TOTAL DOMINATION MULTISUBDIVISION
NUMBER OF A GRAPH

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Abstract

The domination multisubdivision number of a nonempty graph G was defined in [3] as the minimum positive integer k such that there exists an edge which must be subdivided k times to increase the domination number of G. Similarly we define the *total domination multisubdivision number* $\mathrm{msd}_{\gamma_t}(G)$ of a graph G and we show that for any connected graph G of order at least two, $\mathrm{msd}_{\gamma_t}(G) \leq 3$. We show that for trees the total domination multisubdivision number is equal to the known total domination subdivision number. We also determine the total domination multisubdivision number for some classes of graphs and characterize trees T with $\mathrm{msd}_{\gamma_t}(T) = 1$.

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1. Introduction

In this paper we consider connected graphs with $n \geq 2$ vertices and we use $V = V(G)$ and $E = E(G)$ for the vertex set and the edge set of a graph G. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v, the *closed neighbourhood* $N_G[v]$ of a vertex $v \in V(G)$ is $N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = |N_G(v)|$. A vertex v is called *universal* if $d_G(v) = n - 1$.

The *distance* between two vertices a and b, denoted by $d_G(a, b)$, is the length of the shortest ab-path in G. For a subset of vertices $X \subseteq V(G)$, the distance $d(a, X) = \min\{d(a, x) : \ x \in X\}$. The *diameter* $\operatorname{diam}(G)$ of a connected graph G is the maximum distance between two vertices of G.

We say that a vertex v of a graph G is an *end vertex* or a *leaf* if v has exactly one neighbour in G. We denote the set of all leaves in G by $\Omega(G)$. A vertex v is called a *support vertex* if it is adjacent to a leaf. If v is adjacent to more than one leaf, then we call v a *strong support vertex*. The edge incident with a leaf is called a *pendant edge*, in the other case we call it an *inner edge*.

The *private neighbourhood of a vertex u with respect to a set $D \subseteq V(G)$*, where $u \in D$, is the set $PN_G[u, D] = N_G[u] - N_G[D - \{u\}]$. If $v \in PN_G[u, D]$, then we say that v is a *private neighbour* of u with respect to the set D.

A subset D of $V(G)$ is *dominating* in G if every vertex of $V(G) - D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G. A dominating set D in G with $|D| = \gamma(G)$ is called a $\gamma(G)$-set or a *minimum dominating set* of G .

For a graph $G = (V, E)$, subdivision of the edge $e = uv \in E$ with vertex x leads to a graph with vertex set $V \cup \{x\}$ and edge set $(E - \{uv\}) \cup \{ux, xv\}$. Let $G_{e,t}$ denote the graph G with subdivided edge e with t vertices (instead of edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_t, v)$). For $t = 1$ we write G_e. The vertices $\{x_1, x_2, \ldots, x_t\}$ are called *subdivision vertices*.

The *domination subdivision number*, $\operatorname{sd}_\gamma(G)$, of a graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the domination number. We consider subdivision number for connected graphs of order at least 3, since the domination number of the graph K_2 does not increase when its only edge is subdivided. The domination subdivision number was defined in [10] and studied for example in [1, 2, 4].

Let G be a connected graph of order at least 2. By $\operatorname{msd}_\gamma(uv)$ we denote the minimum number of subdivisions of the edge uv such that $\gamma(G)$ increases. In [3], the *domination multisubdivision number* of G, denoted by $\operatorname{msd}_\gamma(G)$, was defined, as

$$\operatorname{msd}_\gamma(G) = \min\{\operatorname{msd}_\gamma(uv) : \ uv \in E(G)\}.$$

A set S of vertices in a graph G is a *total dominating* set of G if every vertex of G is adjacent to a vertex in S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. A total dominating set S in G with $|S| = \gamma_t(G)$ is called a $\gamma_t(G)$-set or a *minimum total dominating set* of G. The *total domination subdivision number* $\operatorname{sd}_{\gamma_t}(G)$ of a graph G (defined in [6]) is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number.

Similarly like above we define the total domination multisubdivision number

of a graph G.

Definition 1. Let $\mathrm{msd}_{\gamma_t}(uv)$ be the minimum number of subdivisions of the edge uv such that $\gamma_t(G)$ increases. The *total domination multisubdivision number* of a graph G of order at least 2, denoted by $\mathrm{msd}_{\gamma_t}(G)$, is defined as

$$\mathrm{msd}_{\gamma_t}(G) = \min\{\mathrm{msd}_{\gamma_t}(uv) : \; uv \in E(G)\}.$$

For any unexplained terms see [5].

2. Preliminary Results

In this section we determine the total domination multisubdivision number for some classes of graphs and we prove that for any connected graph G of order at least 2 we have $\mathrm{msd}_{\gamma_t}(G) \leq 3$. Let G be a graph. It is clear that $\mathrm{sd}_{\gamma_t}(G) = 1$ if and only if $\mathrm{msd}_{\gamma_t}(G) = 1$.

