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ON k-PATH PANCYCLIC GRAPHS

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Abstract

For integers k and n with $2 \leq k \leq n-1$, a graph G of order n is k-path pancyclic if every path P of order k in G lies on a cycle of every length from $k+1$ to n. Thus a 2-path pancyclic graph is edge-pancyclic. In this paper, we present sufficient conditions for graphs to be k-path pancyclic. For a graph G of order $n \geq 3$, we establish sharp lower bounds in terms of n and k for (a) the minimum degree of G, (b) the minimum degree-sum of nonadjacent vertices of G and (c) the size of G such that G is k-path pancyclic

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1. Introduction

A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G and a graph having a Hamiltonian cycle is a *Hamiltonian graph*. The first theoretical result on Hamiltonian graphs occurred in 1952 and is due to Dirac [7].

Theorem 1.1 (Dirac)**.** *If G is a graph of order $n \geq 3$ such that the minimum degree $\delta(G) \geq n/2$, then G is Hamiltonian.*

For a nontrivial graph G that is not complete, define

$$\sigma_2(G) = \min\{\deg u + \deg v : uv \notin E(G)\}$$

where $\deg w$ is the degree of a vertex w in G. For a connected graph G, let $\mathrm{diam}(G)$ denote the diameter of G (the largest distance between two vertices of

G). It is known that if G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n-1$, then G is connected and $\mathrm{diam}(G) \leq 2$. In 1960, Ore [11] obtained a result that generalizes Theorem 1.1.

Theorem 1.2 (Ore)**.** *If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then G is Hamiltonian.*

The following known result gives another sufficient condition for a graph to be Hamiltonian (see [6, p. 136]).

Theorem 1.3. *If G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 2$, then G is Hamiltonian.*

A *Hamiltonian path* in a graph G is a path containing every vertex of G. A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u-v$ path for every pair u, v of distinct vertices of G. Ore [12] also proved the following result in 1963.

Theorem 1.4 (Ore)**.** *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$, then G is Hamiltonian-connected.*

There is now an immediate corollary, similar in statement to the sufficient condition given in Theorem 1.1 for a graph to be Hamiltonian.

Corollary 1.5. *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.*

The following result, also due to Ore [12], is similar to the sufficient condition given in Theorem 1.3 for a graph to be Hamiltonian.

Theorem 1.6 (Ore)**.** *If G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$, then G is Hamiltonian-connected.*

Some 40–50 years ago, there was a great deal of research activity involving Hamiltonian properties of powers of graphs. For a connected graph G and a positive integer k, the *kth power* G^k of G is that graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \leq d_G(u,v) \leq k$ where $d_G(u,v)$ is the distance between two vertices u and v in G (or the length of a shortest $u-v$ path in G). The graph G^2 is called the *square* of G and G^3 is the *cube* of G. In 1960, Sekanina [14] proved that the cube of every connected graph G is Hamiltonian-connected and, consequently, the cube of G is Hamiltonian if its order is at least 3. In the 1960s, it was conjectured independently by Nash-Williams [10] and Plummer (see [6, p.139]) that the square of every 2-connected graph is Hamiltonian. In 1974, Fleischner [9] verified this conjecture. Also, in 1974 and using Fleischner's result, Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [4] proved that the square

of every 2-connected graph is Hamiltonian-connected. Thus the square of every Hamiltonian graph is Hamiltonian-connected.

A graph G of order n is *panconnected* if for every pair u, v of distinct vertices of G, there is a $u-v$ path of length k for every integer k with $d(u, v) \leq k \leq n-1$. It is shown in [1] that if G is a connected graph, then the cube of G is panconnected. For a connected graph G of order $n \geq 4$ and an integer k with $1 \leq k \leq n-3$, the graph G is *k-Hamiltonian* if $G - S$ is Hamiltonian for every set S of k vertices of G and *k-Hamiltonian-connected* if $G - S$ is Hamiltonian-connected for every set S of k vertices of G. If the order of a connected graph G is at least 4, then Chartrand and Kapoor [5] showed that the cube of G is 1-Hamiltonian.

