\mathrm{m}}^{\infty}(T') = 2 < r = \theta(T')$. Hence assume T' is not a star. Since R2 cannot be performed on T', each stem of T' is a branch vertex and T' has at least two branch vertices, hence at least two end-branch vertices. Moreover, each end-branch vertex v is adjacent to $\deg v - 1$ leaves and one non-leaf vertex of T'. Note that each clique partition of T' is a neo-colonization. Consider a minimum clique partition $\Theta = \{U_0, \ldots, U_{\theta-1}\}$ of T' (thus each U_i induces a K_1 or a K_2). We show that there exists a neo-colonization $\mathcal{P}$ of T' with $w(\mathcal{P}) < w(\Theta)$. The result $\theta_c(T') < \theta(T')$ then follows.

Suppose T' has a stem x adjacent to leaves ℓ_1 and ℓ_2 such that $\{\ell_i\}$ is a part of Θ for $i = 1, 2$; without loss of generality say $U_i = \{\ell_i\}$, $i = 1, 2$. See Figure 2. Since Θ is a minimum clique cover, there exists $y \in N(x) - \{\ell_1, \ell_2\}$ such that $\{x, y\}$ is a part of Θ; say $U_0 = \{x, y\}$. Then $w(U_i) = 1$, $i = 0, 1, 2$. Let $U = \bigcup_{i=0}^{2} U_i$ and note that $T'[U] = K_{1,3}$. Let $\mathcal{P}$ be the neo-colonization of T' defined by $\mathcal{P} = (\Theta - \{U_0, U_1, U_2\}) \cup \{U\}$ and note that $w(U) = \gamma_c(T'[U]) + 1 = 2$. Then $w(\mathcal{P}) = w(\Theta) - 3 + 2 = w(\Theta) - 1 < \theta(T')$ and we are done. Hence we may assume that each stem of T' is adjacent to at most one leaf ℓ such that $U_i = \{\ell\}$ for some i. In particular, each end-branch vertex x has degree three and is adjacent to leaves x_1, x_2 such that (say) $\{x_1\}$ and $\{x, x_2\}$ are parts of Θ.

Let x and y be two end-branch vertices of T', with x_1 and x_2 as above, and let y_1, y_2 be the leaves adjacent to y such that $\{y_1\}$ and $\{y, y_2\}$ are parts of Θ. Let

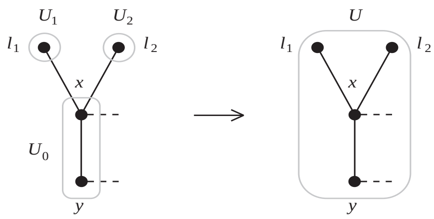

Figure 2. $w(U_0) + w(U_1) + w(U_2) = 3$ and $w(U) = \gamma_c(K_{1,3}) + 1 = 2$.

$Q' : x_1 = v_0, \ldots, v_{t'} = y_1$ be the x_1-y_1 path in T'. (Thus $v_1 = x$ and $v_{t'-1} = y$.) With respect to Q', we consider three types of parts U_i of Θ: a K_1-part $\{u\}$, where $u \in V(Q')$, a part $\{u, u'\}$, where $u, u' \in V(Q')$, which we refer to as a K_2-part, and a part $\{u, u'\}$, where $\{u, u'\} \cap V(Q') = \{u\}$, which we refer to as a P_2-part. Since $\{v_{t'}\}$ is a K_1-part on Q', there exists a smallest integer t, $1 \le t \le t'$, such that $\{v_t\}$ is a K_1-part on Q'. Let $Q : v_0, \ldots, v_t$ be the v_0-v_t subpath of Q'. Note that $\{x, x_2\}$ is a P_2-part. Therefore the parts $\Omega = \{U_i : U_i \cap V(Q) \neq \emptyset\}$ of Θ form a sequence that consists of a K_1-part $\{v_0\} = \{x_1\}$, followed by a number of P_2 parts, followed (possibly) by a number of K_2-parts, then P_2-parts, and so on, finally ending in the K_1-part $\{v_t\}$. We can therefore define a sequence of positive integers $s_1, s_2, \ldots, s_k$ such that the part $\{v_0\}$ is followed by s_1 P_2-parts, the last of which is followed by s_2 K_2-parts, then s_3 P_2-parts, and so on, until the final s_k K_2- or P_2-parts are followed by $\{v_t\}$. See the top graph in Figure 3. Let $\omega = w(\Omega)$. Since each part of Θ is assigned a weight of one when Θ is considered as a neo-colonization,