We start with the next useful observation.

Observation 1. *If G is not a star, then it is always possible to find a $\gamma_t(G)$-set D such that $D \cap \Omega(G) = \emptyset$.*

In [6], it was shown that for any graph G with adjacent support vertices $\mathrm{sd}_{\gamma_t}(G) = 1$.

Similarly like for the domination subdivision number in [2], we have the next result.

Lemma 2. *If G is a graph with an end vertex not belonging to any $\gamma_t(G)$-set, or if G has an inner edge xy such that neither x nor y is in any $\gamma_t(G)$-set, then $\mathrm{sd}_{\gamma_t}(G) = 1$.*

Proof. Let u be an end vertex not belonging to any $\gamma_t(G)$-set and v its neighbour. Let G' be a graph obtained from G by a subdivision of the edge uv with a vertex w. By Observation 1, there exists a minimum total dominating set D' with no end vertex of G'. Then $v, w \in D'$. The set $(D' - \{w\}) \cup \{u\}$ is a total dominating set of G. Since this set contains u, it is not a minimum total dominating set of G. Thus, $\gamma_t(G) < |(D' - \{w\}) \cup \{u\}| = |D'|$ and $\mathrm{sd}_{\gamma_t}(G) = 1$.

Now suppose that there is an inner edge xy in G such that neither x nor y is in any $\gamma_t(G)$-set. Let G' be a graph obtained by subdividing xy with the vertex w and consider any $\gamma_t(G')$-set D'. If $w \notin D'$, then D' is a total dominating set of G containing x or y and by hypothesis $|D'| > \gamma_t(G)$, so we are done.

Now assume $w \in D'$. Then $D' \cap \{x, y\} \neq \emptyset$. Without loss of generality suppose $x \in D'$. Then $D = (D' - \{w\}) \cup \{y\}$ is a total dominating set of G containing x and y. From the assumption, it cannot be minimum and similarly like before $\gamma_t(G) < |D| \leq |D'|$. ∎

The next lemma gives us a sufficient condition for a graph to have the total domination multisubdivision number equal to two.

Lemma 3. *If there is a universal vertex in a graph G with $n \geq 3$ vertices, then $\mathrm{msd}_{\gamma_t}(G) = 2$.*

Proof. If G has a universal vertex v, then $\gamma_t(G) = 2$. If we subdivide an edge $e = vx$ with a subdivision vertex w, then $D = \{v, w\}$ is a minimum total dominating set of G_e. If $e = yz$ with $v \notin \{y, z\}$, then $D = \{v, y\}$ is a minimum total dominating set of G_e. So, $\mathrm{msd}_{\gamma_t}(G) > 1$. For $e = vx$, $\gamma_t(G_{e,2}) = 3$. Therefore, $\mathrm{msd}_{\gamma_t}(G) = 2$. ∎

Corollary 4. *For a complete graph K_n, a star $K_{1,n-1}$ with $n \geq 3$, and for a wheel W_n with $n \geq 4$, we have*

$$\mathrm{msd}_{\gamma_t}(K_n) = \mathrm{msd}_{\gamma_t}(K_{1,n-1}) = \mathrm{msd}_{\gamma_t}(W_n) = 2.$$

In [8] it was shown that for a cycle C_n and a path P_n, $n \geq 3$, we have

$$\mathrm{sd}_{\gamma_t}(C_n) = \mathrm{sd}_{\gamma_t}(P_n) = \begin{cases} 3 & \text{if } n \equiv 2 \pmod 4, \\ 2 & \text{if } n \equiv 3 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Since the cycle (path) with a subdivided edge k times is isomorphic to the cycle (path) with subdivided k edges once, we immediately obtain the following.

Corollary 5. *For a cycle C_n and a path P_n, $n \geq 3$, we have*

$$\mathrm{msd}_{\gamma_t}(C_n) = \mathrm{msd}_{\gamma_t}(P_n) = \begin{cases} 3 & \text{if } n \equiv 2 \pmod 4, \\ 2 & \text{if } n \equiv 3 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

The main result of this section is the next theorem.

Theorem 6. *For a connected graph G, $\mathrm{msd}_{\gamma_t}(G) \leq 3$.*

Proof. We subdivide an edge $e = uv \in E(G)$ with subdivision vertices x_1, x_2, x_3. Let D^* be a minimun total dominating set of $G_{e,3}$. Since D^* is dominating, it contains at least one subdivision vertex. We consider the next three cases.

Case 1. If $|\{x_1, x_2, x_3\} \cap D^*| = 1$, then $u, v \in D^*$ and $D = D^* - \{x_1, x_2, x_3\}$ is a total dominating set of G with $|D| < |D^*|$.