The concepts of Hamiltonian cycles, Hamiltonian paths and Hamiltonian graphs are, of course, named for the famous Irish physicist and mathematician Sir William Rowan Hamilton. Hamilton observed that every path of order 5 on the graph G of the dodecahedron can be extended to a Hamiltonian cycle of G. That is, for every path P of order 5 in G, there exists a Hamiltonian cycle C of G such that P is a path on C. What Hamilton observed for paths of order 5 on the graph of the dodecahedron does not hold for all paths of order 6 as is illustrated in Figure 1 since the path of order 6 (drawn with bold edges) cannot be extended to a Hamiltonian cycle on the graph of the dodecahedron. This led to a concept defined in [3] for all Hamiltonian graphs.

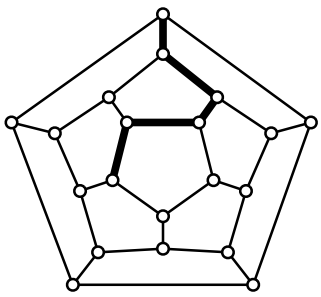

Figure 1. The graph G of the dodecahedron.

A Hamiltonian graph G of order $n \geq 3$ is *k-path Hamiltonian*, $k \geq 1$, if for every path P of order k, there exists a Hamiltonian cycle C of G such that P is a path on C. The *Hamiltonian cycle extension number* $\mathrm{hce}(G)$ of G is the largest integer k such that G is k-path Hamiltonian. So $1 \leq \mathrm{hce}(G) \leq n$. Therefore, if $\mathrm{hce}(G) = k$, then G is a Hamiltonian graph such that

(1) for every path P of order k, there is a Hamiltonian cycle of G containing P as a subgraph;

(2) for $k \leq n-1$, there is some path Q of order $k+1$ for which there is no Hamiltonian cycle of G containing Q as a subgraph.

Among the results obtained in [3] are the following.

Theorem 1.7 (Chartrand, Fujie and Zhang). *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then* $\mathrm{hce}(G) \geq 2\delta(G) - n + 1$.

The lower bound in Theorem 1.7 is sharp.

Theorem 1.8 (Chartrand, Fujie and Zhang). *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq rn$ for some rational number r with $1/2 \leq r < 1$, then* $\mathrm{hce}(G) \geq (2r-1)n + 1$.

The lower bound presented in Theorem 1.8 for the Hamiltonian cycle extension number of a graph is sharp for every rational number r. The following two theorems are extensions of Ore's results in Theorems 1.4 and 1.6. Again, the lower bounds in both Theorems 1.9 and 1.10 are best possible for every positive integer k.

Theorem 1.9 (Chartrand, Fujie and Zhang). *Let k and n be positive integers such that $n \geq k+2$. If G is a graph of order n and size $m \geq \binom{n-1}{2} + k + 1$, then G is k-path Hamiltonian.*

Theorem 1.10 (Chartrand, Fujie and Zhang). *Let k and n be positive integers such that $n \geq k+2$. If G is a graph of order n such that $\sigma_2(G) \geq n + k - 1$, then G is k-path Hamiltonian.*

Inspired by the concept of k-path Hamiltonian graphs, we introduce a concept of k-path pancyclic graphs and path pancyclic graphs. For integers k and n with $2 \leq k \leq n-1$, a graph G of order n is *k-path pancyclic* if every path P of order k in G lies on a cycle of every length from $k+1$ to n. In particular, a 2-path pancyclic graph G of order n is called an *edge-pancyclic* graph, that is, every edge of G lies on a cycle of length from 3 to n. A graph G of order $n \geq 3$ is *path pancyclic* if G is k-path pancyclic for each integer k with $2 \leq k \leq n-1$. In this paper, we present sufficient conditions for a graph to be k-path pancyclic in terms of its order, size, minimum degree as well as the sum of the degrees of every two nonadjacent vertices of the graph. We refer to the book [6] for graph theoretic notation and terminology not described in this paper.

2. A Minimum Degree Condition for k-Path Pancyclic Graphs

In this section, we establish a sufficient condition on the minimum degree of a graph G in terms of its order and a fixed integer k such that G is k-path pancyclic. We saw in Corollary 1.5 that if G is a graph of order $n \geq 4$ such that $\delta(G) \geq \frac{n+1}{2}$, then G is Hamiltonian-connected and therefore G is 2-path Hamiltonian. In fact, more can be said. First, we present a result due to Faudree and Schelp [8].