$$\omega = w(\Omega) = 2 + \sum\nolimits_{i=1}^{k} s_i. \tag{1}$$

We may assume that the parts of Θ that belong to Ω are labeled $U_0 = \{v_0\}, U_1 = \{v_1, x_2\}, \ldots, U_{s_1}, U_{s_1+1}, \ldots, U_{s_1+s_2}, \ldots, U_\omega = \{v_t\}$, in order of their occurrence on Q. Thus $U_1, \ldots, U_{s_1}$ are P_2-parts, $U_{s_1+1}, \ldots, U_{s_1+s_2}$ are K_2 parts, and so on. Let S' be the subgraph of T' induced by $\bigcup_{i=0}^{\omega} U_i$. Since Θ is a clique cover of T and each vertex of Q is contained in a set U_i, $i = 0, \ldots, \omega$, S' is a tree. We define a neo-colonization $\mathcal{P}' = \{V_1, \ldots, V_r\}$ of S' as follows.

As illustrated in Figure 3, we combine each subsequence of consecutive P_2-parts with the last vertex of Q preceding and the first vertex of Q following this subsequence into one part. We also combine the second vertex of each K_2-part with the first vertex of the next K_2-part to form new K_2-parts, ending with v_t belonging to either a K_2-part or a part containing P_2-parts. In order to calculate the weight of $\mathcal{P}'$, we describe the process more formally.

- Let V_1 consist of $\bigcup_{j=0}^{s_1} U_j$ together with the first vertex of Q that belongs to U_{s_1+1}. Then $S'[V_1]$ is connected and $\gamma_c(S'[V_1]) = s_1$.

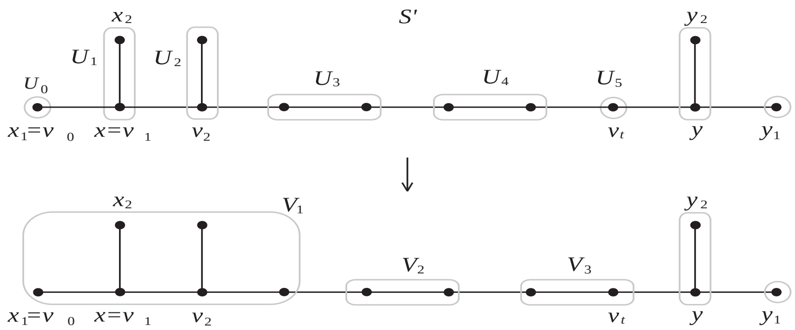

Figure 3. $\sum_{i=0}^{5} w(U_i) = 6$ and $\sum_{i=1}^{3} w(V_i) = 5$.

- For $i = 2, \ldots, s_2$, let V_i consist of the second vertex of Q that belongs to U_{s_1+i-1} and the first vertex of Q that belongs to U_{s_1+i}; each such V_i is a K_2-part.
- If v_t has not been reached above, let V_{s_2+1} consist of $\bigcup_{j=s_2+1}^{s_3} U_j$ together with the last vertex of Q that belongs to U_{s_2} and the first vertex of Q that belongs to U_{s_3+1}. Then $S'[V_{s_2+1}]$ is connected and $\gamma_c(S'[V_{s_2+1}]) = s_3$.
- Continue by splitting and recombining the next K_2-parts, if necessary.
- Finally, V_r either consists of v_t and the last vertex of $U_{\omega-1}$, if $U_{\omega-1}$ is a K_2-part, or of the union of the last s_k consecutive P_2-parts of Ω on Q, together with v_t and the last vertex of Q that belongs to $U_{s_1+\cdots+s_{\omega-2}}$, otherwise.