Case 2. Suppose $|\{x_1, x_2, x_3\} \cap D^*| = 2$. If $u \in D^*$ or $v \in D^*$, then $D = (D^* - \{x_1, x_2, x_3\}) \cup \{u, v\}$ is a total dominating set of G with $|D| < |D^*|$. If $u \notin D^*$ and $v \notin D^*$, then the two subdivision vertices in D^* must be adjacent,

without loss of generality suppose $x_1, x_2 \in D^*$. Then v is dominated by a vertex $z \in D^*$, so $D = D^* - \{x_1, x_2\} \cup \{v\}$ is a total dominating set of G with $|D| < |D^*|$.

Case 3. If $\{x_1, x_2, x_3\} \subset D^*$, then $D = (D^* - \{x_1, x_2, x_3\}) \cup \{u, v\}$ is a total dominating set of G with $|D| < |D^*|$.

In any case, we prove that $\gamma_t(G) \leq |D| < |D^*| = \gamma_t(G_{uv,3})$, which implies that $\mathrm{msd}_{\gamma_t}(G) \leq 3$. ∎
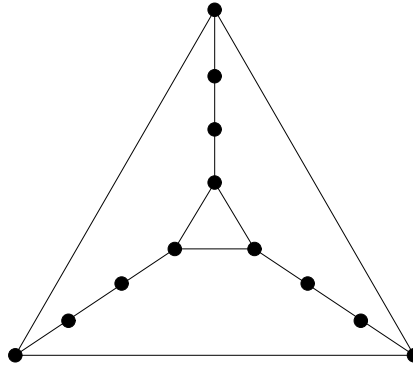

Figure 1. Graph G^*.

In [7] it was proved that for any positive integer k, there exists a graph G such that $\mathrm{sd}_{\gamma_t}(G) = k$. Therefore by the above theorem, in general, the difference between $\mathrm{sd}_{\gamma_t}(G)$ and $\mathrm{msd}_{\gamma_t}(G)$ cannot be bounded by any integer. For small values of sd_{γ_t} ($2 \leq \mathrm{sd}_{\gamma_t}(G) \leq 3$), msd_{γ_t} and sd_{γ_t} are incomparable. For example, for a complete graph K_4 we have $\mathrm{msd}_{\gamma_t}(K_4) = 2$, $\mathrm{sd}_{\gamma_t}(K_4) = 3$. But for the graph G^*, shown in Figure 1, we have $\mathrm{msd}_{\gamma_t}(G^*) = 3$ and $\mathrm{sd}_{\gamma_t}(G^*) = 2$.

3. Total Domination Multisubdivision Number of Trees

Now we consider the total domination multisubdivision number of trees. The main result of this section is the following theorem.

Theorem 7. *For a tree T with $n(T) \geq 3$ we have $\mathrm{sd}_{\gamma_t}(T) = \mathrm{msd}_{\gamma_t}(T)$.*

It was shown by Haynes *et al.* in [6] that the total domination subdivision number of a tree is 1, 2 or 3. The class of trees T with $\mathrm{sd}_\gamma(T) = 3$ was characterized in [8].

Since $\mathrm{sd}_{\gamma_t}(G) = 1$ if and only if $\mathrm{msd}_{\gamma_t}(G) = 1$, in order to prove Theorem 7 it suffices to show that for any tree T of order at least three,

$$\mathrm{sd}_{\gamma_t}(T) = 3 \text{ if and only if } \mathrm{msd}_{\gamma_t}(T) = 3.$$

3.1. Trees with the total domination multisubdivision number equal to 3

The following constructive characterization of the family $\mathcal{F}$ of labeled trees T with $\mathrm{sd}_{\gamma_t}(T) = 3$ was given in [8]. The label of a vertex v is also called the status of v and is denoted by $sta(v)$.

Let $\mathcal{F}$ be the family of labelled trees such that:

- $\mathcal{F}$ contains P_6 where the two leaves have status C, the two support vertices have status B, and the two central vertices have status A; and

- $\mathcal{F}$ is closed under the two operations O_1 and O_2, which extend the tree T by attaching a path to a vertex $y \in V(T)$.

    1. Operation O_1. Assume $sta(y) = A$. Then add a path (x, w, v) and the edge xy. Let $sta(x) = A$, $sta(w) = B$, and $sta(v) = C$.

    2. Operation O_2. Assume $sta(y) \in \{B, C\}$. Then add a path (x, w, v, u) and the edge xy. Let $sta(x) = sta(w) = A$, $sta(v) = B$ and $sta(u) = C$.

In [8] the following observation and theorem was proved.

Observation 8. *If $T \in \mathcal{F}$, then $\mathcal{B} \cup \mathcal{C}$ is a minimum total dominating set of T, where $\mathcal{B}$ and $\mathcal{C}$ are sets of vertices with status B and C, respectively.*

Theorem 9. *For a tree T, $\mathrm{sd}_{\gamma_t}(T) = 3$ if and only if $T \in \mathcal{F}$.*

Operations O_1 and O_2 will be called the *basic operations*. If S is a basic operation of type O_1 or O_2, then denote by V_S and E_S the set of vertices and the set of edges appearing as a result of using the operation S.