Theorem 2.1 (Faudree and Schelp). *If G is a graph of order $n \geq 5$ such that $\sigma_2(G) \geq n+1$, then for every pair u, v of distinct vertices of G, there is a $u-v$ path of length ℓ for every integer ℓ with $4 \leq \ell \leq n-1$.*

Theorem 2.2. *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$. If G is a graph of order n such that $\delta(G) \geq \frac{n+k-1}{2}$, then every path of order k lies on a cycle of length ℓ for each integer ℓ with $k+1 \leq \ell \leq n$ except possibly $k+2$.*

Proof. Let P be a path of order $k \geq 2$ in G, say $P = (u = v_1, v_2, \ldots, v_k = v)$ is a $u-v$ path. We consider two cases, according to whether $k = 2$ or $k \geq 3$.

Case 1. $k = 2$. Then $uv \in E(G)$. If $n = 4$, then $G = K_4$ and the result is true trivially. Thus, we may assume that $n \geq 5$. Since $\delta(G) \geq \frac{n+1}{2}$, it follows that $N(u) \cap N(v) \neq \emptyset$ and so there is $w \in V(G)$ such that (u, v, w, u) is a triangle in G. Hence uv lies on a cycle of length $\ell = 3$. Also, since $\delta(G) \geq \frac{n+1}{2}$, it follows by Theorem 2.1 that there is a $u-v$ path Q_ℓ of length ℓ for every integer ℓ with $4 \leq \ell \leq n-1$. Thus, uv lies on a cycle of length ℓ for each integer $\ell \in \{5, 6, \ldots, n\}$.

Case 2. $k \geq 3$. If $k = 3$ and $n = 4$, then $G = K_4$ and the result is true trivially. Thus, we may assume that $n \geq 5$. For each integer ℓ with $k+1 \leq \ell \leq n$ and $\ell \neq k+2$, we can write $\ell = (k-2) + \ell'$ for some ℓ' with $3 \leq \ell' \leq n-k+2$ and $\ell' \neq 4$. Then the graph $H = G - \{v_2, v_3, \ldots, v_{k-1}\}$ has order $n_H = n - (k-2)$ and the minimum degree

$$\delta(H) \geq \frac{n+k-1}{2} - (k-2) = \frac{[n-(k-2)]+1}{2} = \frac{n_H+1}{2}.$$

First, suppose that uv is an edge of H. It then follows by Case 1 that uv lies on a cycle $C_{\ell'}$ of order ℓ' for each integer ℓ' with $3 \leq \ell' \leq n-(k-2)$ and $\ell' \neq 4$. Then the $u-v$ path $C_{\ell'} - uv$ of $C_{\ell'}$ and P form a cycle of order $\ell = \ell' + (k-2)$ in G that contains P. Next, suppose that uv is not an edge of H. Then the graph $H' = H + uv$ has order $n-(k-2)$ and $\delta(H') \geq \delta(H)$. Again by Case 1, the edge uv lies on a cycle $C_{\ell'}$ of order ℓ' in H' for each integer ℓ' with $3 \leq \ell' \leq n-(k-2)$ and $\ell' \neq 4$. Similarly, the $u-v$ path $C_{\ell'} - uv$ of $C_{\ell'}$ and P form a cycle of order $\ell = \ell' + (k-2)$ in G that contains P. ■

The lower bound for the minimum degree of a graph in Theorem 2.2 cannot be improved. To see this, let n and k be integers of the same parity such that $n \geq k+2$ and let F_0, F_1, F_2 be three vertex-disjoint graphs where $F_0 = K_k$ is the complete graph of order k and $F_1 = F_2 = K_{(n-k)/2}$ are the complete graph of order $(n-k)/2$. The graph G is then constructed from F_0, F_1, F_2 by joining every vertex of F_0 to every vertex in F_1 and F_2. Then the order of G is n and $\delta(G) = \frac{n+k-2}{2}$. Observe that each path of order k in F_0 does not lie on any cycle of order n in G.