The sets V_i are mutually disjoint, each $S'[V_i]$ is connected and $\bigcup_{i=1}^{r} V_i = V(S')$. Hence $\mathcal{P}'$ is a neo-colonization of S'. The weight $w(\mathcal{P}')$ is calculated as follows. If V_i contains s_j P_2-parts, then $w(V_i) = \gamma_c(S'[V_i]) + 1 = s_j + 1$. Each such V_i, $i \neq r$, is followed by $s_{j+1} - 1$ K_2-parts of $\mathcal{P}'$. Therefore, if V_r is a K_2-part of $\mathcal{P}'$, then k is even, $w(V_r) = 1$ and by (1)

$$w(\mathcal{P}') = (s_1+1)+(s_2-1)+\cdots+(s_{k-1}+1)+(s_k-1)+1 = 1+\sum_{i=1}^{k} s_i = w(\Omega)-1,$$

and if V_r contains P_2-parts of Ω, then k is odd and, again using (1),

$$\begin{aligned} w(\mathcal{P}') &= (s_1 + 1) + (s_2 - 1) + \cdots + (s_{k-1} - 1) + (s_k + 1) = 1 + \sum_{i=1}^{k} s_i \\ &= w(\Omega) - 1. \end{aligned}$$

Let $\mathcal{P} = \mathcal{P}' \cup \{U_i \in \Theta : U_i \cap V(S') = \emptyset\}$. Then $\mathcal{P}$ is a neo-colonization of T' and

$$w(\mathcal{P}) = w(\mathcal{P}') + w(\Theta - \Omega) \leq w(\Omega) - 1 + w(\Theta) - w(\Omega) < w(\Theta).$$

Therefore $\theta_c(T') < \theta(T')$, hence $\gamma_{\mathrm{m}}^{\infty}(T) = \theta_c(T') < \theta(T')$. ■

The next result follows immediately from Theorem 13.

Corollary 14. *If T is a tree with at least two vertices, then $\gamma_{\mathrm{m}}^{\infty}(T) = \theta(T)$ if and only if T can be obtained from K_2 or P_3 by successively adding a new K_2, joining one of its leaves to any vertex of the previously constructed tree.*

6. Clique Covering Numbers of Graphs with $\gamma = \gamma^{\infty}$

There are many graphs with $\gamma(G) = \theta(G)$, including C_4, and two K_n's connected by one edge, though the two parameters may also differ by any arbitrary amount, for example in $K_{1,m}$. There does not exist a meaningful characterization of the graphs G with $\gamma(G) = \theta(G)$ and this complicates the issue of characterizing graphs with $\gamma^{\infty}(G) = \theta(G)$. The results of this section are motivated by an error discovered in [11], where it was claimed that if $\gamma(G) = \gamma^{\infty}(G)$, then $\gamma(G) = \theta(G)$. The proof given in [11] is incorrect, as the initial set of cliques consisting of the vertices in dominating set D and their private neighbors cannot, in fact, be extended to other vertices of G. We determine two classes of graphs G such that $\gamma(G) = \gamma^{\infty}(G) = \theta(G)$. The following fact was proved in [11], and will be needed below.

Fact 15. *Let D be an EDS of a graph G. For each $v \in D$, $G[\{v\} \cup \mathrm{epn}(v, D)]$ is a clique, and if $v \in D$ defends $u \in V(G) - D$, then $G[\{u, v\} \cup \mathrm{epn}(v, D)]$ is a clique.*

As shown in [2], every graph without isolated vertices has a minimum dominating set D such that $\mathrm{epn}(v, D) \neq \emptyset$ for each $v \in D$. A similar result does not hold for minimum EDS's—consider P_3, for example. We now prove a corresponding result under restricted conditions. If D is an EDS of a graph G, and $w \in V(G) - D$ is adjacent to more than one vertex in D, we say that w is a *shared vertex*.