Observation 10. *Let $T \in \mathcal{F}$ and S, S' be two basic operations. Consider $S'(S(T))$; if the path added by S' is attached to a vertex $v \in V(T)$, then $S'(S(T)) = S(S'(T))$.*

Lemma 11. *Let $T \in \mathcal{F}$ with $|V(T)| > 6$. Then there exist $T', T'' \in \mathcal{F}$ and basic operations S', S'' such that $T = S'(T') = S''(T'')$ and $V_{S'} \cap V_{S''} = \emptyset$. Additionally, $E_{S'} \cap E_{S''} = \emptyset$.*

Proof. We use induction on n, the number of vertices of T. Any $T \in \mathcal{F}$ with $n > 6$ has at least 9 or 10 vertices. For $n = 9$, $T = S'(T')$ where T' is the path $(v_1, v_2, v_3, v_4, v_5, v_6)$ and S' is the operation of type O_1 of adding a path (x, w, v) attached to vertex v_3; then $T = S''(T'')$ where T'' is the path (v_1, v_2, v_3, x, w, v) and S'' is the operation of type O_1 of adding a path (v_4, v_5, v_6) attached to vertex v_3. Obviously $V_{S'} \cap V_{S''} = \emptyset$. For $n = 10$ we have two cases, $T = S'(T')$ where T' is the path $(v_1, v_2, v_3, v_4, v_5, v_6)$ and S'—the operation of type O_2 of

adding a path (x, w, v, u) attached to vertex v_5; then $T = S''(T'')$ where T'' is the path (v_6, v_5, x, w, v, u) and S''—the operation of type O_2 of adding a path (v_4, v_3, v_2, v_1) attached to vertex v_5. The second case is $T = S'(T')$ where T' is the path $(v_1, v_2, v_3, v_4, v_5, v_6)$ and S' is the operation of type O_2 of adding a path (x, w, v, u) attached to vertex v_6; then $T = S''(T'')$ where T'' is the path (v_5, v_6, x, w, v, u) and S''—the operation of type O_2 of adding a path (v_4, v_3, v_2, v_1) attached to vertex v_5. In both cases, $V_{S'} \cap V_{S''} = \emptyset$.

Let $T \in \mathcal{F}$ with $n > 10$, and suppose the result holds for every tree of $\mathcal{F}$ with less than n vertices. By definition of the family $\mathcal{F}$ we know $T = S(\hat{T})$, for some $\hat{T} \in \mathcal{F}$ and a basic operation S. By induction hypothesis, there exist $T', T'' \in \mathcal{F}$ and basic operations S', S'' such that $\hat{T} = S'(T') = S''(T'')$, $V_{S'} \cap V_{S''} = \emptyset$, and then $T = S(S'(T')) = S(S''(T''))$. The path added by S is attached to a vertex $v \in \hat{T}$, and since $V_{S'} \cap V_{S''} = \emptyset$, v does not belong to both $V_{S'}$ and $V_{S''}$, without loss of generality, $v \notin V_{S''}$, so by Observation 10, $S(S''(T'')) = S''(S(T''))$. Then $T = S(S'(T')) = S''(S(T''))$, with $V_S \cap V_{S''} = \emptyset$.   ∎

With the above result we can prove the next lemma.

Lemma 12. *If T is a tree with* $\mathrm{sd}_{\gamma_t}(T) = 3$, *then* $\mathrm{msd}_{\gamma_t}(T) = 3$.

Proof. From Theorem 9, it is enough to prove that if $T \in \mathcal{F}$, then $\mathrm{msd}_{\gamma_t}(T) = 3$. We prove that for any edge e of $T \in \mathcal{F}$, $\gamma_t(T_{e,2}) = \gamma_t(T)$. We use induction on n, the number of vertices of T.

By Corollary 6, the result is true for a path P_6. Assume that for every tree T' with $n' < n$ vertices belonging to the family $\mathcal{F}$, the equality $\gamma_t(T'_{e,2}) = \gamma_t(T')$ holds for any edge e of T'.

Let $T \in \mathcal{F}$ be a tree with $n > 6$ vertices and let e be any edge of T. Since $T \in \mathcal{F}$, $T = T_j$ and is constructed from P_6 by applying $j - 1$ basic operations. By Lemma 11 we can assume that $e \in E(T_{j-1})$. Since $|V(T_{j-1})| < |V(T_j)|$, from the induction hypothesis, $\gamma_t((T_{j-1})_{e,2}) = \gamma_t(T_{j-1})$. Using Observation 8 we know that $\gamma_t(T) = \gamma_t(T_{j-1}) + 2$.