Recall that a graph G of order n is *panconnected* if for every pair u, v of distinct vertices of G, there is a $u-v$ path of length k for every integer k with $d(u,v) \leq k \leq n-1$. The following result was established by Williamson [15] in 1977.

Theorem 2.3 (Williamson). *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+2)/2$, then G is panconnected.*

With the same minimum degree condition, Randerath, Schiermeyer, Tewes and Nolkmann [13] showed that those graphs are edge-pancyclic in 2002.

Theorem 2.4 (Randerath, Schiermeyer, Tewes and Nolkmann). *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+2)/2$, then G is edge-pancyclic.*

For two vertices u and v in a connected graph G, a $u-v$ *geodesic* is a $u-v$ path of length $d(u,v)$ in G or a shortest $u-v$ path in G. A graph G of order n is defined in [2] to be *geodesic-pancyclic* if for each pair u, v of G, every $u-v$ geodesic lies on a cycle of length k for every k with $\max\{2d_G(u,v), 3\} \leq k \leq n$. In particular, a geodesic-pancyclic graph is edge-pancyclic. The following result is due to Chan, Chang, Wang and Horng (see [2]).

Theorem 2.5 (Chan, Chang, Wang and Horng). *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+2)/2$, then G is geodesic-pancyclic.*

Observe that if G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+2)/2$, then $\mathrm{diam}(G) \leq 2$ and so a $u-v$ geodesic in G is either the edge uv or a $u-v$ path of length 2. Therefore, the following result is an extension of the three theorems above.

Theorem 2.6. *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$. If G is a graph of order n such that $\delta(G) \geq \frac{n+k}{2}$, then G is k-path pancyclic.*

Proof. Let P be any path of order $k \geq 2$ in G, say $P = (u = v_1, v_2, \ldots, v_k = v)$ is a $u-v$ path. We consider two cases, according to whether $k = 2$ or $k \geq 3$.

Case 1. $k = 2$. Then $uv \in E(G)$. For each integer ℓ with $3 \leq \ell \leq n$, we can write $\ell = \ell' + 1$ for some $\ell' \geq 2$. Since $\delta(G) \geq (n+2)/2$, it follows by Theorem 2.3 that G is panconnected and so G contains a $u-v$ path $Q_{\ell'}$ of length ℓ' for each integer ℓ' with $1 = d_G(u,v) < \ell' \leq n-1$. Then $Q_{\ell'} + uv$ is a cycle of order ℓ.

Case 2. $k \geq 3$. For each integer ℓ with $k+1 \leq \ell \leq n$, we can write $\ell = \ell' + (k-1)$ for some integer ℓ' with $2 \leq \ell' \leq n-k+1$. Then the graph $H = G - \{v_2, v_3, \ldots, v_{k-1}\}$ has order $n_H = n - (k-2)$ and

$$\delta(H) \geq \frac{n+k}{2} - (k-2) = \frac{[n-(k-2)]+2}{2} = \frac{n_H+2}{2}.$$

Thus H is panconnected and furthermore $d_H(u,v) \leq 2$. Therefore, H contains a $u-v$ path $Q_{\ell'}$ of length ℓ' for each integer ℓ' with $2 \leq \ell' \leq n_H - 1$. Then $Q_{\ell'}$ and P form a cycle of order $\ell = \ell' + 1 + (k-2) = \ell' + (k-1)$ in G that contains P. ■

The lower bound for the minimum degree of a graph in Theorem 2.6 cannot be improved. To see this, let $k \geq 2$ be an integer and let $G = kK_k \vee \overline{K}_{k^2-k+1}$ be the join of kK_k and $\overline{K}_{k^2-k+1}$, where kK_k is the union of k vertex-disjoint copies of K_k. Then G is a k^2-regular graph of order $n = 2k^2 - k + 1$. Observe that $\delta(G) = \frac{n+k-1}{2} = k^2$. However, each path of order k in any subgraph K_k in G does not lie on a cycle of order $k+2$ in G.