Lemma 16. *If G is a graph without isolated vertices such that $\gamma(G) = \gamma^{\infty}(G)$ and $\Delta(G) \leq 3$, then G has a minimum EDS D such that $\mathrm{epn}(v, D) \neq \emptyset$ for each $v \in D$.*

Proof. Let D be a minimum EDS of G that maximizes the number of edges in $G[D]$. We first show that

(1) If $u \in D$ and $\mathrm{epn}(u, D) = \emptyset$, then u does not defend any vertex of $G - D$.

Suppose $u \in D$ and $\mathrm{epn}(u, D) = \emptyset$. Since $\gamma(G) = \gamma^{\infty}(G)$, D is a minimum dominating set, hence u is isolated in $G[D]$ (because $\mathrm{pn}(u, D) \neq \emptyset$). Suppose, to the contrary, that u defends $w \in V(G) - D$. Then $D' = (D - \{u\}) \cup \{w\}$ is an EDS. Moreover, w is adjacent to a vertex in D', so that $G[D']$ has more edges than $G[D]$, contrary to the choice of D.

Now we show that

(2) Each $w \in V(G) - D$ is adjacent to at most two vertices in D.

Suppose $w \in V(G) - D$ is adjacent to more than two vertices in D. Since $\Delta(G) \leq 3$, w is adjacent to exactly three vertices $v_1, v_2, v_3 \in D$ and nonadjacent to all external private neighbors of v_i, $i = 1, 2, 3$. But D is an EDS, and some $v \in \{v_1, v_2, v_3\}$ defends w. By Fact 15, $\mathrm{epn}(v, D) = \emptyset$. This contradicts (1).

We also show that

(3) If $u, v \in D$, $\mathrm{epn}(u, D) = \emptyset$, and u and v have a shared neighbor in $G - D$, then they have exactly two shared neighbors in $G - D$.

Suppose $N(u) \cap N(v) \cap (V - D) = \{w\}$. By (2), $N(w) \cap D = \{u, v\}$. Since D is an EDS and u does not defend w by (1), v defends w, $\mathrm{epn}(v, D) \neq \emptyset$, and w is adjacent to each vertex in $\mathrm{epn}(v, D)$ (Fact 15). But then $(D - \{u, v\}) \cup \{w\}$ dominates G, a contradiction because D is a minimum dominating set. On the other hand, suppose $N(u) \cap N(v) \cap (V - D) = \{w_1, w_2, w_3\}$. Then $N(u) \cap N(v) = \{w_1, w_2, w_3\}$ because $\Delta(G) \leq 3$, hence $\mathrm{epn}(v, D) = \mathrm{epn}(u, D) = \emptyset$, and by (1), neither u nor v defends w_i, $i = 1, 2, 3$. But by (2), $N(w_i) \cap D = \{u, v\}$ and so no vertex in D defends w_i, a contradiction.

Now consider $u \in D$ such that $\mathrm{epn}(u, D) = \emptyset$. As in the proof of (1), u is isolated in $G[D]$. Since $\delta(G) \geq 1$, u has at least one neighbor in $G - D$. By (3) there exists $v \in D$ such that $N(u) \cap N(v) \cap (V - D) = \{w_1, w_2\}$, say. As in the proof of (3), v defends w_1 and w_2, and $\mathrm{epn}(v, D) \neq \emptyset$. Now v is adjacent to three vertices of $G - D$, hence v is isolated in $G[D]$. Since v defends w_1, $D'' = (D - \{v\}) \cup \{w_1\}$ is an EDS. However, w_1 is adjacent to u in D'', which implies that $G[D'']$ has more edges than $G[D]$, a contradiction. ■

We use Fact 15 and Lemma 16 to prove the main result of this section.