We consider two cases:

Case 1. If $T = T_j = O_1(T_{j-1})$ then we added a path (x, w, v) to a vertex of T_{j-1} with status A. If D' is a minimum total dominating set of $(T_{j-1})_{e,2}$, then $D_1 = D' \cup \{v, w\}$ is a total dominating set of $T_{e,2}$ with $|D_1| = \gamma_t(T_{j-1}) + 2 = \gamma_t(T)$, so $\gamma_t(T_{e,2}) \leq \gamma_t(T)$. Then $\gamma_t(T_{e,2}) = \gamma_t(T)$.

Case 2. If $T = T_j = O_2(T_{j-1})$ then we added a path (x, w, v, u) to a vertex of T_{j-1} with status B or C. If D' is a minimum total dominating set of $(T_{j-1})_{e,2}$, then $D_1 = D' \cup \{w, v\}$ is a total dominating set of $T_{e,2}$ with $|D_1| = \gamma_t(T_{j-1}) + 2 = \gamma_t(T)$, so $\gamma_t(T_{e,2}) \leq \gamma_t(T)$. Then $\gamma_t(T_{e,2}) = \gamma_t(T)$.   ∎

The next observation and lemmas are necessaries in order to finish the proof of Theorem 7.

Observation 13. *If T is a tree with $\mathrm{msd}_{\gamma_t}(T) = 3$, then T does not have a strong support vertex.*

Proof. Suppose $\mathrm{msd}_{\gamma_t}(T) = 3$ and T has a strong support vertex v adjacent to a leaf u. Let us subdivide the edge $e = uv$ with two vertices a, b and let D' be a minimum total dominating set with no end vertex of $T_{e,2}$. It is clear that $a, b \in D'$. Since v is a support in $T_{e,2}$, $v \in D'$. Hence, $(D' - \{a, b\}) \cup \{u\}$ is a total dominating set in T, what implies $\gamma_t(T) \leq |D'| - 1 < \gamma_t(T_{e,2})$, a contradiction with $\mathrm{msd}_{\gamma_t}(T) = 3$. ∎

Lemma 14. *Let T be a tree with $n > 6$ vertices such that $\mathrm{msd}_{\gamma_t}(T) = 3$. Let $P = (v_0, \ldots, v_l)$ be a longest path of T ($l \geq 5$) and let D be a minimum total dominating set with no end vertex of T. Then*
(1) $d_T(v_1) = d_T(v_2) = 2$,
(2) *v_3 is not a support vertex. Moreover, if $d_T(v_3) > 2$, then outside the path P, only one P_2 path or some P_3 paths may be attached to v_3 and for $T' = T - \{v_0, v_1, v_2\}$, $\gamma_t(T) = \gamma_t(T') + 2$.*

Proof. Let D be a minimum total dominating set with no end vertex of T.

(1) It is clear that $v_1, v_2 \in D$. By Observation 13, $d_T(v_1) = 2$. Suppose $d_T(v_2) > 2$. For the edge $e = v_0 v_1$ consider the tree $T_{e,2}$, where we subdivide e with two vertices a, b. If D' is a minimum total dominating set with no end vertex of $T_{e,2}$, then $a, b \in D'$. If v_2 is a support vertex, then $v_2 \in D'$. If v_2 is not a support vertex, then it is a neighbour of a support vertex of degree two and in this case also $v_2 \in D'$. Then $(D' - \{a, b\}) \cup \{v_1\}$ is a total dominating set of T, a contradiction with $\mathrm{msd}_{\gamma_t}(T) = 3$. Thus, $d_T(v_2) = 2$.

(2) Suppose v_3 is a support vertex adjacent to a leaf y. Consider $T_{e,2}$, where $e = v_3 y$ and denote the two vertices on the subdivided edge by a, b. If D' is a minimum total dominating set with no end vertex of $T_{e,2}$, then $a, b, v_1, v_2 \in D'$. Then $(D' - \{a, b\}) \cup \{v_3\}$ is a total dominating set of T, a contradiction with $\mathrm{msd}_{\gamma_t}(T) = 3$.

Suppose $d_T(v_3) > 2$. If $d_T(v_3, \Omega(T)) = 2$, then v_3 is adjacent to a support vertex x which is a neighbour of a leaf y. By Observation 13, x is not a strong support vertex, if $d_T(x) > 2$ then x belongs to a longest path of T and by (1), $d_T(x) = 2$, a contradiction. Since $\mathrm{msd}_{\gamma_t}(T) = 3$ outside the path P, only one P_2 path may be attached to v_3. Now, if $d_T(v_3, \Omega(T)) = 3$, then there are vertices x, y, z such that $(z, y, x, v_3, \ldots, v_l)$ is a longest path of T and by (1), $d_T(x) = d_T(y) = 2$. Hence, outside the path P, only P_3's may be attached to v_3.