3. On the Degree-Sum and Size Conditions for k-Path Pancyclic Graphs

In this section, we establish sufficient conditions on the degree-sum of nonadjacent vertices and the size of a graph G (in terms of its order and a fixed integer k) such that G is k-path pancyclic. We begin with the degree-sum condition. We saw in Theorem 1.4 that if G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$, then G is Hamiltonian-connected. It is known, however, that there are non-panconnected graphs G of order n such that $\sigma_2(G) \geq n+2$ (see [6, p. 133]). We illustrate this fact with the following example. Let $n = 2p+2$, where $p \geq 3$, and let $H = K_{2p}$ be the complete graph of order $2p$. Partition $V(H)$ into V_1 and V_2 with $|V_1| = |V_2| = p$. Define G to be the graph obtained by adding two adjacent vertices x and y to H and joining (1) x to every vertex in V_1 and (2) y to every vertex in V_2. Then $\deg x = \deg y = p+1$ and $\deg u = 2p$ for all $u \in V(G) - \{x,y\}$. Thus if u and v are two nonadjacent vertices in G, then $\deg u + \deg v = 2p+p+1 = (2p+2)+(p-1) \geq n+2$ since $p \geq 3$. However, there is no $x-y$ path of length 2 in the graph G. Therefore, G is not 2-path pancyclic since xy does not lie on a cycle of order 3 in G. In addition, if P is an $x-y$ path of order k for some integer k with $4 \leq k \leq n-1$, then P does not lie on a cycle of order $k+1$ in G. Thus, G is not k-path pancyclic. This example also illustrates the fact that there is no constant c such that if G is a graph of order n with $\sigma_2(G) \geq n+c$, then G is panconnected. Similarly, this example provides the following.

Proposition 3.1. *For any two integers k and n with $n \geq 4$ and $2 \leq k \leq n-1$, there is no constant c such that if G is a graph of order n with $\sigma_2(G) \geq n+c$, then G is k-path pancyclic.*

The two following results provide sufficient conditions on $\sigma_2(G)$ in terms of the order of a graph G such that G is panconnected and geodesic-pancyclic,

respectively (see [2, 15]).

Theorem 3.2 (Williamson). *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then G is panconnected.*

Theorem 3.3 (Chan, Chang, Wang and Horng). *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then G is geodesic-pancyclic.*

It can be shown that if G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then $\mathrm{diam}(G) \leq 2$. Therefore, the following result is an extension of these two theorems.

Theorem 3.4. *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$. If G is a graph of order n such that $\sigma_2(G) \geq \frac{3n+k-4}{2}$, then G is k-path pancyclic.*

Proof. By Theorem 3.2, the statement is true for $k = 2$. Thus, we may assume that $k \geq 3$. Let P be a path of order k in G, say $P = (x = v_1, v_2, \ldots, v_k = y)$ is an $x-y$ path. Let $H = G - \{v_2, v_3, \ldots, v_{k-1}\}$. The order of H is $n_H = n-(k-2) = n-k+2$. If u and v are any two nonadjacent vertices of H, then

$$\deg_H u + \deg_H v \geq \frac{3n+k-4}{2} - 2(k-2) = \frac{3(n-k+2)-2}{2} = \frac{3n_H - 2}{2}.$$

Thus H is panconnected by Theorem 3.2 and furthermore $d_H(x,y) \leq 2$. Therefore, H contains an $x-y$ path $Q_{\ell'}$ of length ℓ' for each integer ℓ' with $2 \leq \ell' \leq n_H - 1 = n-k+1$. Then $Q_{\ell'}$ and P form a cycle of order $\ell = \ell' + 1 + (k-2) = \ell' + (k-1)$ in G that contains P for each ℓ with $k+1 \leq \ell \leq n$. ■

If $2 \leq k \leq n-2$, then Theorem 3.4 can also be verified with the aid of Theorem 2.6 as follows. Assume, to the contrary, that G is not k-path pancyclic. It then follows by Theorem 2.6 that there is a vertex u in G such that $\deg u < \frac{n+k}{2} \leq n-1$ (since $k \leq n-2$). Thus there is a vertex v in G such that u and v are nonadjacent and so $\deg_G v \leq n-2$. However then, $\deg_G u + \deg_G v < \frac{n+k}{2} + (n-2) = \frac{3n+k-4}{2}$, which is a contradiction.