Theorem 17. *Let G be a graph with $\gamma(G) = \gamma^{\infty}(G)$ and $\Delta(G) \leq 3$. Then $\gamma^{\infty}(G) = \theta(G)$.*

Proof. We may assume without loss of generality that G has no isolated vertices. Let D be a minimum EDS of G such that $\mathrm{epn}(v, D) \neq \emptyset$ for each $v \in D$; such an EDS exists by Lemma 16. If $\gamma(G) = \gamma^{\infty}(G) = 1$, then G is complete and the statement holds. Hence we assume $\gamma(G) > 1$.

If each vertex of $G - D$ is an external private neighbor of a vertex in D, then, by Fact 15, $\{\{x\} \cup \mathrm{epn}(x, D) : x \in D\}$ is a clique cover of G and the result follows. Hence assume some vertex of $G - D$ is a shared vertex. For each $x \in D$, let S_x denote the set of shared vertices defended by x. If $|S_x| \leq 1$ for each $x \in D$, then

$R_x = \{x\} \cup S_x \cup \operatorname{epn}(x, D)$ forms a clique (Fact 15) and $\{R_x : x \in D\}$ is a clique partition of G into $\gamma(G)$ parts.

Therefore we assume that $w, w' \in S_u$ for some $u \in D$. Say w and w' are also adjacent to v and v', respectively, where possibly $v = v'$. Let $y \in \operatorname{epn}(v', D)$ and $z \in \operatorname{epn}(u, D)$. By Fact 15, w and w' are adjacent to z. Since $\Delta(G) \leq 3$, $N(w) = \{u, v, z\}$ and $N(w') = \{u, v', z\}$; note that w, w' are not adjacent to each other or to y. Since u defends w, $D' = (D - \{u\}) \cup \{w\}$ is an EDS, and $\{w', y\} \subseteq \operatorname{epn}(v', D')$. Since w' is not adjacent to y, this contradicts Fact 15. ∎

Corollary 18. *Let G be a triangle-free graph such that $1 \leq \delta(G) \leq \Delta(G) \leq 3$. Then $\gamma(G) = \gamma^\infty(G)$ if and only if $\gamma(G) = n/2$.*

Proof. Since G has no isolated vertices, $\gamma(G) \leq n/2$. Suppose $\gamma(G) = \gamma^\infty(G)$. By Theorem 17, $\theta(G) = \gamma(G)$, and since G is triangle-free, $\theta(G) \geq n/2$. Conversely, suppose $\gamma(G) = n/2$ and let D be a minimum dominating set such that $\operatorname{epn}(v, D) \neq \emptyset$ for each $v \in D$. Then $|\operatorname{epn}(v, D)| = 1$ for each $v \in D$; say $\operatorname{epn}(v, D) = \{v'\}$. Then $\mathcal{P} = \{\{v, v'\} : v \in D\}$ is a clique partition of G. Since G is triangle-free, $\mathcal{P}$ is a minimum clique partition and so $\theta(G) = n/2$. ∎

The graphs with $\gamma = n/2$ are known; they are coronas or unions of 4-cycles, see [8]. If the corona of H is triangle-free, then so is H. Thus a connected triangle-free graph G such that $\Delta(G) \leq 3$ satisfies $\gamma(G) = \gamma^\infty(G)$ if and only if $G = C_4$, or G is the corona of P_n, $n \geq 1$, or of C_n, $n \geq 4$. We improve this result for triangle-free graphs. Again we need a lemma about the existence of an EDS in which every vertex has an external private neighbor.