Observe that for any minimum total dominating set with no end vertex D of T, $D - \{v_1, v_2\}$ is a total dominating set of T'. Similarly, for any minimum total dominating set with no end vertex D' of T', $D' \cup \{v_1, v_2\}$ is a total dominating set of T and $\gamma_t(T) \leq \gamma_t(T') + 2$. Therefore, $\gamma_t(T) = \gamma_t(T') + 2$. ∎

As a consequence of the last case, if $d_T(v_3) > 2$, then we can observe that every minimum total dominating set with no end vertex D of T has the form $D = D' \cup \{v_1, v_2\}$, where D' is a minimum total dominating set with no end vertex of T'. Equivalently, every D' has the form $D' = D - \{v_1, v_2\}$.

Lemma 15. *If T is a tree with $\mathrm{msd}_{\gamma_t}(T) = 3$, then $\mathrm{sd}_{\gamma_t}(T) = 3$.*

Proof. From Theorem 9, it is enough to prove that if T is a tree with $\mathrm{msd}_{\gamma_t}(T) = 3$, then T belongs to the family $\mathcal{F}$. We use induction on n, the number of vertices of a tree T. The smallest tree T such that $\mathrm{msd}_{\gamma_t}(T) = 3$ is a path P_6 and $P_6 \in \mathcal{F}$. Assume that every tree T' with less than n vertices such that $\mathrm{msd}_{\gamma_t}(T') = 3$ belongs to the family $\mathcal{F}$.

Let T be a tree with $\mathrm{msd}_{\gamma_t}(T) = 3$ and $n > 6$ vertices. Consider $P = (v_0, \ldots, v_l)$ a longest path of T, $l \geq 5$, and let D be a minimum total dominating set with no end vertex of T.

By Lemma 14, $d_T(v_1) = d_T(v_2) = 2$. So we consider the next two cases.

Case 1. $d_T(v_3) > 2$. By Lemma 14, v_3 is not a support vertex. We have the following subcases.

Subcase 1.1. $d_T(v_3, \Omega(T)) = 2$. By Lemma 14, outside the path P only one P_2 path may be attached to v_3. Let us denote x, y the vertices of that path, where y is a leaf of T. Again by Lemma 14, for $T' = T - \{v_0, v_1, v_2\}$, $\gamma_t(T') = \gamma_t(T) - 2$.

For any $e \in E(T') - \{xy, xv_3\}$, $\gamma_t(T'_{e,2}) = \gamma_t(T_{e,2}) - 2 = \gamma_t(T) - 2 = \gamma_t(T')$. In order to see that also for $e \in \{xy, xv_3\}$, $\gamma_t(T'_{e,2}) = \gamma_t(T')$, we claim that there exists a $\gamma_t(T')$-set D^* with no end vertex such that $v_4 \in D^*$ and $|N_{T'}(v_4) \cap D^*| \geq 2$.

Proof of the claim: Consider $T_{e,2}$, where $e = v_3 v_4$, and denote the two subdivision vertices by a, b. If D' is a minimum total dominating set with no end vertex of $T_{e,2}$, then $\{v_1, v_2, x, v_3\} \subset D'$. If $\{a, b\} \cap D' \neq \emptyset$, then $D = D' - \{a, b\}$ is a total dominating set of T with $|D| < \gamma_t(T_{e,2})$, which is a contradiction with $\gamma_t(T) = \gamma_t(T_{e,2})$. Therefore, there exists $z \in N_{T_{e,2}}(v_4)$, $z \neq b$, such that $\{v_4, z\} \subset D'$, and then $D^* = D' - \{v_1, v_2\}$ is a $\gamma_t(T')$-set with no end vertex such that $v_4 \in D^*$ and $|N_{T'}(v_4) \cap D^*| \geq 2$.

Now, without loss of generality, consider $e = xy$ and subdivision of the edge xy with vertices c, d. We know that $(D^* - \{x, v_3\}) \cup \{c, d\}$ is a total dominating set in $T'_{xy,2}$, so $\gamma_t(T'_{e,2}) = \gamma_t(T')$.

Finally, for any edge $e \in E(T')$ we have $\gamma_t(T') = \gamma_t(T'_{e,2})$. Thus, $\mathrm{msd}_{\gamma_t}(T') = 3$ and from the induction hypothesis $T' \in \mathcal{F}$. Since $sta(v_3) = A$, it is possible to obtain T from T' by Operation O_1. It implies that $T \in \mathcal{F}$.

Subcase 1.2. $d_T(v_3, \Omega(T)) = 3$. Thus, by Lemma 14, outside the path P, only P_3's may be attached to v_3. Let us denote x, y, z the vertices of one of such paths, where z is a leaf of T. Define $T' = T - \{v_0, v_1, v_2\}$.