The lower bound $(3n+k-4)/2$ in Theorem 3.4 for the sum of the degrees of two nonadjacent vertices of a graph cannot be replaced by $(3n+k-6)/2$. For example, let $H = K_{2p}$ be the complete graph of order $2p$ for some integer $p \geq 3$, let $F = K_k$ be the complete graph of order $k \geq 3$ and let $P = (x = v_1, v_2, \ldots, v_k = y)$ be an $x-y$ Hamiltonian path in F. Partition $V(H)$ into V_1 and V_2 with $|V_1| = |V_2| = p$. Define G to be the graph obtained by (1) joining x to every vertex in V_1, (2) joining y to every vertex in V_2, and (3) joining each vertex v_i $(2 \leq i \leq k-1)$ to every vertex in H. Then the order of G is $n = 2p+k$. Furthermore, $\deg_G x = \deg_G y = p + (k-1)$ and $\deg z = (2p-1) + (k-1) = 2p+k-2$ for all $z \in V(G) - V(F)$. Thus if u and v are two nonadjacent vertices

in G, then $\deg_G u + \deg_G v = (2p+k-2)+(p+k-1) = 3p+2k-3 = \frac{3n+k-6}{2}$. Observe that the path P of order k does not lie on a cycle of order $k+1$ in G and so G is not k-path pancyclic. On the other hand, it is not known if the lower bound $(3n+k-4)/2$ in Theorem 3.4 for the sum of the degrees of two nonadjacent vertices of a graph cannot be replaced by $(3n+k-5)/2$ when n and k are of opposite parity.

The following two results provide sufficient conditions on the size of a graph G such that G is panconnected and geodesic-pancyclic, respectively (see [2, 15]).

Theorem 3.5 (Williamson)**.** *If G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2}+3$, then G is panconnected.*

Theorem 3.6 (Chan, Chang, Wang and Horng)**.** *If G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2}+3$, then G is geodesic-pancyclic.*

Since the diameter of a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2}+3$ is at most 2, the following is an extension of Theorems 3.5 and 3.6 as well as Theorem 1.9.

Theorem 3.7. *Let k and n be positive integers such that $n \geq k+2$. If G is a graph of order n and size $m \geq \binom{n-1}{2}+k+1$, then G is k-path pancyclic.*

Proof. By Theorem 3.5, the statement is true for $k=2$. Thus, we may assume that $k \geq 3$. Let G be a graph of order $n \geq k+2$ and size $m \geq \binom{n-1}{2}+k+1$ and let $P = (u = v_1, v_2, \ldots, v_k = v)$ be a path of order k in G. Let $H = G - \{v_2, v_3, \ldots, v_{k-1}\}$. Thus H has order $n_H = n-k+2$ and size

$$\begin{aligned} m_H &\geq \binom{n-1}{2} + k + 1 - [(n-1)+(n-2)+\cdots+(n-k+2)] \\ &= \binom{n-k+1}{2} + 3 = \binom{n_H-1}{2} + 3. \end{aligned}$$

Thus H is panconnected by Theorem 3.5 and furthermore $d_H(x,y) \leq 2$. Therefore, H contains a $u-v$ path $Q_{\ell'}$ of length ℓ' for each integer ℓ' with $2 \leq \ell' \leq n_H - 1 = n-k+1$. Then $Q_{\ell'}$ and P form a cycle of order $\ell = \ell'+1+(k-2) = \ell'+(k-1)$ in G that contains P for each ℓ with $k+1 \leq \ell \leq n$. ■

The bound on the size m of a graph in Theorem 3.7 cannot be improved. To see this, let G be a graph of order $n \geq k+2 \geq 4$ consisting of a complete subgraph G' of order $n-1$, where $V(G') = \{v_1, v_2, \ldots, v_{n-1}\}$ and another vertex v adjacent to $v_1, v_2, \ldots, v_k$. Then the size of G is $m = \binom{n-1}{2}+k$. However, the path $P = (v_1, v_2, \ldots, v_k)$ of order k lies on no Hamiltonian cycle of G. Hence P cannot be extended to a cycle of order n in G. Thus G is not k-path pancyclic.

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