Lemma 19. *If G is a triangle-free graph without isolated vertices such that $\gamma^\infty(G) = \gamma(G)$, then G has a minimum EDS D such that $\operatorname{epn}(v, D) \neq \emptyset$ for each $v \in D$.*

Proof. Let D be a minimum EDS of G that maximizes the number of edges in $G[D]$. Suppose $\operatorname{epn}(u, D) = \emptyset$ for some $u \in D$. Since D is a minimum dominating set, u is isolated in $G[D]$. Since $\deg u \geq 1$, u is adjacent to a shared vertex w. If u defends w, then $D' = (D - \{u\}) \cup \{w\}$ is an EDS such that $G[D']$ has more edges than $G[D]$, a contradiction. Therefore w is defended by $v \in D$ such that $\operatorname{epn}(v, D) \neq \emptyset$. By Fact 15, $G[\{v, w\} \cup \operatorname{epn}(v, D)] \cong K_n$ for some $n \geq 3$, which is impossible in a triangle-free graph. ∎

Theorem 20. *Let G be a triangle-free graph with $\gamma^\infty(G) = \gamma(G)$. Then $\gamma^\infty(G) = \theta(G)$.*

Proof. Assume without loss of generality that G has no isolated vertices and let D be a minimum EDS such that $\operatorname{epn}(v, D) \neq \emptyset$ for each $v \in D$; such a set D exists by Lemma 19. By Fact 15, $\{v\} \cup \operatorname{epn}(v, D)$ forms a clique. Since G is triangle-free, $|\operatorname{epn}(v, D)| = 1$ for each $v \in D$; say $\operatorname{epn}(v, D) = \{v'\}$. Let C be the set of all shared vertices. If $C = \emptyset$, then we are done, so assume $C \neq \emptyset$; say $w \in C$. Since D is an EDS, w is defended by some vertex $v \in D$. But then Fact 15 implies that w is adjacent to v', that is, $\{v, v', w\}$ forms a triangle, a contradiction. ∎

7. Open Problems

We consider Questions 21 and 22 to be fundamental questions in the study of eternal domination.

Question 21. *Does there exist a graph G such that $\gamma(G) = \gamma^{\infty}(G)$ and $\gamma(G) < \theta(G)$?*

Question 22. *Does there exist a triangle-free graph G such that $\gamma^{\infty}(G) = \alpha(G) < \theta(G)$?*

We do not know of similar questions to Questions 21 and 22 in the m-eternal domination problem. For example, $\gamma(C_n) = \gamma_{\mathrm{m}}^{\infty}(C_n) = \alpha(C_n) < \theta(C_n)$ when $n \in \{5, 7\}$ (and, of course, C_n is triangle-free for $n > 3$).

There exist triangle-free graphs G with $\theta(G) = \gamma^{\infty}(G)$ and $\alpha(G) < \theta(G)$; C_5 is one example. Infinitely many graphs that are not triangle-free with the property that $\alpha(G) = \gamma^{\infty}(G) < \theta(G)$ are described in [11], as well as graphs with $\alpha(G) < \gamma^{\infty}(G) < \theta(G)$. It remains open to characterize all graphs having $\gamma(G) = \gamma^{\infty}(G)$.

Question 23. *Is it true for all planar graphs G that $\gamma(G) = \gamma^{\infty}(G)$ implies $\gamma(G) = \theta(G)$?*

Determining additional classes of graphs for which $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G)$, $\gamma_{\mathrm{m}}^{\infty}(G) = \alpha(G)$, or $\gamma_{\mathrm{m}}^{\infty}(G) = \theta(G)$ is also an interesting direction for future work. As mentioned in Section 3, if $\alpha(G) = 2$, then $\gamma_{\mathrm{m}}^{\infty}(G) = 2$, and if $\alpha(G) = 3$, then $\gamma_{\mathrm{m}}^{\infty}(G) \in \{2, 3\}$. Proposition 7 gives a sufficient condition for $\gamma_{\mathrm{m}}^{\infty}(G)$ to equal 2 while $\alpha(G) = 3$. The following problem could be a starting point for an investigation into graphs that satisfy $\gamma_{\mathrm{m}}^{\infty}(G) = \alpha(G)$.