For any $e \in E(T') - \{xy, yz, xv_3\}$, $\gamma_t(T'_{e,2}) = \gamma_t(T_{e,2}) - 2 = \gamma_t(T) - 2 = \gamma_t(T')$. Since $\mathrm{msd}_{\gamma_t}(T) = 3$ and by Lemma 14, $\gamma_t(T') = \gamma_t(T) - 2$, there exists

a $\gamma_t(T')$-set D^* with no end vertex such that $\{x, y, v_3, v_4\} \subset D^*$ (if not, then $\gamma_t(T_{v_3v_4,2}) > \gamma_t(T)$, a contradiction). It is enough to consider subdivision of the edge yz with vertices a, b. Hence $(D^* - \{x, y\}) \cup \{a, b\}$ is a total dominating set in $T'_{yz,2}$. Finally, for any edge $e \in E(T')$ we have $\gamma_t(T') = \gamma_t(T'_{e,2})$. Thus, $\mathrm{msd}_{\gamma_t}(T') = 3$ and from the induction hypothesis $T' \in \mathcal{F}$. Since $sta(v_3) = A$, it is possible to obtain T from T' by Operation O_1. Hence, $T \in \mathcal{F}$.

Case 2. $d_T(v_3) = 2$. We have two subcases.

Subcase 2.1. $d_T(v_4) = 2$ or $(d_T(v_4) > 2$ and $d_T(v_4, \Omega(T)) \in \{1, 4\})$. It is clear that $v_1, v_2 \in D$ for any minimum total dominting set without end vertex of T. Without lost of generality we can suppose that $v_3 \notin D$. If we consider $T' = T - \{v_0, v_1, v_2, v_3\}$, then $\gamma_t(T') = \gamma_t(T) - 2$ and for any $e \in E(T')$, $\gamma_t(T'_{e,2}) = \gamma_t(T_{e,2}) - 2 = \gamma_t(T) - 2 = \gamma_t(T')$. Thus, $\mathrm{msd}_{\gamma_t}(T') = 3$, from the induction hypothesis $T' \in \mathcal{F}$ and by the definition of the family $\mathcal{F}$, the status of the vertex v_4 is B or C. So T can be obtained from T' by Operation O_2, what implies $T \in \mathcal{F}$.

Subcase 2.2. $d_T(v_4, \Omega(T)) \in \{2, 3\}$. Suppose $d_T(v_4, \Omega(T)) = 2$. Then v_4 is adjacent to a support vertex y. Consider $T_{e,2}$, where $e = v_3v_4$, and denote the two subdivision vertices by a, b. If D' is a minimum total dominating set with no end vertex of $T_{e,2}$, then $v_1, v_2, y, v_4 \in D'$. Since D' is total dominating, there exist $z \in D \cap \{b, v_3\} \neq \emptyset$ such that $D' - \{z\}$ is a total dominating set of T, a contradiction with $\mathrm{msd}_{\gamma_t}(T) = 3$. The case of $d_T(v_4, \Omega(T)) = 3$ is similar. ■

3.2. Trees with the total domination multisubdivision number equal to 1

In [9] we can find a characterization of trees with total domination subdivision number equal to one. In this section we give a different characterization of trees T of order at least three with $\mathrm{sd}_{\gamma_t}(T) = \mathrm{msd}_{\gamma_t}(T) = 1$. In order to prove the main Theorem 18 we need the next technical lemmas.

Lemma 16. *Let T be a tree of order $n \geq 3$ such that*

(1) *for any end-vertex u there exists a $\gamma_t(T)$-set D such that $u \in D$ and*

(2) *for any inner edge uv there is a $\gamma_t(T)$-set D such that*
   (a) *$|\{u, v\} \cap D| = 1$, say $u \in D$, and $v \notin PN_T[u, D]$ or*
   (b) *$|\{u, v\} \cap D| = 2$ and at least one of the following conditions holds:*
      (b1) *$|N_T(u) \cap D| \geq 2$ and $|N_T(v) \cap D| \geq 2$;*
      (b2) *$N_T(u) \cap D = \{v\}$ and $(PN_T[u, D] = \emptyset$ or $(PN_T[v, D] = \emptyset$ and $|N_T(x) \cap D| \geq 2$ for any vertex $x \in (N_T(v) \cap D) - \{u\}))$;*
      (b3) *$N_T(v) \cap D = \{u\}$ and $(PN_T[v, D] = \emptyset$ or $(PN_T[u, D] = \emptyset$ and $|N_T(x) \cap D| \geq 2$ for any vertex $x \in (N_T(u) \cap D) - \{v\}))$.*

Then $\mathrm{sd}_{\gamma_t}(T) > 1$.

Proof. Let $e = uv$ be an edge of the tree T. Let us subdivide the edge e with a vertex w. If $u \in \Omega(T)$, then there is a $\gamma_t(T)$-set D containing u and v. Thus, $(D - \{u\}) \cup \{w\}$ is a $\gamma_t(T_{uv})$-set and $\gamma_t(T) = \gamma_t(T_{uv})$.

Suppose that $\{u, v\} \cap \Omega(T) = \emptyset$.

If (a) holds, then D is also a $\gamma_t(T_{uv})$-set and again we obtain $\gamma_t(T) = \gamma_t(T_{uv})$.

Assume now (b) holds.