Problem 24. *Characterize the class of graphs G such that $2 = \gamma_{\mathrm{m}}^{\infty}(G) < \alpha(G) = 3$ (equivalently $\gamma_{\mathrm{m}}^{\infty}(G) = \alpha(G) = 3$).*

Sixty one Cayley graphs of nonabelian groups for which $\gamma_{\mathrm{m}}^{\infty}(G) = \gamma(G) + 1$ were discovered by Braga *et al.* in [3]. Disjoint unions of these graphs give examples of Cayley graphs for which the difference $\gamma_{\mathrm{m}}^{\infty}(G) - \gamma(G)$ can be an arbitrary positive integer, but at present there is no similar result for connected Cayley graphs.

Question 25. *Does there exist a connected Cayley graph G such that $\gamma_{\mathrm{m}}^{\infty}(G) > \gamma(G)+1$? Can the difference $\gamma_{\mathrm{m}}^{\infty}(G) - \gamma(G)$ be arbitrary for connected Cayley graphs?*

Problem 26. *Find an infinite class of connected Cayley graphs such that $\gamma_{\mathrm{m}}^{\infty}(G) > \gamma(G)$.*

The next question relates to Fact 5 and Proposition 6.

Question 27. *For $k \geq 3$, which graphs G satisfy $\gamma^{\infty}(G) = k$, or $\gamma_{\mathrm{m}}^{\infty}(G) = k$, if and only if every vertex of G is in a dominating set of size k?*

Let $G \square H$ denote the Cartesian product of G and H. An interesting conjecture is that of Finbow and Klostermeyer [13], who conjectured there exists a constant c such that $\gamma_{\mathrm{m}}^{\infty}(P_n \square P_n) \leq \gamma(P_n \square P_n) + c$, for all n. We state another conjecture.

Conjecture 28. *Let G be a graph such that $\theta(G) = \gamma^{\infty}(G)$. Then $\theta(G \square K_2) = \gamma^{\infty}(G \square K_2)$.*

Perhaps Conjecture 28 is also true if K_2 is replaced with any tree. Similar statements for $\gamma_{\mathrm{m}}^{\infty}(G)$ do not seem to be true. For example, let G be a graph such that $\gamma(G) = \gamma_{\mathrm{m}}^{\infty}(G)$. In many cases, $\gamma(G \square K_2) = \gamma_{\mathrm{m}}^{\infty}(G \square K_2)$. But $\gamma(K_{2,3} - e \square K_2) = 3 < \gamma_{\mathrm{m}}^{\infty}(K_{2,3} - e \square K_2) = 4$. Likewise, if we replace γ with θ in this, we find the following example: $\theta(C_4 \square K_2) = 4 > \gamma_{\mathrm{m}}^{\infty}(C_4 \square K_2) = 3$.

One might consider Vizing-like conjectures by asking whether $\gamma_{\mathrm{m}}^{\infty}(G \square H) \geq \gamma_{\mathrm{m}}^{\infty}(G) \cdot \gamma_{\mathrm{m}}^{\infty}(H)$, for all G, H. But this is not true in general, as $\gamma_{\mathrm{m}}^{\infty}(P_3 \square P_3) = 3 < \gamma_{\mathrm{m}}^{\infty}(P_3) \cdot \gamma_{\mathrm{m}}^{\infty}(P_3) = 4$. A proof that $\gamma_{\mathrm{m}}^{\infty}(P_3 \square P_3) = 3$ can be found in [7]. Perhaps $\gamma_{\mathrm{m}}^{\infty}(G \square H) \geq \max\{\gamma_{\mathrm{m}}^{\infty}(G) \cdot \gamma(H), \gamma(G) \cdot \gamma_{\mathrm{m}}^{\infty}(H)\}$, for all G, H?

However, the Vizing-like problem for eternal domination seems challenging.

Question 29. *Is it true for all graphs G, H that $\gamma^{\infty}(G \square H) \geq \gamma^{\infty}(G) \cdot \gamma^{\infty}(H)$?*

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