If condition (b1) holds, then D is also a $\gamma_t(T_{uv})$-set.

If condition (b2) holds, we have two cases. If $N_T(u) \cap D = \{v\}$ and $PN_T[u, D] = \emptyset$, then $(D - \{u\}) \cup \{w\}$ is a $\gamma_t(T_{uv})$-set. If $N_T(u) \cap D = \{v\}$ and $PN_T[v, D] = \emptyset$, and for any vertex $x \in (N_T(v) \cap D) - \{u\}$ we have $|N_T(x) \cap D| \geq 2$, then $(D - \{v\}) \cup \{w\}$ is a $\gamma_t(T_{uv})$-set.

Similarly, if condition (b3) holds.

In all the cases we have found a $\gamma_t(T_{uv})$-set of cardinality $\gamma_t(T)$. This implies that $\mathrm{sd}_{\gamma_t}(T) > 1$. ∎

Lemma 17. *Let T be a tree of order $n \geq 3$ having an inner edge $uv \in E(T)$ such that for any $\gamma_t(T)$-set D we have:*

(1) *if $|\{u, v\} \cap D| = 1$, let us say $u \in D$, then $v \in PN_T[u, D]$ and*

(2) *if $|\{u, v\} \cap D| = 2$, then $N_T(u) \cap D = \{v\}$ or $N_T(v) \cap D = \{u\}$, and if $N_T(u) \cap D = \{v\}$, then $PN_T[u, D] \neq \emptyset$ and $(PN_T[v, D] \neq \emptyset$ or $N_T(x) \cap D = \{v\}$ for a vertex $x \in (N_T(v) \cap D) - \{u\})$. Similarly if $N_T(v) \cap D = \{u\}$.*

Then $\mathrm{sd}_{\gamma_t}(T) = 1$.

Proof. We subdivide the edge uv with a vertex w. Let D' be a γ_t-set of T_{uv}.

Case 1. If $w \in D'$, then at least one of u, v belongs to D'.

Suppose $\{u, w, v\} \subseteq D'$, then $D' - \{w\}$ is a total dominating set of T and $\gamma_t(T) < \gamma_t(T_{uv})$.

Assume that $|\{u, v\} \cap D'| = 1$ and without loss of generality suppose $\{u, w\} \subseteq D'$. Thus, if $|N_{T_{uv}}(u) \cap D'| \geq 2$, then $D' - \{w\}$ is a total dominating set of T and $\gamma_t(T) < \gamma_t(T_{uv})$. In the other case, if $N_{T_{uv}}(u) \cap D' = \{w\}$, then $D = (D' - \{w\}) \cup \{v\}$ is a total dominating set of T such that $PN_T[v, D] = \emptyset$ and for any vertex $x \in (N_T(v) \cap D) - \{u\}$ we have $|N_T(x) \cap D| \geq 2$, so by hypothesis (2), $\gamma_t(T_{uv}) = |D| > \gamma_t(T)$.

Case 2. If $w \notin D'$, then we have two possibilities.

Assume that $|\{u, v\} \cap D'| = 1$ and, without loss of generality, $u \in D'$. Then D' is a total dominating set in T such that $v \notin PN_T[u, D']$ and by hypothesis (1), $\gamma_t(T_{uv}) = |D'| > \gamma_t(T)$.

If $\{u, v\} \subseteq D'$, then D' is total dominating set of T such that $|N_T(u) \cap D'| \geq 2$ and $|N_T(v) \cap D'| \geq 2$; again we have that $|D'| > \gamma_t(T)$. In all cases we obtained $\gamma_t(T_{uv}) > \gamma_t(T)$, what implies $\mathrm{sd}_{\gamma_t}(T) = 1$. ∎

It is straightforward that from Lemmas 2, 16 and 17 we have the next theorem.

Theorem 18. *Let T be a tree of order $n \geq 3$. Then $\mathrm{sd}_{\gamma_t}(T) = 1$ if and only if T has*

(1) *a leaf which does not belong to any $\gamma_t(T)$-set or*

(2) *an inner edge $uv \in E(T)$ such that for any $\gamma_t(T)$-set D*

    (i) *if $|\{u,v\} \cap D| = 1$, let us say $u \in D$, then $v \in PN_T[u,D]$ and*

    (ii) *if $|\{u,v\} \cap D| = 2$, then $N_T(u) \cap D = \{v\}$ or $N_T(v) \cap D = \{u\}$, and if $N_T(u) \cap D = \{v\}$, then $PN_T[u,D] \neq \emptyset$ and $\big(PN_T[v,D] \neq \emptyset$ or $N_T(x) \cap D = \{v\}$ for a vertex $x \in (N_T(v) \cap D) - \{u\}\big)$. For the case $N_T(v) \cap D = \{u\}$, conclusions are similar.*

In this paper we concentrated in the study of trees. The characterization of other infinity families of graphs with multisubdivision number equal 1, 2 or 3 remains an open problem.

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